Stability:

We already know that a system is stable iff all its poles are in the open left half plane. The poles are the roots of the denominator polynomial of the transfer function. They are also the eigenvalues of the system matrix, A.

Sometimes, engineers would like to know whether or not a system is stable, <u>without</u> having to compute the pole locations. Of course, this question has become less important lately, since tools, such as Matlab, can be carried around on a laptop.

A simple <u>necessary</u> condition for stability is that all coefficients of the denominator polynomial have the same sign:

$$D(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

If there exists any

$$a_i < \emptyset$$

$\Rightarrow$ there is at least pole in the right half plane (RHP).

<u>Example</u>:

$$D(s) = s^3 - 2s^2 + s + 7$$

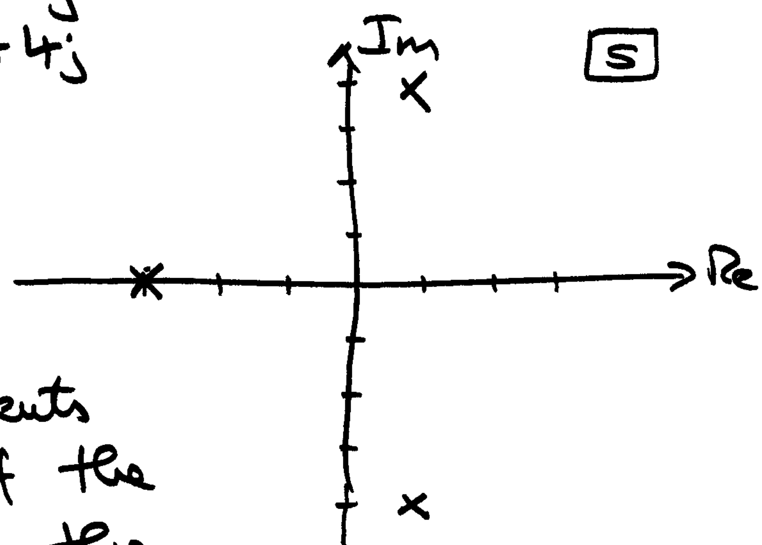
<u>unstable</u>

Unfortunately, the condition is only <u>necessary</u>, but not <u>sufficient</u>.

Example:

$$D(s) = s^3 + s^2 + 11s + 51 = (s+3)(s-1+4j)(s-1-4j)$$

$$\Rightarrow \quad s_1 = -3$$
$$s_2 = +1-4j$$
$$s_3 = +1+4j$$



Although all coefficients are positive, two of the three poles are in the right half plane.

We shall now formulate a condition that is both necessary and sufficient (without proof).

We decompose $D(s)$ into two polynomials:

$$D(s) = D_1(s) + D_2(s)$$

as follows:

$$D_1(s) = s^n + a_{n-2} \cdot s^{n-2} + a_{n-4} \cdot s^{n-4} + \dots$$
$$D_2(s) = a_{n-1} \cdot s^{n-1} + a_{n-3} \cdot s^{n-3} + \dots$$

If the order of $D(s)$ is even, then $D_1(s)$ contains only the even powers of s, and $D_2(s)$ contains only the odd powers.

If the order of $D(s)$ is odd, then $D_1(s)$ contains only the odd powers of s, and $D_2(s)$ contains only the even powers.

We construct the transfer function:

$$G_{12}(s) = \frac{D_1(s)}{D_2(s)}$$

We find a continued fraction expansion of $G_{12}(s)$:

$$G_{12}(s) = \alpha_1 \cdot s + \cfrac{1}{\alpha_2 \cdot s + \cfrac{1}{\alpha_3 \cdot s + \cfrac{1}{\ddots \, \cfrac{1}{\alpha_{n-1} \cdot s + \cfrac{1}{\alpha_n \cdot s}}}}}$$

Example:

$$D(s) = s^4 + 2s^3 + 6s^2 + 4s + 1$$

$$\Rightarrow D_1(s) = s^4 + 6s^2 + 1$$

$$D_2(s) = 2s^3 + 4s$$

$$\Rightarrow G_{12}(s) = \frac{s^4 + 6s^2 + 1}{2s^3 + 4s}$$

$$
\begin{array}{l}
(s^4 + 6s^2 + 1) : (2s^3 + 4s) = \frac{1}{2}s \\
\underline{-\; s^4 + 2s^2 \qquad} \\
\quad\; 4s^2 + 1
\end{array}
$$

$$\Rightarrow G_{12}(s) = \frac{1}{2}s + \frac{4s^2+1}{2s^3+4s}$$

$$= \frac{1}{2}s + \frac{1}{\left(\frac{2s^3+4s}{4s^2+1}\right)}$$

$$
\begin{array}{l}
(2s^3 + 4s) : (4s^2 + 1) = \frac{1}{2}s \\
\underline{-\; 2s^3 + \frac{1}{2}s \qquad} \\
\quad\; 3\frac{1}{2}s
\end{array}
$$

$$\Rightarrow G_{12}(s) = \frac{1}{2}s + \frac{1}{\frac{1}{2}s + \frac{3\frac{1}{2}s}{4s^2+1}}$$

$$= \frac{1}{2}s + \frac{1}{\frac{1}{2}s + \frac{1}{\left(\frac{4s^2+1}{3\frac{1}{2}s}\right)}}$$

$$(4s^2+1) : (3\tfrac{1}{2}s) = \tfrac{8}{7}s$$

$$\begin{array}{l} -\,4s^2 \\ \hline \quad +1 \end{array}$$

$$\Rightarrow G_{12}(s) = \frac{1}{2}s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{\frac{8}{7}s + \cfrac{1}{\frac{7}{2}s}}}$$

This is the continued fraction expansion (CFE) of $G_{12}(s)$.

Lemma (without proof):

A necessary and sufficient condition for stability of $D(s)$ is that all α_i of the CFE of $G_{12}(s)$ have equal sign.

The Routh scheme offers a convenient algorithm for computing the CFE of a transfer function.

$\alpha_1 = \frac{a_n}{a_{n-1}}$	s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
	s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
$\alpha_2 = \frac{a_{n-1}}{b_{n-2}}$	s^{n-2}	b_{n-2}	b_{n-4}	b_{n-6}	$\cdots$
	s^{n-3}	c_{n-3}	c_{n-5}	c_{n-7}	$\cdots$
$\alpha_3 = \frac{b_{n-2}}{c_{n-3}}$					
$\vdots$	$\vdots$	$\vdots$			
	s^1	k_1			
$\alpha_n = \frac{k_1}{l_0}$	s^0	l_0			

where:

$$b_{n-2} = \frac{a_{n-1} \cdot a_{n-2} - a_n \cdot a_{n-3}}{a_{n-1}}$$

$$b_{n-4} = \frac{a_{n-1} \cdot a_{n-4} - a_n \cdot a_{n-5}}{a_{n-1}}$$

$$\vdots$$

$$c_{n-3} = \frac{b_{n-2} \cdot a_{n-3} - a_{n-1} \cdot b_{n-4}}{b_{n-2}}$$

$$\vdots$$

Example:

$$D(s) = s^4 + 2s^3 + 6s^2 + 4s + 1$$

	s^4	1	6	1
$\alpha_1 = \frac{1}{2}$	s^3	2	4	
$\alpha_2 = \frac{1}{2}$	s^2	4	1	
$\alpha_3 = \frac{8}{7}$	s^1	$3\frac{1}{2}$		
$\alpha_4 = \frac{7}{2}$	s^0	1		

$$\Rightarrow G_{12}(s) = \frac{s^4+6s^2+1}{2s^3+4s} = \frac{1}{2}s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{\frac{8}{7}s + \cfrac{1}{\frac{7}{2}s}}}$$

$\Rightarrow$ $D(s)$ is stable