Functional Algebra: Function Spaces, Orthonormality & Bases

Let $f(t)$ be a periodic function with period T_0 (or in any other period T, such that T_0 is a multiple of T).

Example: $\sin(t)$ is periodic in $T_0 = 2\pi$

$\sin(2t)$ is periodic in $T = \pi$, but it is also periodic in $T_0 = 2\pi$.

etc.

A function is called normal, iff

$$\int_{T_0} |f(t)|^2 dt = 1$$

The <u>inner product</u> of two such functions is defined as:

$$\langle f(t), g(t) \rangle = \int_{T_0} f^*(t) \cdot g(t)\, dt$$

It is only defined for functions that are periodic in the same period T_0.

<u>Rules</u>: $\langle f(t), g(t) \rangle \equiv \langle g(t), f(t) \rangle$

$$\langle f(t), f(t) \rangle \equiv \int_{T_0} |f(t)|^2 dt$$

Two functions are called <u>orthogonal</u>, if their inner product is zero:

$$\langle f(t), g(t) \rangle = \int_{T_0} f^*(t) \cdot g(t)\, dt \equiv 0$$

$$\Leftrightarrow \quad f(t) \perp g(t)$$

Two functions are called <u>orthonormal</u>, if they are orthogonal to each other and normal.

Let $\{\varphi_i(t)\}$ be a set of orthonormal functions, i.e.

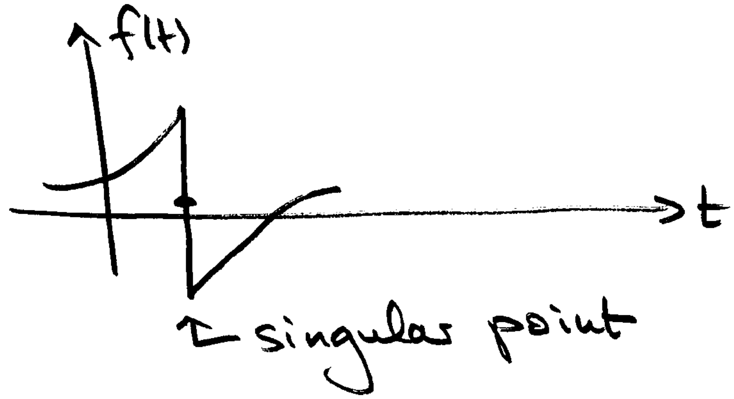
$$\langle \varphi_i(t), \varphi_j(t) \rangle = \delta_{ij}$$

$\{\varphi_i(t)\}$ form a <u>base</u>, iff the set is complete, i.e., if it is impossible to find another normal function that is orthogonal to all $\varphi_i(t)$ without concluding that this function is already contained in the set $\{\varphi_i(t)\}$.

Lemma: Given any function $f(t)$ that is periodic in T_0. $f(t)$ can be approximated arbitarily well by a weighted sum of any base of functions periodic in T_0:

$$f(t) = \sum_{i=\emptyset}^{\infty} \alpha_i \cdot \varphi_i(t)$$

possibly with the exception of a finite number of singular points:



Let us find α_i :

$$\langle f(t), \varphi_i(t) \rangle = \int_{T_0} f^*(t) \cdot \varphi_i(t)\, dt$$

$$= \int_{T_0} \left(\sum_{j=\emptyset}^{\infty} \alpha_j^* \cdot \varphi_j^*(t) \right) \cdot \varphi_i(t)\, dt$$

$$= \sum_{j=\emptyset}^{\infty} \alpha_j^* \cdot \int_{T_0} \varphi_j^*(t) \cdot \varphi_i(t)\, dt$$

$$= \sum_{j=\emptyset}^{\infty} \alpha_j^* \cdot \langle \varphi_j, \varphi_i \rangle = \sum_{j=\emptyset}^{\infty} \alpha_j^* \cdot \delta_{ij}$$

$$= \underline{\underline{\alpha_i^*}}$$

$\Rightarrow$ We can find all coefficients α_i by computing the inner product of $f(t)$ with the corresponding $\varphi_i(t)$.

Example:

$$f(t) = \cos(t)$$

Normalization:

$$\int_{T_0} \cos^2(t)\,dt = \int_{T_0} \frac{\cos(2t)+1}{2}\,dt$$

$$= \underbrace{\int_{T_0} \frac{1}{2}\,dt}_{\frac{T_0}{2}} + \underbrace{\int_{T_0} \frac{\cos(2t)}{2}\,dt}_{\emptyset} = \frac{T_0}{2}$$

$$\Rightarrow \quad \varphi_{1_a}(t) = \sqrt{\frac{2}{T_0}} \cdot \cos(t)$$

Similarly:
$$\varphi_{1b}(t) = \sqrt{\frac{2}{T_0}} \cdot \sin(t) \perp \varphi_{1a}(t)$$

$$\varphi_0(t) = \sqrt{\frac{1}{T_0}} \perp \varphi_{1a}(t),\ \varphi_{1b}(t)$$

Similarly:

$$\varphi_{i_a}(t) = \sqrt{\frac{2}{T_0}} \cdot \cos(i \cdot t) \quad ; \quad \varphi_{i_b}(t) = \sqrt{\frac{2}{T_0}} \cdot \sin(i \cdot t)$$

The set $\{\varphi_0(t), \{\varphi_{ia}(t)\}, \{\varphi_{ib}(t)\}\}$ forms a base of all 2π-periodic functions, since no additional function can be found that is orthogonal to all these functions and that is not already contained in the set.

⇒ Given any 2π-periodic function $f(t)$, we can approximate this function as:

$$f(t) = \alpha_0 \varphi_0(t) + \sum_{i=1}^{\infty} \alpha_{ia} \cdot \varphi_{ia}(t) + \alpha_{ib} \cdot \varphi_{ib}(t)$$

i.e.

$$f(t) = \sqrt{\frac{1}{T_0}} \cdot \alpha_0 + \sqrt{\frac{2}{T_0}} \cdot \sum_{i=1}^{\infty} \alpha_{ia} \cdot \cos(it) + \alpha_{ib} \cdot \sin(it)$$

This looks exactly like the real Fourier series with:

$$\left|\begin{array}{l} \alpha_0 = \sqrt{T_0}\cdot a_0 \\ \alpha_{ia} = \sqrt{\frac{T_0}{2}}\cdot a_i \\ \alpha_{ib} = \sqrt{\frac{T_0}{2}}\cdot b_i \end{array}\right|$$

Example:

$$f(t) = e^{j\omega_0 t}$$

Normalization:

$$\int_{T_0} f^*(t)\cdot f(t)\,dt = \int_{T_0} e^{-j\omega_0 t}\cdot e^{+j\omega_0 t}\,dt$$

$$= \int_{T_0} 1\cdot dt = T_0$$

$$\Rightarrow \varphi_1(t) = \sqrt{\frac{1}{T_0}}\cdot e^{j\omega_0 t}$$

$$\varphi_n(t) = \sqrt{\frac{1}{T_0}} \cdot e^{jn\omega_0 t} \quad ; \quad n \in (-\infty, +\infty)$$

is a set of orthonormal functions that is complete, i.e.

$\{\varphi_n(t)\}$ form a base.

$\Rightarrow$ Given and $\frac{2\pi}{\omega_0}$ - periodic function $f(t)$, this function can be approximated by:

$$f(t) = \sum_{n=-\infty}^{+\infty} \alpha_n \cdot \varphi_n(t)$$

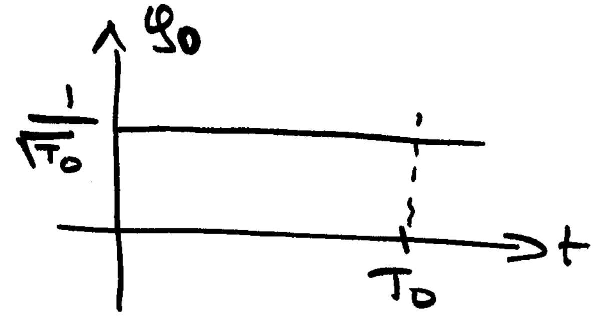
$$= \sqrt{\frac{1}{T_0}} \cdot \sum_{n=-\infty}^{+\infty} \alpha_n \cdot e^{jn\omega_0 t}$$

This looks exactly like the complex Fourier series, with

$$\alpha_n = \sqrt{T_0} \cdot X_n$$

Example:

$$\varphi_0(t) = \frac{1}{\sqrt{T_0}}$$

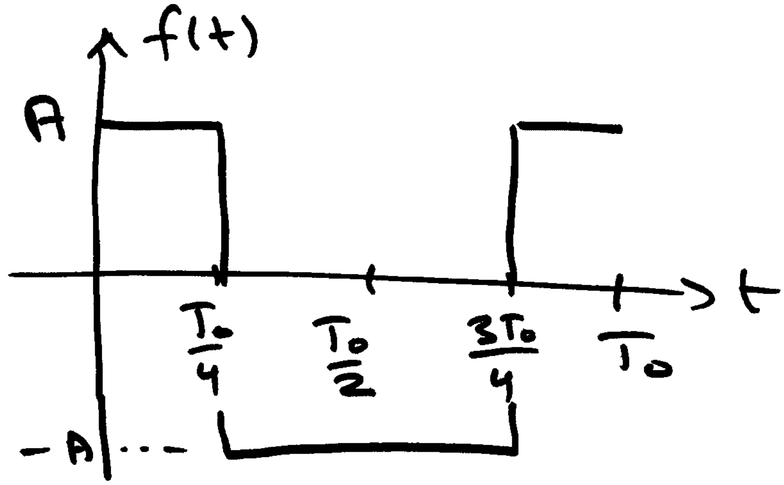


Normalization:

$$\int_{T_0} \varphi_0^2(t)\, dt = \int_{T_0} \frac{1}{\sqrt{T_0}} \cdot \frac{1}{\sqrt{T_0}}\, dt$$

$$= \frac{1}{T_0} \int_{T_0} 1\, dt = 1 \quad \checkmark$$

$f(t) =$



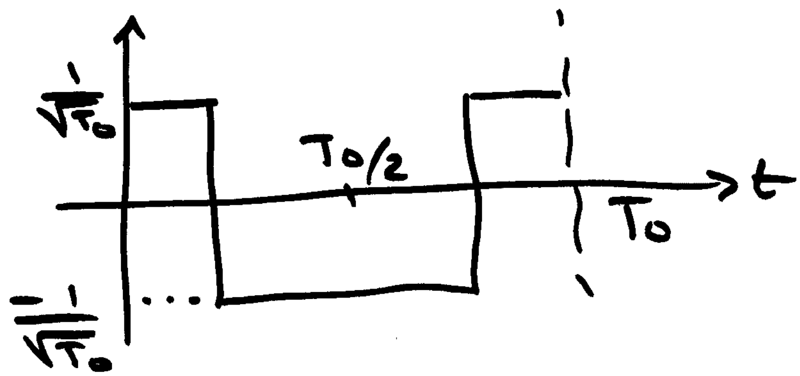
<u>Normalization</u>:

$$\int_{T_0} f^*(t) \cdot f(t)\, dt = \int_{T_0} A^2 dt = A^2 \cdot T_0$$

$$\text{Let } A^2 = \frac{1}{T_0} \Rightarrow A = \frac{1}{\sqrt{T_0}}$$

$$\Rightarrow \int_{T_0} f^*(t) \cdot f(t)\, dt \equiv 1$$

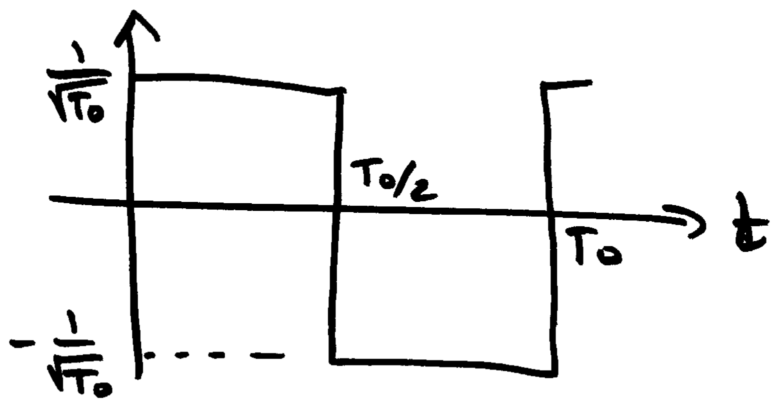
$\Rightarrow \quad \varphi_{1a}(t) =$



is normal.

Looks like a "botched" cos(t).

$\varphi_{1_b}(t) =$

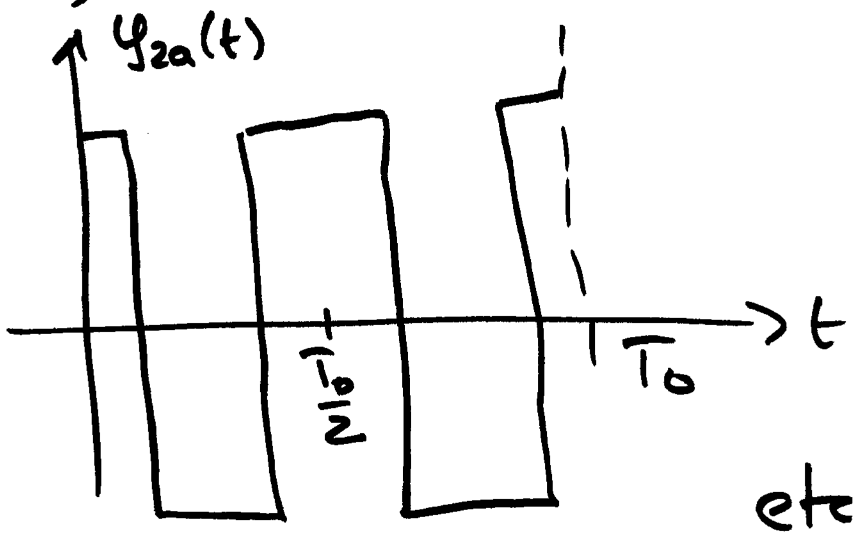


is also normal. Looks like a "botched" sin(t).

It is easy to verify that

$$\varphi_0(t) \perp \varphi_{1_a}(t) \perp \varphi_{1_b}(t)$$

Similarly, we build a "botched" cos(2t), etc.



<u>etc</u>.

All these functions form a base.

$\Rightarrow$ Given any $\frac{2\pi}{\omega_0}$-periodic function $f(t)$, we can approximate as:

$$f(t) = \alpha_0 \varphi_0(t) + \sum_{i=1}^{\infty} \alpha_{ia} \varphi_{ia}(t) + \alpha_{ib} \cdot \varphi_{ib}(t)$$

Parseval's Theorem:

$$\frac{1}{T_0} \int_{T_0} |f(t)|^2 dt = \frac{1}{T_0} \int_{T_0} \Big| \Big(\alpha_0 \varphi_0(t) + \sum_{i=1}^{\infty} \alpha_{ia} \varphi_{ia}(t) + \alpha_{ib} \cdot \varphi_{ib}(t) \Big) \Big(\alpha_0 \varphi_0(t) + \sum_{i=1}^{\infty} \alpha_{ia} \varphi_{ia}(t) \Big) \Big| dt$$

$$= \frac{1}{T_0} \int_{T_0} |\alpha_0|^2 dt + \frac{1}{T_0} \sum_{i=1}^{\infty} \int_{T_0} |\alpha_{ia}|^2 dt + \frac{1}{T_0} \sum_{i=1}^{\infty} \int_{T_0} |\alpha_{ib}|^2 dt$$

because all the crossterms drop out due to orthogonality.

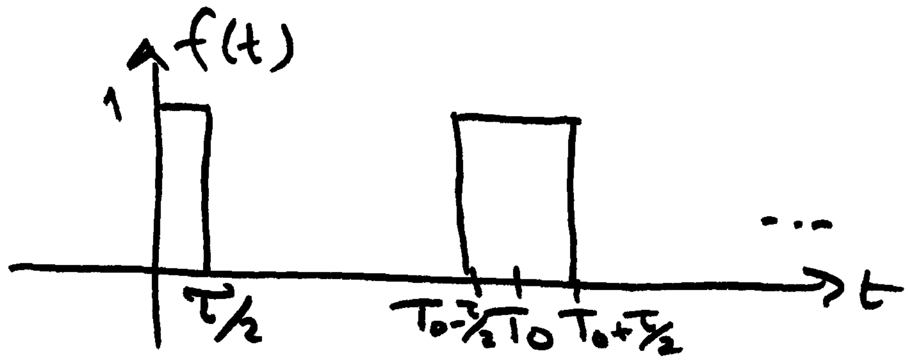
Hence:

$$P_{av} = \frac{1}{T_0}\int_{T_0} |f(t)|^2 dt =$$

$$|\alpha_0|^2 + \sum_{i=1}^{\infty}\left(|\alpha_{ia}|^2 + |\alpha_{ib}|^2\right)$$

The Parseval Theorem works just as well for this approximation, i.e., using a set of square-wave base functions.

Example:



$$P_{av} = \frac{1}{T_0}\int_{T_0} |f(t)|^2 dt = \frac{1}{T_0}\left[\int_0^{\tau/2} 1\,dt + \int_{T_0-\tau/2}^{T_0} 1\,dt\right]$$

$$\Rightarrow \underline{\underline{P_{av} = \frac{\tau}{T_0}}}$$

$$\alpha_0 = \langle f(t), \varphi_0(t) \rangle = \int_{T_0} f(t) \cdot \frac{1}{\sqrt{T_0}}\, dt$$

$$= \frac{1}{\sqrt{T_0}} \int_{T_0} f(t)\, dt = \frac{1}{\sqrt{T_0}} \cdot \frac{\tau}{T_0} = \frac{\tau}{T_0^{3/2}}$$

$$\alpha_{1a} = \langle f(t), \varphi_{1a}(t) \rangle = \int_{T_0} f(t) \cdot \varphi_{1a}(t)\, dt$$

<u>if $\tau < T_0/2$</u> :

$$\alpha_{1a} = \frac{1}{\sqrt{T_0}} \cdot \frac{\tau}{T_0} = \frac{\tau}{T_0^{3/2}}$$

<u>if $\tau \in \left[\frac{T_0}{2}, T_0 \right]$</u> :

$$\alpha_{1a} = \frac{1}{\sqrt{T_0}} \cdot \frac{T_0 - \tau}{T_0} = \frac{T_0 - \tau}{T_0^{3/2}}$$

$\alpha_{1b} \equiv \alpha_{ib} \equiv \phi$ as $f(t)$ is <u>even</u>.

$$\alpha_{2a} = \begin{cases} \dfrac{\tau}{T_0^{3/2}} & ; \ \tau < T_0/4 \\[2ex] \dfrac{T_0 - 2\tau}{2T_0^{3/2}} & ; \ \tau \in \left[\dfrac{T_0}{4}, \dfrac{3T_0}{4} \right] \\[2ex] \dfrac{\tau - T_0}{T_0^{3/2}} & ; \ \tau \in \left[\dfrac{3T_0}{4}, T_0 \right] \end{cases}$$

<u>etc</u>.

- It is possible to define a "line spectrum" and even a "power spectrum" also in terms of the new base.

- How many terms are needed depends on the function to be approximated. Smooth functions are most likely easier to approximate using smooth base functions, i.e., will require less coefficients, whereas discontinuous functions may be easier to approximate using square-wave base functions.

- What makes the Fourier approximation special? Electric power distribution is done mostly using AC. In this context, the Fourier spectrum indeed assumes a special role ("crosstalk" on frequency-multiplexed analog communication lines; aliasing across power lines, etc.).

- Yet, it is important to realize that the theory is <u>much</u> more general than Fourier analysis! It is based on linear function spaces (functional algebra) and contains Fourier analysis only as <u>one</u> special case.