Differentiation Theorem:

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = \int_{-\infty}^{+\infty} \frac{dx}{dt} \cdot e^{-j2\pi ft}\, dt$$

$$= \underbrace{\left[x(t)\cdot e^{-j2\pi ft}\right]_{-\infty}^{+\infty}}_{\substack{=\emptyset \text{ for any function} \\ x(t) \text{ that has a} \\ \text{Fourier transform}}} - \int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi ft}\cdot(-j2\pi f)\cdot dt$$

$$\Rightarrow \mathcal{F}\left\{\frac{dx}{dt}\right\} = j2\pi f \int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi ft}\, dt$$

$$\Rightarrow \boxed{\mathcal{F}\left\{\frac{dx}{dt}\right\} = j2\pi f \cdot X(f)}$$

Similarly for higher derivatives.

Integration Theorem

What is the Fourier transform of

$$\int_{-\infty}^{t} x(\tau)\, d\tau \qquad ?$$

Let us look at the convolution of $x(t)$ with $\varepsilon(t)$:

$$x(t) * \varepsilon(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot \varepsilon(t-\tau)\, d\tau$$

$\varepsilon(t-\tau)$ contributes to the integral for $\tau \in (-\infty, t]$.

$$\Rightarrow \quad x(t) * \varepsilon(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$\Rightarrow \mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\} \equiv \mathcal{F}\left\{ x(t) * \varepsilon(t) \right\}$$

$$\equiv \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{\varepsilon(t)\}$$

$$\equiv X(f) \cdot \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \right]$$

$$\Rightarrow \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{1}{j2\pi f}\cdot X(f) + \frac{1}{2}X(f)\delta(f)$$

$$\equiv \frac{1}{j2\pi f}\cdot X(f) + \frac{1}{2}X(\emptyset)\cdot\delta(f)$$

Unfortunately, it was necessary to make use of a generalized Fourier transform in order to derive the integration theorem.

<u>Consistency:</u>

It is not easy to show that the three proposed extensions are indeed consistent with each other and satisfy all the theorems.

There is one degree of freedom, i.e., we can for example <u>define</u> that

$$1 \circ\!\!-\!\!\bullet \delta(f)$$

There is no need to prove this. However, at this point, everything is fixed, and the other extensions can no longer be chosen freely.

For example, we can write:

$$1 = \varepsilon(t) + \varepsilon(-t)$$

Thus, we should expect that

$$\mathcal{F}\{1\} = \mathcal{F}\{\varepsilon(t)\} + \mathcal{F}\{\varepsilon(-t)\}$$

$$\mathcal{F}\{1\} = \delta(f)$$

$$\mathcal{F}\{\varepsilon(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\mathcal{F}\{\varepsilon(-t)\} = \frac{1}{j2\pi(-f)} + \frac{1}{2}\delta(-f)$$

$$= \frac{-1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\Rightarrow \mathcal{F}\{\varepsilon(t)\} + \mathcal{F}\{\varepsilon(-t)\} =$$

$$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f) - \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \equiv \delta(f)$$

✓

You will have to trust me that the three extensions, as they have been formulated, are indeed fully consistent among themselves <u>and</u> satisfy all the theorems.

As long as you rely on the

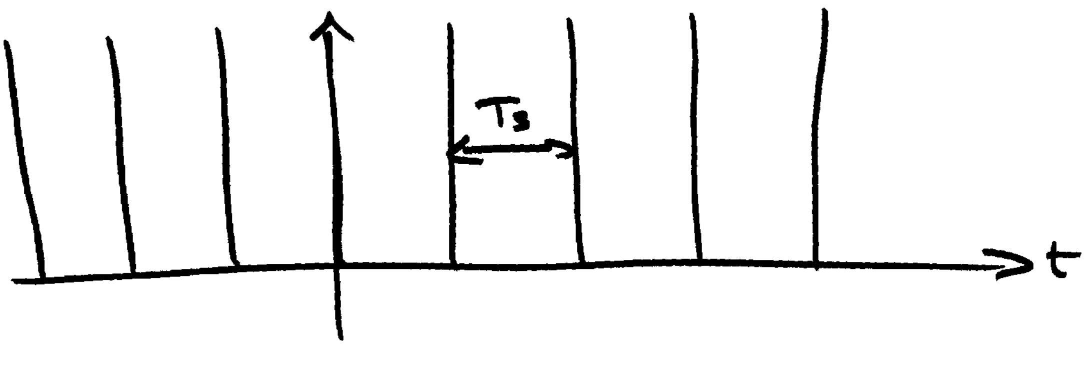
→ definition of the Fourier transform,

→ rules (theorems) of the Fourier transform,

→ three extensions

solely, you are safe, and you can compute with the generalized Fourier transform without problems, but any further generalization would have first to be shown to be consistent

Impulse Trains:



$$x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT_s)$$

is a train of impulses separated by T_s time units.

Using the time delay theorem, we can write:

$$X(f) = \sum_{m=-\infty}^{+\infty} e^{-j2\pi m f T_s}$$

Each impulse contributes one spectrum of amplitude 1, but of different phase.

We still don't know what the spectrum $X(f)$ looks like.

x(t) is a periodic function.
Hence it has a Fourier series:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \, e^{j2\pi n f_s t}$$

<u>where</u>:

$$X_n = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{+\frac{T_s}{2}} \delta(t) \cdot e^{-j2\pi n f_s t} \cdot dt$$

$$= \frac{1}{T_s} \cdot e^{-j2\pi n f_s \cdot \emptyset} = \frac{1}{T_s} = f_s$$

$$\Rightarrow \quad x(t) = \sum_{n=-\infty}^{+\infty} f_s \cdot e^{j2\pi n f_s t}$$

$$\Rightarrow X(f) = f_s \cdot \sum_{n=-\infty}^{+\infty} \mathcal{F}\left\{ e^{j2\pi n f_s t} \right\}$$

$$= f_s \cdot \sum_{n=-\infty}^{+\infty} \delta(f - nf_s)$$

$$\Rightarrow \sum_{m=-\infty}^{+\infty} \delta(t - mT_s) \circ\!\!-\!\!\bullet \sum_{n=-\infty}^{+\infty} \delta(f - nf_s)$$

Although each impulse contributes a spectrum of amplitude 1, because of the different phases, the sums of amplitudes average out to zero everywhere, except at multiples of f_s, where they al add up.

$\Rightarrow$ An impulse train transforms into an impulse train.