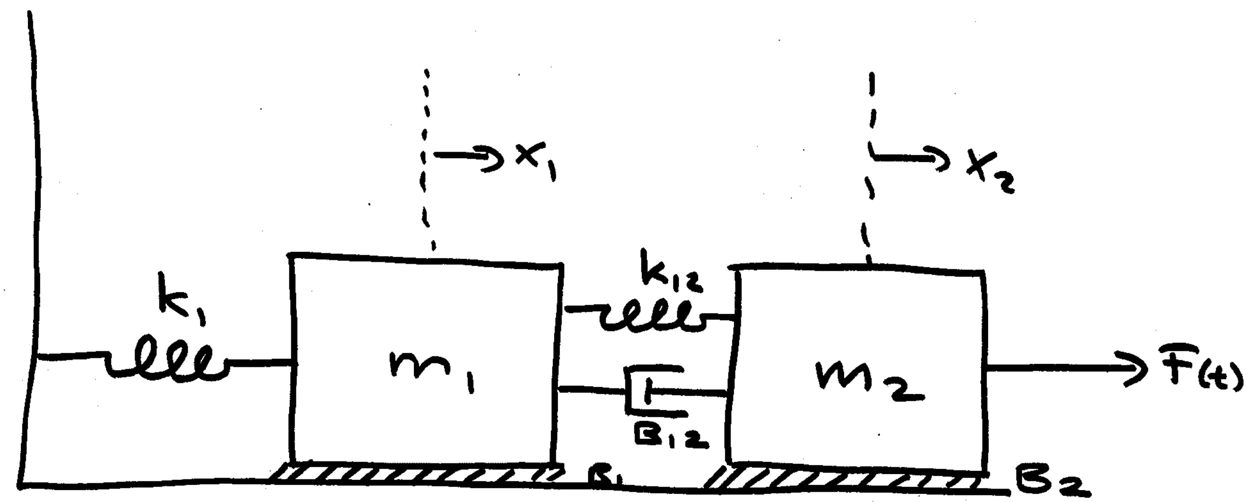
State-space Models:

Example:



Differential Equations:

$$m_2 \frac{d^2 x_2}{dt^2} = F(t) - B_2 \cdot \frac{dx_2}{dt} - B_{12}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) - k_{12}(x_2 - x_1)$$

$$m_1 \frac{d^2 x_1}{dt^2} = B_{12}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + k_{12}(x_2 - x_1) - B_1 \frac{dx_1}{dt} - k_1 x_1$$

State Variables:

Always choose the outputs of all integrators as your state variables:

$$\xrightarrow{\frac{d^2x_1}{dt^2}} \boxed{\int} \xrightarrow{\frac{dx_1}{dt}} \boxed{\int} \xrightarrow{x_1}$$

$\Rightarrow$ x_1 and $\frac{dx_1}{dt}$ are state variables

$\frac{d^2x_1}{dt^2}$ is <u>not</u> a state variable.

Thus, for the given example:

$$\left| \begin{array}{l} z_1 = x_1 \\ z_2 = x_2 \\ z_3 = \frac{dx_1}{dt} \\ z_4 = \frac{dx_2}{dt} \end{array} \right|$$

are the natural (physical) state variables that characterize the system.

- If you know the values of the state variables, all other variables can be computed.
- The state variables are those variables that must be given initial conditions. In the example: We need to know the initial positions and velocities of the two bodies, then everything else can be computed.

State Equations:

We find one equation for each first state derivative:

$$\dot{z}_1 = \frac{dz_1}{dt} = \frac{dx_1}{dt} = z_3$$

$$\dot{z}_2 = \frac{dz_2}{dt} = \frac{dx_2}{dt} = z_4$$

$$\dot{z}_3 = \frac{dz_3}{dt} = \frac{d^2x_1}{dt^2} = \frac{B_{12}}{m_1}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + \frac{k_{12}}{m_1}(x_2 - x_1) - \frac{B_1}{m_1}\cdot\frac{dx_1}{dt} - \frac{k_1}{m_1}\cdot x_1$$

$$= \frac{B_{12}}{m_1}(z_4 - z_3) + \frac{k_{12}}{m_1}(z_2 - z_1) - \frac{B_1}{m_1}\cdot z_3 - \frac{k_1}{m_1}\cdot z_1$$

$$\Rightarrow \dot{z}_3 = -\frac{k_1 + k_{12}}{m_1}\cdot z_1 + \frac{k_{12}}{m_1}\cdot z_2 - \frac{B_1 + B_{12}}{m_1}\cdot z_3 + \frac{B_{12}}{m_1}\cdot z_4$$

$$\dot{z}_4 = \frac{dz_4}{dt} = \frac{d^2x_2}{dt^2} = \frac{1}{m_2}F(t) - \frac{B_2}{m_2}\cdot\frac{dx_2}{dt} - \frac{B_{12}}{m_2}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) - \frac{k_{12}}{m_2}(x_2 - x_1)$$

$$= \frac{1}{m_2}\cdot F(t) - \frac{B_2}{m_2}\cdot z_4 - \frac{B_{12}}{m_2}(z_4 - z_3) - \frac{k_{12}}{m_2}(z_2 - z_1)$$

$$\Rightarrow \dot{z}_4 = \frac{k_{12}}{m_2}\cdot z_1 - \frac{k_{12}}{m_2}\cdot z_2 + \frac{B_{12}}{m_2}\cdot z_3 - \frac{B_2 + B_{12}}{m_2}\cdot z_4 + \frac{1}{m_2}\cdot F(t)$$

We can rewrite these equations in a matrix/vector form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & 1 \\ -\frac{k_1+k_{12}}{m_1} & \frac{k_{12}}{m_1} & -\frac{B_1+B_{12}}{m_1} & \frac{B_{12}}{m_1} \\ \frac{k_{12}}{m_2} & -\frac{k_{12}}{m_2} & \frac{B_{12}}{m_2} & -\frac{B_2+B_{12}}{m_2} \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \emptyset \\ \emptyset \\ \emptyset \\ \frac{1}{m_2} \end{bmatrix} F(t)$$

The output equations compute the outputs, i.e., any variable(s) that we choose to measure.

For example:

$$y = x_2$$

$$\Rightarrow y = \begin{bmatrix} \emptyset & 1 & \emptyset & \emptyset \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \emptyset \end{bmatrix} \cdot F(t)$$

Let:

$$\underline{x} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

be the state vector, and

$$\dot{\underline{x}} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}^{\bullet} \equiv \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix}$$

the state-derivative vector.

Then:

$$\left| \begin{aligned} \dot{\underline{x}} &= A \cdot \underline{x} + \underline{b} \cdot u \\ y &= \underline{c}' \cdot \underline{x} + d \cdot u \end{aligned} \right|$$

is the state-space model, where u is the input, and y is the output.

If there are multiple inputs and outputs:

$$\left| \begin{array}{l} \dot{\underline{x}} = A \cdot \underline{x} + B \cdot \underline{u} \\ \underline{y} = C \cdot \underline{x} + D \cdot \underline{u} \end{array} \right|$$

is the state space model.

If we have m inputs, n states, and p outputs:

$$\left| \begin{array}{l} \underline{x} \in \mathbb{R}^n \\ \underline{u} \in \mathbb{R}^m \\ \underline{y} \in \mathbb{R}^p \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} A \in \mathbb{R}^{n \times n} \\ B \in \mathbb{R}^{n \times m} \\ C \in \mathbb{R}^{p \times n} \\ D \in \mathbb{R}^{p \times m} \end{array} \right|$$

Block diagrams:

- We always start with the integrators, then work ourselves backwards to the inputs, and finally forwards to the outputs.
- We first introduce auxilliary variables to group common expressions together.

For the given example:

$$\left|\begin{array}{l}
F_{d_2} = B_2 \cdot \frac{dx_2}{dt} \\
F_{d_{12}} = B_{12} \cdot \left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) \\
F_{s_{12}} = k_{12} \cdot (x_2 - x_1) \\
F_{d_1} = B_1 \cdot \frac{dx_1}{dt} \\
F_{s_1} = k_1 \cdot x_1
\end{array}\right|$$

$$F_{12} = F_{d_{12}} + F_{s12}$$

$$\frac{d^2x_2}{dt^2} = \frac{1}{m_2}\left[F(t) - F_{d_2} - F_{12}\right]$$

$$\frac{d^2x_1}{dt^2} = \frac{1}{m_1}\left[F_{12} - F_{d_1} - F_{s_1}\right]$$