Vector Calculus: Vector Spaces, Orthonormality & Bases

Given two vectors

$$\underline{v}, \underline{w} \in \mathbb{R}^n$$

i.e.

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad ; \quad \underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

The inner product between these two vectors, written as:

$$\underline{v} \bullet \underline{w} \quad \text{or} \quad \langle \underline{v}, \underline{w} \rangle \quad \text{or} \quad \underline{v}' \cdot \underline{w}$$

is defined as:

$$\langle \underline{v}, \underline{w} \rangle = \sum_{i=1}^{n} v_i \cdot w_i \tag{1}$$

I won't use the "dot-notation" or "dot-product", because I already use the dot to denote the normal matrix/vector product.

In handwriting, it is too difficult to distinguish between a bold-face • and a regular · .

Usually, I would use the normal vector notation (transpose of $\underline{v}$ multiplied by $\underline{w}$), but in this section, I prefer to use the bracket notation.

Rules:

$$\langle \underline{x}, \underline{y} \rangle \equiv \langle \underline{y}, \underline{x} \rangle$$

follows from the definition (1).

$$\langle \underline{x}, \underline{x} \rangle \equiv |\underline{x}|^2$$

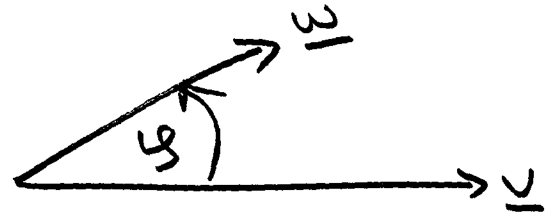
follows from the definition (1).

$$\langle A \cdot \underline{x}, \underline{y} \rangle \equiv \langle \underline{x}, A' \cdot \underline{y} \rangle$$

Proof: $\langle A\underline{x}, \underline{y} \rangle = (A\underline{x})' \cdot \underline{y}$

$= \underline{x}' \cdot A' \cdot \underline{y} = \underline{x}' \cdot (A' \cdot \underline{y})$

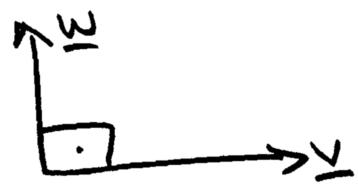
Geometric interpretation:



$$\langle \underline{v}, \underline{w} \rangle = |\underline{v}| \cdot |\underline{w}| \cdot \cos(\varphi)$$

- Two vectors are called orthogonal, if their inner product is zero.

$$\langle \underline{v}, \underline{w} \rangle = 0 \iff \underline{v} \perp \underline{w}$$



$$\langle \underline{v}, \underline{w} \rangle = |\underline{v}| \cdot |\underline{w}| \cdot \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0} = 0$$

- A vector is called <u>normal</u>, if its length is 1.Ø.

- Two vectors are called <u>orthonormal</u>, if they are orthogonal to each other and of length 1.Ø.

- A <u>vector base</u> is a set of orthonormal vectors that is <u>complete</u>.

Let $\{\underline{n}_i\}$ be a set of orthonormal vectors. $\{\underline{n}_i\}$ form a base, iff for any normal vector $\underline{x}$; $|\underline{x}|=1$ that is orthogonal to all $\underline{n}_i$

$$\langle \underline{x}, \underline{n}_i \rangle = \emptyset \quad ; \quad \forall \underline{n}_i$$

it can be concluded that

$$\underline{x} \in \pm \{\underline{n}_i\} .$$

In other words, no additional vector can be added to the set $\{\underline{n}_i\}$, i.e., $\{\underline{n}_i\}$ is complete.

<u>Lemma</u>: Given a vector $\underline{x} \in \mathbb{R}^n$, and $\{\underline{n}_i\}$, a base spanning $\mathbb{R}^n$.

$$\Rightarrow \quad \underline{x} = \sum_{i=1}^{n} k_i \cdot \underline{n}_i$$

i.e., any vector $\in \mathbb{R}^n$ can be written as a linear combination of base vectors.

<u>Special case</u>:

$$\text{Let} \quad \underline{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{\text{th}} \text{ row}$$

be a <u>unit vector</u> in the i direction.

Obviously, $\{\underline{e}_i\}$ form a base in $\mathbb{R}^n$, since

$$\langle \underline{e}_i, e_j \rangle = \delta_{ij}$$

← Kronecker symbol

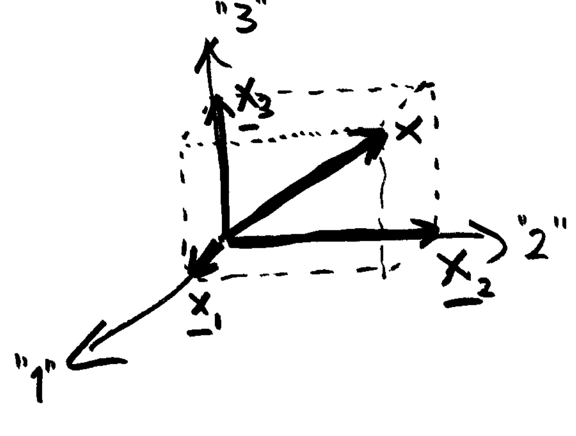
$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and $\{\underline{e}_i\}$ is complete.

Given: $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\Rightarrow \quad \underline{x} = x_1 \cdot \underline{e}_1 + x_2 \cdot \underline{e}_2 + \cdots + x_n \cdot \underline{e}_n$$

$$= \sum_{i=1}^{n} x_i \cdot \underline{e}_i$$



$$\underline{x}_i \equiv x_i \cdot \underline{e}_i$$