

Numerical Simulation of Dynamic Systems: Hw4 - Solution

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[H3.15] Backinterpolation With Step-Size Control

We want to repeat Hw.[H3.14] once more, this time using a step-size controlled algorithm. The step-size control to be used is the following. On the *explicit semi-step*, compute now both correctors, and find ε_{rel} according to the formula:

$$\varepsilon_{\text{rel}} = \frac{\|\mathbf{x}_1 - \mathbf{x}_2\|_{\infty}}{\max(\|\mathbf{x}_1\|_2, \|\mathbf{x}_2\|_2, \delta)}$$

[H3.15] Backinterpolation With Step-Size Control

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If $\varepsilon_{\text{rel}} \leq 10^{-4}$, use the Gustafsson algorithm to compute the step size to be used in the next step:

$$h_{\text{new}} = \left(\frac{0.8 \cdot 10^{-4}}{\varepsilon_{\text{rel}_{\text{now}}}} \right)^{0.06} \cdot \left(\frac{\varepsilon_{\text{rel}_{\text{last}}}}{\varepsilon_{\text{rel}_{\text{now}}}} \right)^{0.08} \cdot h_{\text{old}}$$

except during the first step, when we use:

$$h_{\text{new}} = \left(\frac{0.8 \cdot 10^{-4}}{\varepsilon_{\text{rel}_{\text{now}}}} \right)^{0.2} \cdot h_{\text{old}}$$

[H3.15] Backinterpolation With Step-Size Control II

However, if $\epsilon_{\text{rel}} > 10^{-4}$, we reject the step at once, i.e., we never even proceed to the implicit semi-step, and compute a new step size in accordance with the same equation as during the first step.

[H3.15] Backinterpolation With Step-Size Control II

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If a step was repeated, the step size for the immediately following next step is also computed according to that equation.

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Apply this step-size control algorithm to the same problem as before, and determine the largest global relative error by comparing the solution with the analytical solution of this linear time-invariant system.

[H3.15] Backinterpolation With Step-Size Control III

We start out by implementing the step-size control in the forward semi-step of the *BI4/5_{0.45}* code:

```
function [xnew, err, tnew, hnew] = bi45tv_step(x, t, h, errl, theta, tol)
    err = 2 * tol;
    rep = 0;
    while err > tol,
        [x_left4, x_left5] = rkf45_step(x, t, theta * h);
        err = norm(x_left4 - x_left5, 'inf') / max([norm(x_left4), norm(x_left5), 1.0e - 10]);
        if err > tol,
            h = (0.8 * tol / err) ^ (0.2) * h;
            rep = 1;
        else
            if errl > 0 & rep == 0,
                hnew = (0.8 * tol / err) ^ (0.06) * (errl / err) ^ (0.08) * h;
            else
                hnew = (0.8 * tol / err) ^ (0.2) * h;
            end,
        end,
    end,
    end
    if rep == 1,
        err = -err;
    end
    tnew = t + h;
    xnew = x_left4;
```

[H3.15] Backinterpolation With Step-Size Control IV

```

err2 = tol;
H = hessian(x, t, (theta - 1) * h, 5);
while err2 > 0.1 * tol,
    [dummy, x_right] = rkf45_step(xnew, t + h, (theta - 1) * h);
    xnew = xnew - H \ (x_right - x_left4);
    err2 = norm(x_right - x_left4, 'inf') / max([norm(x_left4), norm(x_right), 1.0e - 10]);
end
return

```

The backward semi-step is still the same as before.

[H3.15] Backinterpolation With Step-Size Control V

We also need the analytical solution of this linear system. We compute this in the *frequency domain*, i.e., using *Laplace transform*.

We start by building a system in the time domain:

$$S = \text{ss}(A, b, c, d);$$

We convert to the frequency domain by computing the *transfer function*:

$$G = \text{tf}(S);$$

The Laplacian of a step input is $\frac{1}{s}$:

$$\begin{aligned} Pu &= 1; \\ Qu &= [1 \ 0]; \\ U &= \text{tf}(Pu, Qu); \end{aligned}$$

The *input response* in the frequency domain is the product of the transfer function and the input signal:

$$Y = G * U;$$

[H3.15] Backinterpolation With Step-Size Control VI

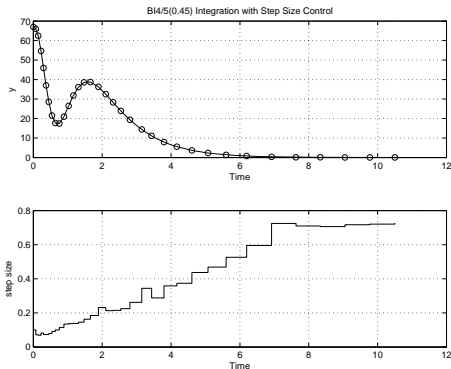
We need to convert the input response back into the time domain. To this end, we perform a *partial fraction expansion*:

```
[Py, Qy] = tfdata(Y, 'v');  
[r2, l2] = residue(Py, Qy);
```

The analytical solution is the superposition of the *input response* and the *initial state response*:

```
ycorr = zeros(size(yvec));  
for i = 1 : nm,  
    t = tvec(i);  
    y = sum( r2 .* exp(l2 * t) ) + c*expm(A * t) * x0;  
    ycorr(i) = y;  
end
```

[H3.15] Backinterpolation With Step-Size Control VII



[H3.15] Backinterpolation With Step-Size Control VIII

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This is quite good. We wanted to get $tol = 1.0e - 005$, i.e., the actual *global error* is about three times larger than the desired error, but then again, all we do is to control the *local error*. We simply assumed that the local error would be roughly one order of magnitude smaller than the global error (rule of thumb), and therefore, we checked for $err > 0.1 * tol$.

[H3.19] Order Star

Find the *damping order star* for **B14/5**_{0.45}, and plot it together with the pole and zero locations. Compare with the damping order star of **B14** that was shown in class.

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Find the *frequency order star* for **BI4**/**5**_{0.45}, and plot it together with the pole and zero locations. Compare with the frequency order star **BI4** that was shown in class.

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Find the *frequency order star* for **BI4/5**_{0.45}, and plot it together with the pole and zero locations. Compare with the frequency order star **BI4** that was shown in class.

Finally, compute and plot the *order star accuracy domain* of this method.

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Find the *frequency order star* for **B14**/**5**_{0.45}, and plot it together with the pole and zero locations. Compare with the frequency order star **B14** that was shown in class.

Finally, compute and plot the *order star accuracy domain* of this method.

For this problem, it may be easier to use MATLAB's *contour* plot, than your own domain tracking routine.

[H3.19] Order Star II

We need to compute the three-dimensional function $\hat{\sigma}_d(\sigma_d, \omega_d)$. We place a roster of values on the two-dimensional plane spanned by σ_d and ω_d , and compute $\hat{\sigma}_d$ at each roster point:

```

j = sqrt(-1);
svec = zeros(161, 201);
s = [-10 : 0.1 : 10];
w = [-8 : 0.1 : 8];
for i = 1 : 201,
    ss = s(i);
    for k = 1 : 161,
        ww = w(k);
        lambd = ss + j * ww;
        shat = -damp(-lambd, algor);
        svec(k, i) = ss - shat;
    end
end

```

[H3.19] Order Star III

The function **damp** computes the numerical damping:

```
function shat = damp(a, algor)
    f = ff(-a, 1, algor);
    shat = -log(abs(f));
return
```

whereby the function **ff** computes the **F**-matrix of the ODE solver:

```
function [F] = ff(A, h, algor)
    I = eye(size(A));
    Ah = A * h;
    theta = 0.45;
    Af = Ah * theta;
    Af2 = Af * Af; Af3 = Af2 * Af;
    Af4 = Af3 * Af; Af5 = Af4 * Af;
    Ff = I + Af + Af2/2 + Af3/6 + Af4/24 + Af5/120;
    Ab = Ah * (1 - theta);
    Ab2 = Ab * Ab; Ab3 = Ab2 * Ab; Ab4 = Ab3 * Ab;
    Ab5 = Ab4 * Ab; Ab6 = Ab5 * Ab;
    Fb = I - Ab + Ab2/2 - Ab3/6 + Ab4/24 - Ab5/120 + Ab6/720;
    F = Fb\Ff;
return
```

based on the $f_4(q)$ and $f_5(q)$ functions that I presented in class for the *RKF4/5* algorithm.

[H3.19] Order Star IV

We also need to compute the poles and zeros of $f(q)$ for the $BI4/5_{0.45}$ algorithm:

```
function [p, z] = pz(algor)
    z = tf('z', 1);
    theta = 0.45;
    zf = z * theta;
    zf2 = zf * zf;  zf3 = zf2 * zf;
    zf4 = zf3 * zf;  zf5 = zf4 * zf;
    Gf = 1 + zf + zf2/2 + zf3/6 + zf4/24 + zf5/104;
    zb = z * (1 - theta);
    zb2 = zb * zb;  zb3 = zb2 * zb;  zb4 = zb3 * zb;
    zb5 = zb4 * zb;  zb6 = zb5 * zb;
    Gb = 1 - zb + zb2/2 - zb3/6 + zb4/24 - zb5/120 + zb6/2080;
    G = Gb \ Gf;
    G = zpk(G);
    [z, p, k] = zpkmdata(G, 'v');
return
```

[H3.19] Order Star V

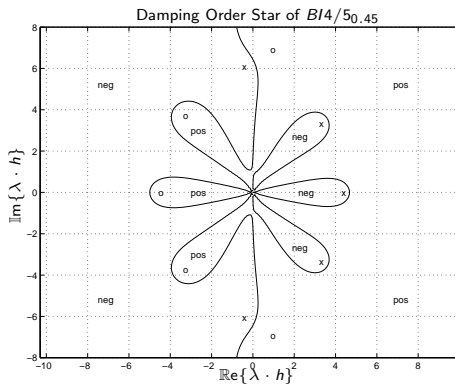
We could now cut the three-dimensional function $\hat{\sigma}_d(\sigma_d, \omega_d)$ with a horizontal plane going through the origin. This would give us another way to plot the stability domain of the method. Matlab's **contour** function will do just that:

```
contour( $\sigma_d, \omega_d, \hat{\sigma}_d, [0 \ 0], 'k-')$ 
```

However, we want to get the damping order star, which can be computed in the same fashion:

```
contour( $\sigma_d, \omega_d, \varepsilon_\sigma, [0 \ 0], 'k-')$ 
```

[H3.19] Order Star VI

Figure: Damping order star of $BI4/5_{0.45}$ method

[H3.19] Order Star VII

The rational function $f(q)$ has six poles and five zeros, located at:

$p =$

21.8413
 $-0.5210 + 6.0863i$
 $-0.5210 - 6.0863i$
 4.2886
 $3.2137 + 3.3423i$
 $3.2137 - 3.3423i$

and:

$z =$

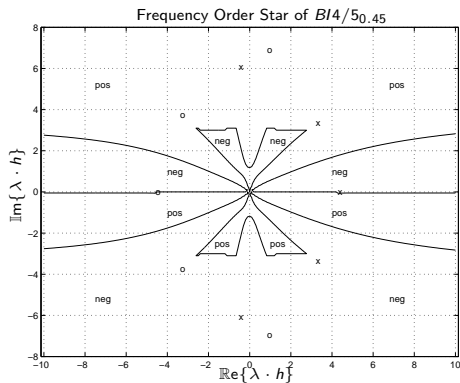
$0.8496 + 6.9172i$
 $0.8496 - 6.9172i$
 -4.5705
 $-3.3792 + 3.7377i$
 $-3.3792 - 3.7377i$

[H3.19] Order Star VIII

The frequency order star can be computed in the same fashion. We now need to compute the three-dimensional function $\hat{\omega}_d(\sigma_d, \omega_d)$. To this end, we need a function **freq**:

```
function ohat = freq(a, algor)
    f = ff(a, 1, algor);
    ohat = atan2(imag(f), real(f));
return
```

[H3.19] Order Star IX

Figure: Frequency order star of $B/4/5_{0.45}$ method

[H3.19] Order Star X

The order star accuracy domain is once again computed in the same fashion. Now, we compute OS_{err} at each roster point:

```
shat = -damp(-lambd, algor);
ohat = atan2(imag(f), real(f));
errvec(k, i) = abs(ss - shat) + abs(ww - ohat);
```

and plot:

```
tol = 1.0e - 4;
contour(s, w, errvec, [tol tol], 'k-')
hold on
tol = 1.0e - 3;
contour(s, w, errvec, [tol tol], 'k-')
tol = 1.0e - 2;
contour(s, w, errvec, [tol tol], 'k-')
```

[H3.19] Order Star XI

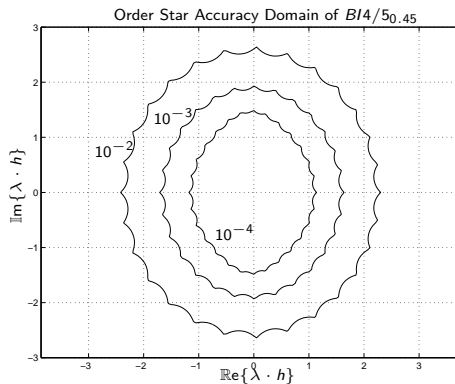


Figure: Order star accuracy domain of $BI4/5_{0.45}$ method