Solution of Differential Equations in the Time Domain

1) Linear first-order Ordinary Differential Equation (ODE) with known input function

Example: Mechanical system

$$\left| \begin{array}{l} J \cdot \dot{\omega} + H_r \cdot \omega = T(t) \\ \omega(t=\emptyset) = \omega_0 \\ T(t) = \begin{cases} \sin(2t) & ; t \geq \emptyset \\ \emptyset & ; t < \emptyset \end{cases} \end{array} \right.$$

Solution:

$$\omega(t) = \omega_p(t) + C_0 \cdot \omega_h(t)$$

$\omega_p(t) =$ particular solution of the inhomogeneous system without taking initial condition into account.

$C_0 \cdot \omega_h(t) =$ general solution of the homogeneous system for all initial conditions.

Ansatz:

$$\omega_h(t) = C_0 \cdot e^{\lambda t} \cdot \varepsilon(t)$$

where λ is the solution of the characteristic equation:

$$J \cdot \lambda + H_r = \emptyset \Rightarrow \lambda = -\frac{H_r}{J}$$

$$\Rightarrow \underline{\omega_h(t) = C_0 \cdot e^{-\frac{H_r}{J} t} \cdot \varepsilon(t)}$$

Verification:

$$\dot{\omega}_h(t) = -C_0 \cdot \frac{H_r}{J} \cdot e^{-\frac{H_r}{J} \cdot t} \quad ; \quad t \geq \emptyset$$

homogeneous system:

$$J \cdot \dot{\omega}_h + H_r \cdot \omega_h = \emptyset$$

Plug in:

$$\cancel{J} \cdot \left(-C_0 \cdot \frac{H_r}{\cancel{J}} \cdot e^{-\frac{H_r}{J} \cdot t}\right) + H_r \cdot \left(C_0 \cdot e^{-\frac{H_r}{J} \cdot t}\right) \stackrel{?}{=} \emptyset$$

$$\Rightarrow \underbrace{\left(H_r \cdot C_0 - C_0 \cdot H_r\right)}_{\emptyset} \cdot e^{-\frac{H_r}{J} t} \stackrel{?}{=} \emptyset$$

$\underline{\checkmark}$ q.e.d.

<u>Ansatz</u>:

$$\omega_p(t) = \left(C_1 \cdot \sin(2t) + C_2 \cdot \cos(2t)\right) \varepsilon(t)$$

<u>Verification</u>:

$$\dot{\omega}_p(t) = 2C_1 \cdot \cos(2t) - 2C_2 \cdot \sin(2t); \quad t \geq \emptyset$$

inhomogeneous system:

$$J \cdot \dot{\omega}_p + H_r \cdot \omega_p = \sin(2t) \quad ; \quad \forall t \geq \emptyset$$

$$\Rightarrow J\left(2C_1 \cdot \cos(2t) - 2C_2 \cdot \sin(2t)\right) + H_r \cdot \left(C_1 \cdot \sin(2t) + C_2 \cdot \cos(2t)\right) \overset{?}{=} \sin(2t)$$

In order for this equation to be true, the coefficients for sin(2t) and cos(2t) must be independently equal to the left and right of the equal sign:

$$\Rightarrow \left| \begin{array}{l} 2JC_1 + H_r \cdot C_2 \overset{!}{=} \emptyset \\ -2JC_2 + H_r \cdot C_1 \overset{!}{=} 1 \end{array} \right|$$

$$\Rightarrow C_1 = -\frac{\lambda}{J(\lambda^2+4)} \quad ; \quad C_2 = -\frac{2}{J(\lambda^2+4)}$$

$$\Rightarrow \omega(t) = \left(C_0 e^{\lambda t} + C_1 \cdot \sin(2t) + C_2 \cdot \cos(2t)\right) \cdot \varepsilon(t)$$

Initial conditions:

$$\omega(t = \phi_+) = \omega_0 = C_0 + C_2$$

↑ This means: immediately after time ϕ.

$$\begin{bmatrix} \varepsilon(t = \phi_-) = \phi.\phi \\ \varepsilon(t = \phi) = \phi.5 \\ \varepsilon(t = \phi_+) = 1.\phi \end{bmatrix}$$

$$\Rightarrow C_0 = \omega_0 - C_2 = \omega_0 + \frac{2}{J(\lambda^2+4)}$$

$$\Rightarrow \omega(t) = \underbrace{\left[\omega_0 + \frac{2}{J(\lambda^2+4)}\right] e^{\lambda t}}_{\text{homogeneous solution}} \underbrace{- \frac{\lambda}{J(\lambda^2+4)} \cdot \sin(2t) - \frac{2}{J(\lambda^2+4)} \cdot \cos(2t)}_{\text{particular solution}}$$

OS:

$$\omega(t) = \underbrace{\omega_0 e^{\lambda t}}_{\substack{\text{state} \\ \text{response}}} + \underbrace{\frac{1}{J(\lambda^2+4)}\left(2e^{\lambda t} - \lambda \sin(2t) - 2\cos(2t)\right)}_{\text{input response}}$$

(depends on initial condition)

(depends on input function)

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2) Linear 1^{st}-order ODE with unknown input function

Example:

$$J \cdot \dot{\omega} + H_r \cdot \omega = T(t)$$

$$\omega(t = 0_+) = \omega_0$$

$$T(t) = 0 \; ; \; \forall t < 0$$

The homogenous system hasn't changed, thus:

$$\omega_h(t) = C_0 \cdot e^{\lambda t} \cdot \varepsilon(t) \quad \text{(as before)}$$

Ansatz:

$$\omega_p(t) = \gamma(t) \cdot e^{\lambda t} \cdot \varepsilon(t)$$

Verification:

$$\dot{\omega}_p(t) = \dot{\gamma}(t) \cdot e^{\lambda t} + \gamma(t) \cdot \lambda \cdot e^{\lambda t}$$

$$\Rightarrow J\left(\dot{\gamma}(t) e^{\lambda t} + \gamma(t) \cdot \lambda \cdot e^{\lambda t}\right) + H_r \left(\gamma(t) \cdot e^{\lambda t}\right) = T(t)$$

$$\lambda = -\frac{H_r}{J}$$

$$\Rightarrow \dot{\gamma}(t) \cdot J \cdot e^{\lambda t} \cancel{- \gamma(t) \cdot H_r \cdot e^{\lambda t}} + \cancel{\gamma(t) \cdot H_r \cdot e^{\lambda t}} = T(t)$$

$$\Rightarrow \dot{\gamma}(t) = \frac{1}{J} \cdot e^{-\lambda t} \cdot T(t)$$

$$\Rightarrow \gamma(t) = \int_{0-}^{t} e^{-\lambda \tau} \cdot \frac{T(\tau)}{J} \cdot d\tau$$

$$\Rightarrow \omega_p(t) = e^{\lambda t} \cdot \int_{0-}^{t} e^{-\lambda \tau} \cdot \frac{T(\tau)}{J} \, d\tau$$

$$= \int_{0-}^{t} e^{\lambda (t-\tau)} \cdot \frac{T(\tau)}{J} \cdot d\tau$$

$$\Rightarrow w(t) = C_0 \cdot e^{\lambda t} + \int_{0-}^{t} e^{\lambda(t-\tau)} \cdot \frac{T(\tau)}{J}\, d\tau$$

Initial conditions:

$$w(t=0_+) = \omega_0 = C_0 + \underbrace{\int_{0-}^{0+} e^{\lambda(t-\tau)} \cdot \frac{T(\tau)}{J}\, d\tau}_{=0}$$

unless $T(t)$ contains a Dirac at time 0.

$$\Rightarrow \omega_0 = C_0$$

$$\Rightarrow w(t) = \underbrace{\omega_0 \cdot e^{\lambda t}}_{\substack{\text{state response} \\ = f(\omega_0)}} + \underbrace{\int_{0-}^{t} e^{\lambda(t-\tau)} \cdot \frac{T(\tau)}{J}\, d\tau}_{\substack{\text{input response} \\ = f(T(t))}}$$