Jordan-Canonical Form:

Idea: $G(s) = \frac{p(s)}{q(s)}$; $\text{ord}\{p\} < \text{ord}\{q\}$

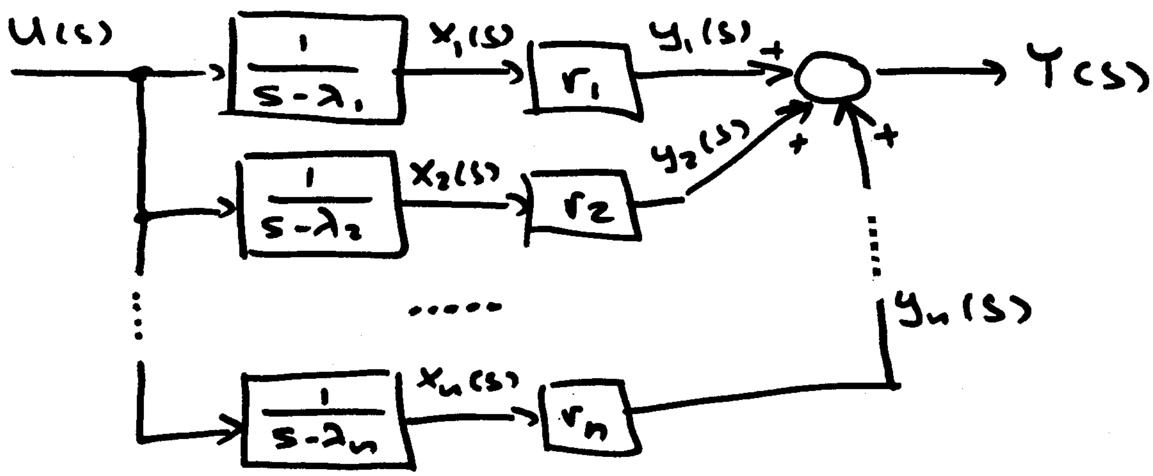
can be developed into a partial fraction expansion. We want to distinguish 3 cases:

(1) All roots of $q(s)$ are distinct and real:

$$\Rightarrow \quad \frac{Y(s)}{U(s)} = G(s) = \sum_{i=1}^{n} \frac{r_i}{s-\lambda_i}$$

λ_i := roots of $q(s)$
r_i := residua belonging to these roots

In a block-digram:



$$\Rightarrow \quad X_i(s) = \frac{1}{s-\lambda_i} U(s)$$

$$\Rightarrow \quad sX_i(s) - \lambda_i X_i(s) = U(s)$$

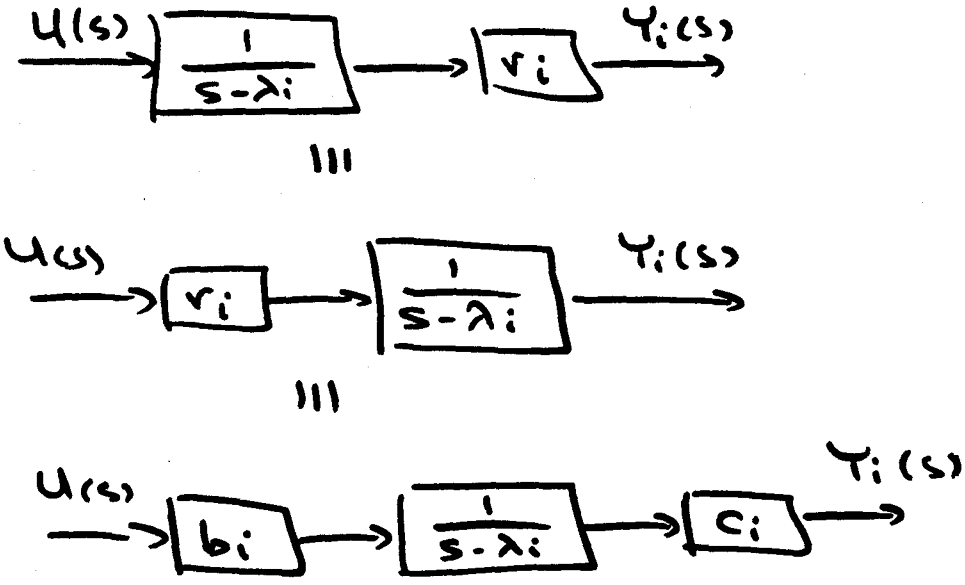
$$\dot{x}_i(t) - \lambda_i x_i(t) = u(t)$$

$$\Rightarrow \boxed{\dot{x}_i = \lambda_i x_i + u} \quad ; \forall i \in [1, n]$$

$$\Rightarrow \boxed{\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} \lambda_1 & \phi & \cdots & - & \phi \\ \phi & \lambda_2 & \phi & \cdots & \phi \\ & & \ddots & & \phi \\ & \phi & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y &= [r_1 \ r_2 \ \cdots \ r_n] \underline{x} + [d] u \end{aligned}}$$

is called the Jordan-canonical form.

- We realize that we could have moved the r_i also to the left side:



$r_i = b_i \cdot c_i$

$\Rightarrow$ The Jordan-canonical form is not totally unique, b_i and c_i are only determined up to their product:

$$\underline{\underline{r_i = b_i \cdot c_i}}$$

thus, e.g.

$$\underline{\dot{x}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \Phi \\ & & \ddots & \\ & \Phi & & \lambda_n \end{bmatrix} \underline{x} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} u$$

$$y = [1 \ 1 \ \cdots \ 1] \, \underline{x} + [d] u$$

would be another such representation.

- In the Jordan-canonical form, we can determine the influence of the input on each state (or the influence of each state on the output) separately by tuning b_i and c_i.

- The Jordan-canonical form provides for complete input/output decoupling. Each state variable depends only on itself and the input:

$$\dot{x}_i = \lambda_i x_i + u$$

but not on any other state:

$$x_i \neq f(x_j) \,; \quad j \neq i \quad .$$

- We see at once that uncontrollable states will show up as ϕ-entries into the $\underline{b}$-vector, while unobservable modes will appear as ϕ-entries into the $\underline{c}'$-vector (by inspection)

Question:

Can we find a similarity transformation that will get us into the Jordan-canonical form?

Answer:

Let us look at the definition of Eigenvalues and Eigenvectors:

$$A\underline{v}_i = \lambda_i \underline{v}_i \quad ; \forall\, i \in \{1, n\}$$

Let us create a matrix V consisting of a concatenation of all eigenvectors:

$$V = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n]$$

This $(n \times n)$-matrix is often referred to as the <u>right Modal matrix</u>.

$$\Longrightarrow \quad A \cdot V = [\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2, \ldots, \lambda_n \underline{v}$$

e.g.: $n=3$:

$$A \cdot V = [\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2, \lambda_3 \underline{v}_3]$$

$$= \begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} & \lambda_3 v_{13} \\ \lambda_1 v_{21} & \lambda_2 v_{22} & \lambda_3 v_{23} \\ \lambda_1 v_{31} & \lambda_2 v_{32} & \lambda_3 v_{33} \end{bmatrix}$$

$$\equiv \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{bmatrix}$$

In general:

$$A \cdot V \equiv V \cdot \Lambda$$

where: $\Lambda = \begin{bmatrix} \lambda_1 & & & \phi \\ & \lambda_2 & & \\ & & \ddots & \\ \phi & & & \lambda_n \end{bmatrix}$

We notice that Λ is the new system matrix in Jordan canonical form, thus:

$$\hat{A} = \Lambda = V^{-1} \cdot A \cdot V$$
$$= T \cdot A \cdot T^{-1}$$

$\Longleftrightarrow$ $\boxed{T = V^{-1}}$

- Application of a similarity transformation with the inverse Modal matrix will get us into Jordan form

- The length of the eigen= vectors does not matter (though normalization helps with respect to numerical properties). The Λ-matrix will not change, but the distribution of the residua onto $\underline{b}$ and $\underline{c}'$ will.

(1) $\Longrightarrow$ Normalize $\underline{v}_i \Longleftrightarrow |\underline{v}_i| = 1$

$$\Longleftrightarrow \|V\|_2 = 1$$

(L2-norm of matrix V

(2) $\Longrightarrow$ $T = V^{-1}$

(3) $\Longrightarrow$ $\Lambda = T \cdot A / T \;;\; \hat{\underline{b}} = T \cdot \underline{b} \;;$
$\hat{\underline{c}}' = \underline{c}' / T$

(4) $\Longrightarrow$ $r_i = \hat{b}_i \cdot \hat{c}_i$

(5) $\Longrightarrow$ $\hat{\hat{b}}_i = 1 \;;\; \hat{\hat{c}}_i = r_i$
except for any ϕ-entries

In Matlab:

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[V, Lambda] = eig(A);
T = inv(V);
Ah = Lambda;
bh = T*b;
ch = c*V;
chh = ch .* conj(bh');        <- real transpose
            ^ elementwise multiplicati
bhh = ones(size(bh));
n = length(b);
for i = 1:n,
    if bh(i) == 0,
       bhh(i) = 0;
       chh(i) = ch(i);
    end,
end
```

We could have operated equally well on the left eigenvectors:

$$\underline{w}_i' A = \lambda_i \cdot \underline{w}_i'$$

We build a <u>left</u> <u>Modal Matrix</u> :

$$W = \left[\underline{w}_1' ; \underline{w}_2' ; \cdots ; \underline{w}_n' \right]$$

$$\Rightarrow W \cdot A = \left[\lambda_1 \underline{w}_1' ; \lambda_2 \underline{w}_2' ; \cdots ; \lambda_n \underline{w}_n' \right]$$

$$= \Lambda \cdot W$$

$$\Rightarrow \Lambda = \hat{A} = W \cdot A \cdot W^{-1}$$

$$= T \cdot A \cdot T^{-1}$$

$$\Rightarrow \boxed{T = W}$$

works equally well.

$$\Rightarrow \boxed{W \cong V^{-1}}$$

except for the length of the eigenvectors.

- Notice : If $\|V\|_2 = 1$, you cannot conclude that

$$\|V^{-1}\|_2 = 1$$

$$\Rightarrow \boxed{W \approx V^{-1}}$$

Spectral Decomposition :

We can turn the argument over:

$$A \cdot V = V \cdot \Lambda$$

$$\Rightarrow \boxed{A = V \cdot \Lambda \cdot V^{-1}}$$

is called the spectral decomposition of matrix A into its Modal matrix and its eigenvalue matrix which is often called the Jordan matrix and the inverse modal matrix.

Similarly:

$$W \cdot A = \Lambda \cdot W$$

$$\Rightarrow \boxed{A = W^{-1} \cdot \Lambda \cdot W}$$

Notice:

$$A \neq V \cdot \Lambda \cdot W$$

due to the normalization problem !!!

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(2) All roots are distinct, but some appear as conjugate complex pairs:

$$\lambda_j = \overline{\lambda_i} = \mathrm{conj}(\lambda_i)$$

Let us regroup the eigenvalues such that conjugate complex pairs appear next to each other:

$$\begin{aligned} \lambda_i &= \pm\alpha + j\beta \\ \lambda_{i+1} &= \pm\alpha - j\beta \end{aligned}$$

Of course, the transformation from before works just as well, but presents us now with complex state-variables:

$$\dot{\underline{x}} = \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & (\alpha+j\beta) & & \\ & & & (\alpha-j\beta) & \\ & & & & \ddots \\ & & & & & \lambda_n \end{bmatrix} \underline{x} + \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} u$$

$$y = \begin{bmatrix} r_1 & \cdots & (\gamma+j\delta) & (\gamma-j\delta) & \cdots & r_n \end{bmatrix} \underline{x} + [d]\, u$$

Obviously, also x_i and x_{i+1} will be conjugate complex pairs, thus:

$$\left| \begin{array}{l} x_i = \xi + j\eta \\ x_{i+1} = \xi - j\eta \end{array} \right|$$

$\Rightarrow$ $\dot{x}_i = \lambda_i x_i + u$

can be written as:

$$\dot{\xi} + j\dot{\eta} = (\alpha + j\beta)(\xi + j\eta) + u$$
$$= (\alpha\xi - \beta\eta) + j(\alpha\eta + \beta\xi).$$

$$\Rightarrow \left| \begin{array}{l} \dot{\eta} = \alpha\eta + \beta\xi \\ \dot{\xi} = -\beta\eta + \alpha\xi + u \end{array} \right|$$

We can thus replace the two complex state-variables

$$\begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix}$$

by the two <u>real</u> state-variables

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

and find:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{\eta} \\ \dot{\xi} \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \alpha & \beta & \\ & & & -\beta & \alpha & \\ & & & & & \ddots \\ & & & & & & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \eta \\ \xi \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} r_1 & r_2 & \cdots & -2\delta & +2\gamma & \cdots & r_n \end{bmatrix} \cdot \underline{x} + [d]$$

$\Longrightarrow$ Thus, we can replace the complex state-space representation by a real state-space representation for the price of having the two new state-equations coupled rather than decoupled. $\Rightarrow \Lambda$ is no longer diagonal, but block-diagonal.

Notice: This is not needed, just possibly more convenient. CTRL-C does not care whether your numbers are real or complex.

Question: Which similarity transformation gets us from the complex to the real Jordan form?

$$\begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} j\eta + \xi \\ -j\eta + \xi \end{bmatrix} = T_i^{-1} \cdot \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

$$\Rightarrow T_i^{-1} = \begin{bmatrix} j & 1 \\ -j & 1 \end{bmatrix}$$

$\Rightarrow$ The complex similarity transformation with:

$$T^{-1} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \emptyset \\ & & \ddots & & & \\ & & & \boxed{\begin{matrix} j & 1 \\ -j & 1 \end{matrix}} & & \\ & \emptyset & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

(block-diagonal) will transfer our system from the complex Jordan form to the real Jordan form

- Of course, there will be one such diagonal block for each of the conjugate complex pairs.

- <u>In Matlab</u>:

$$j = \text{sqrt}(-1)$$

exists as a system-defined variable either referrable to as $\hat{i}$ or as $\hat{j}$, depending on your preferences.

- Ordering state equations:

e.g.
$$\left|\begin{array}{l} \xi_1 = x_5 \\ \xi_2 = x_3 \\ \xi_3 = x_1 \\ \xi_4 = x_4 \\ \xi_5 = x_2 \end{array}\right|$$

can be achieved by a similarity transformation with a permutation matrix:

$$\underline{\xi} = T \cdot \underline{x} \quad ; \quad T = \begin{bmatrix} \phi & \phi & \phi & \phi & 1 \\ \phi & \phi & 1 & \phi & \phi \\ 1 & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & 1 & \phi \\ \phi & 1 & \phi & \phi & \phi \end{bmatrix}$$

Warning: In Matlab, the '-operator actually denotes <u>not</u> the transpose, but the conjugate complex transpose, thus:

$$A = \begin{bmatrix} (5+j) & (-3+2j) \\ (-j) & 27 \end{bmatrix}$$

$$\Rightarrow \quad A' = \begin{bmatrix} (5-j) & (j) \\ (-3-2j) & 27 \end{bmatrix}$$

which is often called the Hermitian transpose. In mathematics, it is usually written as: A^*.

$\Rightarrow$ The real transpose can be written as:

$$\text{Atrans} = \text{conj}(A')$$

(3) Some roots are multiple, but there are still n eigenvectors to be found.

Example:

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = [c_1 \ c_2 \ c_3] \underline{x}$$

$$\Rightarrow V = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

(according to page 110), or written in the normalized version:

$$V = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

This is a nonsingular matrix

$$\Longrightarrow T = V^{-1} \text{ exists.}$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{\xi}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\xi} + \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} u \\ y &= [\hat{c}_1 \;\; \hat{c}_2 \;\; \hat{c}_3]\, \underline{\xi} \end{aligned} \right|$$

is the Jordan-canonical form.

- The length of the eigenvectors will not effect the Λ-matrix, but it will effect the elements $\hat{b}_i$ and $\hat{c}_i$ which are only determined through their products $\hat{b}_i \cdot \hat{c}_i$.
- As long as none of the $\hat{b}_i$ is zero, we can normalize the $\underline{\hat{b}}$-vector to one:

$$\Rightarrow \left| \begin{aligned} \dot{\underline{\eta}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\eta} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \\ y &= [\hat{\hat{c}}_1 \;\; \hat{\hat{c}}_2 \;\; \hat{\hat{c}}_3]\, \underline{\eta} \end{aligned} \right|$$

Let us take a more general case:

$$A = \begin{bmatrix} \boxed{\begin{matrix}\lambda_1 & \\ & \lambda_1\end{matrix}} & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \lambda_3 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \underline{\underline{\text{same}}} \\ \\ \\ \end{matrix}$$

- Q_c of a diagonal form is a <u>Van-der-Monde matrix</u> over the set of eigenvalues.
- If any eigenvalue occurs more than once $\Longrightarrow$ a row will be repeated $\Longrightarrow$ Q_c has not full rank $\Longrightarrow$ System is not controllable

We could, of course, also have normalized the output vector:

$$\left|\begin{aligned} \underline{\dot{v}} &= \begin{bmatrix} \lambda_1 & & & \phi \\ & \lambda_1 & & \\ \phi & & \lambda_2 & \\ & & & \lambda_3 \end{bmatrix} \underline{v} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \underline{v} \end{aligned}\right.$$

$$\Rightarrow Q_o = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix}$$

$\underbrace{\hspace{3em}}_{\text{same}}$ (first two columns)

- Q_o of any diagonal form is a Van-der-Monde matrix over the set of eigenvalues.
- If any eigenvalue is repeated $\Longrightarrow$ a column is repeated $\Longrightarrow$ Q_o has not full rank $\Rightarrow$ System is