Similarity Transformation:

Given a SISO system in the time domain:

$$\left| \begin{array}{l} \underline{\dot{x}} = A \cdot \underline{x} + \underline{b} \cdot u \\ y = \underline{c}' \cdot \underline{x} + d \cdot u \end{array} \right| \quad ; \quad \underline{x} \in \mathbb{R}^n$$

with the transfer function

$$G(s) = \underline{c}'(sI - A)^{-1}\underline{b} + d$$

$G(s)$ is at least proper, i.e.

$$G(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

The transfer function is <u>unique</u>, i.e., a system of n^{th} order with a given I/O behavior is characterized by exactly one transfer function.

The number of parameters of the transfer function is $(2n+1)$, i.e., there are $(2n+1)$ degrees of freedom.

The number of parameters of the state-space description is

$n^2 + 2n + 1 > 2n + 1$, i.e.,

we expect that the state-space description is not unique. There are many different state-space descriptions of the same system (the same I/O behavior).

Given a non-singular matrix

$$T \in \mathbb{R}^{n \times n},$$

we can introduce a new state vector:

$$\boxed{\underline{\xi} = T \cdot \underline{x}}$$

For any given T, if $\underline{x}$ is known, $\underline{\xi}$ is known as well.

It is also true that if $\underline{\xi}$ is known, $\underline{x}$ is known.

Proof:

$$T^{-1} \cdot \underline{\xi} = T^{-1} \cdot (T \cdot \underline{x})$$
$$= (T^{-1} \cdot T) \cdot \underline{x}$$
$$= I^{(n)} \cdot \underline{x} = \underline{x}$$

$$\Rightarrow \boxed{\underline{x} = T^{-1} \cdot \underline{\xi}}$$

$$\Rightarrow \dot{\underline{x}} = T^{-1} \cdot \dot{\underline{\xi}}$$

Plug into state-space description:

$$T^{-1} \cdot \dot{\underline{\xi}} = A \cdot (T^{-1} \cdot \underline{\xi}) + \underline{b} \cdot u$$

Expand with T:

$$T\cdot(T^{-1}\cdot\dot{\underline{\xi}}) = T\cdot\left[A\cdot T^{-1}\cdot\underline{\xi}\right] + T\cdot(\underline{b}\cdot u)$$

$$\Rightarrow \underbrace{(T\cdot T^{-1})}_{I^{(n)}}\cdot\dot{\underline{\xi}} = (T\cdot A\cdot T^{-1})\cdot\underline{\xi} + (T\cdot\underline{b})\cdot u$$

$$\Rightarrow \left|\begin{array}{l} \dot{\underline{\xi}} = (T\cdot A\cdot T^{-1})\,\underline{\xi} + (T\cdot\underline{b})\cdot u \\ y = (\underline{c}'\cdot T^{-1})\cdot\underline{\xi} + d\cdot u \end{array}\right|$$

is a new state-space description with the same structure as the previous one:

$$\left|\begin{array}{l} \dot{\underline{\xi}} = \hat{A}\cdot\underline{\xi} + \hat{\underline{b}}\cdot u \\ y = \hat{\underline{c}}'\cdot\underline{\xi} + \hat{d}\cdot u \end{array}\right| ; \quad \underline{\xi}\in\mathbb{R}^n$$

where:

$$\left| \begin{array}{l} \hat{A} = T \cdot A \cdot T^{-1} \\ \hat{\underline{b}} = T \cdot \underline{b} \\ \hat{\underline{c}}' = \underline{c}' \cdot T^{-1} \\ \hat{d} = d \end{array} \right|$$

Any non-singular matrix T leads to a new (similar) state-space description.

This is called similarity transformation.

The transfer function does not change. Proof:

$$\hat{G}(s) = \hat{\underline{c}}' \cdot (sI - \hat{A})^{-1} \hat{\underline{b}} + \hat{d}$$

$$= \underline{c}'\cdot T^{-1}\cdot\left(sI - T\cdot A\cdot T^{-1}\right)^{-1}\cdot T\cdot \underline{b} + d$$

$$= \underline{c}'\cdot T^{-1}\cdot\left(s\cdot T\cdot T^{-1} - T\cdot A\cdot T^{-1}\right)^{-1}\cdot T\cdot \underline{b} + d$$

$$= \underline{c}'\cdot T^{-1}\cdot\left(T\cdot sI\cdot T^{-1} - T\cdot A\cdot T^{-1}\right)^{-1}\cdot T\cdot \underline{b} + d$$

$$= \underline{c}'\cdot T^{-1}\cdot\left[T(sI-A)T^{-1}\right]^{-1}\cdot T\cdot \underline{b} + d$$

$$= \underline{c}'\cdot \underbrace{T^{-1}\cdot T}_{I^{(n)}}\cdot (sI-A)^{-1}\cdot \underbrace{T^{-1}\cdot T}_{I^{(n)}}\cdot \underline{b} + d$$

$$= \underline{c}'\cdot (sI-A)^{-1}\cdot \underline{b} + d$$

$$\equiv G(s)$$

q.e.d.

Duality Transformation:

Given a SISO system with a transfer function $G(s)$ with real coefficients. Obviously:

$$G'(s) \equiv G(s)$$

[*Notice*: This is <u>only</u> true for SISO systems!]

$$\begin{aligned}
G'(s) &= \left[\underline{c}' \cdot (sI-A)^{-1} \cdot \underline{b} + d\right]' \\
&= \left[\underline{c}' \cdot (sI-A)^{-1} \cdot \underline{b}\right]' + d' \\
&= \underline{b}' \cdot \left[(sI-A)^{-1}\right]' \cdot (\underline{c}')' + d' \\
&= \underline{b}' \cdot \left[(sI-A)'\right]^{-1} \cdot \underline{c} + d' \\
&= \underline{b}' \cdot (sI-A')^{-1} \cdot \underline{c} + d' \\
&= \underline{b}' \cdot (sI-A')^{-1} \cdot \underline{c} + d
\end{aligned}$$

Obviously, the state-space description with:

$$\left|\begin{array}{l} \hat{A} = A' \\ \underline{\hat{b}} = \underline{c} \\ \underline{\hat{c}}' = \underline{b}' \\ \hat{d} = d \end{array}\right|$$

is similar to the original state-space description. This is called the <u>dual</u> representation.

Example:

$$G(s) = \frac{2s+7}{s^3+5s^2+3s+9}$$

$$\Rightarrow \left|\begin{array}{l} \underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -3 & -5 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ \\ y = \begin{bmatrix} 7 & 2 & 0 \end{bmatrix} \underline{x} \end{array}\right|$$

is a state-space representation. This is a special representation called the <u>controller-canonical form</u>.

The dual representation is:

$$\left| \begin{array}{l} \dot{\underline{\xi}} = \begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix} \underline{\xi} + \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \underline{\xi} \end{array} \right|$$

This is also a special representation. It is called the <u>observer-canonical form</u>.

<u>Warning</u>: Only if there is no pole-zero cancellation of $G(s)$, will the dual representation be related to the original one by a similarity transform.

$\Rightarrow$ cf. ECE 441