Solution of Linear Systems

A) In the time domain

Given: $\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$; $\underline{x}(t=\phi_+) = \underline{x}_0$, $\underline{u}(t) = \phi, \ \forall t < \phi$

$$\Rightarrow \underline{x}(t) = e^{At} \cdot \underline{x}_0 + \int_{0-}^{t} e^{A(t-\tau)} \cdot B \cdot \underline{u}(\tau)\, d\tau$$

$$\Rightarrow \underline{y}(t) = C \cdot e^{At} \cdot \underline{x}_0 + C \int_{0-}^{t} e^{A(t-\tau)} \cdot B \cdot \underline{u}(\tau)\, d\tau + D \cdot \underline{u}(t)$$

B) In the frequency domain

$$\dot{\underline{x}} \circ\!\!-\!\!\bullet \ s \cdot \underline{X}(s) - \underline{x}_0$$

$$\Rightarrow \ s \cdot \underline{X}(s) - \underline{x}_0 = A \cdot \underline{X}(s) + B \cdot \underline{U}(s)$$

$$\Rightarrow \ s \cdot \underline{X}(s) - A \cdot \underline{X}(s) = \underline{x}_0 + B \cdot \underline{U}(s)$$

$$\Rightarrow \ s \cdot I^{(n)} \cdot \underline{X}(s) - A \cdot \underline{X}(s) = \underline{x}_0 + B \cdot \underline{U}(s)$$

$I^{(n)}$: Identity matrix of dimensions $n \times n$

$$\Rightarrow [s \cdot I^{(n)} - A] \cdot \underline{X}(s) = \underline{x}_0 + B \cdot \underline{U}(s)$$

$$\Rightarrow \underline{X}(s) = [sI^{(n)} - A]^{-1} \cdot \underline{x}_0 + [sI^{(n)} - A]^{-1} \cdot B \cdot \underline{u}(s)$$

$$\Rightarrow \underline{Y}(s) = C \cdot [sI^{(n)} - A]^{-1} \cdot \underline{x}_0 + C [sI^{(n)} - A]^{-1} \cdot B \cdot \underline{u}(s) + D \cdot \underline{u}(s)$$

c) Comparison of time- and frequency-domain solutions:

$$C \cdot e^{At} \cdot \underline{x}_0 \circ\!\!-\!\!\bullet C \cdot [sI^{(n)} - A]^{-1} \cdot \underline{x}_0$$

$$\Rightarrow \quad e^{At} \circ\!\!-\!\!\bullet [sI^{(n)} - A]^{-1}$$

$$\Rightarrow \quad \underline{\underline{e^{At} = \mathcal{L}^{-1}\{(sI^{(n)} - A)^{-1}\}}}$$

Example:

$$A = \begin{bmatrix} -8 & 2 \\ -15 & 3 \end{bmatrix}$$

$$\Rightarrow sI^{(2)} - A = \begin{bmatrix} s & \emptyset \\ \emptyset & s \end{bmatrix} - \begin{bmatrix} -8 & 2 \\ -15 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (s+8) & -2 \\ 15 & (s-3) \end{bmatrix}$$

$$\left[sI^{(2)} - A\right]^{-1} = \frac{\left[sI^{(2)} - A\right]^{\dagger}}{\left| sI^{(2)} - A \right|}$$

$$= \frac{\operatorname{adj}\left(sI^{(2)} - A\right)}{\det\left(sI^{(2)} - A\right)}$$

$$\left[sI^{(2)} - A\right]^{\dagger} = \begin{bmatrix} (s-3) & 2 \\ -15 & (s+8) \end{bmatrix}$$

$$\begin{aligned} \left| sI^{(2)} - A \right| &= (s+8)(s-3) - 15 \cdot (-2) \\ &= s^2 + 5s - 24 + 30 \\ &= s^2 + 5s + 6 \\ &= (s+2)(s+3) \end{aligned}$$

$$\Rightarrow \left[sI^{(2)} - A\right]^{-1} = \begin{bmatrix} \dfrac{s-3}{(s+2)(s+3)} & \dfrac{2}{(s+2)(s+3)} \\ \dfrac{-15}{(s+2)(s+3)} & \dfrac{s+8}{(s+2)(s+3)} \end{bmatrix}$$

We need to use partial fraction expansion on all four terms:

$$\frac{s-3}{(s+2)(s+3)} = \frac{a_{11}}{s+2} + \frac{b_{11}}{s+3}$$

$$a_{11} = \lim_{s \to -2} (s+2) \cdot \frac{s-3}{(s+2)(s+3)}$$

$$= \lim_{s \to -2} \frac{s-3}{s+3} = \frac{-5}{1} = -5$$

$$b_{11} = \lim_{s \to -3} (s+3) \cdot \frac{s-3}{(s+2)(s+3)}$$

$$= \lim_{s \to -3} \frac{s-3}{s+2} = \frac{-6}{-1} = 6$$

<u>similarly:</u>

$$\frac{2}{(s+2)(s+3)} \equiv \frac{2}{s+2} + \frac{-2}{s+3}$$

$$\frac{-15}{(s+2)(s+3)} \equiv \frac{-15}{s+2} + \frac{15}{s+3}$$

$$\frac{s+8}{(s+2)(s+3)} \equiv \frac{6}{s+2} + \frac{-5}{s+3}$$

$$\Rightarrow \left[sI^{(2)} - A\right]^{-1} = \begin{bmatrix} \left(\frac{-5}{s+2} + \frac{6}{s+3}\right) & \left(\frac{2}{s+2} + \frac{-2}{s+3}\right) \\ \left(\frac{-15}{s+2} + \frac{15}{s+3}\right) & \left(\frac{6}{s+2} + \frac{-5}{s+3}\right) \end{bmatrix}$$

$$\Rightarrow e^{At} = \mathcal{L}^{-1}\left\{ \left[sI^{(2)} - A \right]^{-1} \right\}$$

$$= \begin{bmatrix} \left(-5e^{-2t} + 6e^{-3t}\right) & \left(2e^{-2t} - 2e^{-3t}\right) \\ \left(-15e^{-2t} + 15e^{-3t}\right) & \left(6e^{-2t} - 5e^{-3t}\right) \end{bmatrix}$$

$$\neq \begin{bmatrix} e^{-8t} & e^{2t} \\ e^{-15t} & e^{3t} \end{bmatrix}$$

In Matlab:

A=[-8, 2; -15, 3]

e^A= expm(A)

↑ matrix exponential

SISO Systems & Transfer Functions:

Given a single-input/single-output (SISO) system:

$$\left| \begin{array}{l} \dot{\underline{x}} = A \cdot \underline{x} + \underline{b} u \\ y = \underline{c}' \cdot \underline{x} + d u \end{array} \right| ; \quad \begin{array}{l} \underline{x}(t = 0_+) = \underline{x}_0 \\ u(t) = 0, \ \forall t < 0 \end{array}$$

$$y(t) = \underline{c}' \cdot e^{At} \cdot \underline{x}_0 + \underline{c}' \cdot \int_{0-}^{t} e^{A(t-\tau)} \cdot \underline{b} \cdot u(\tau) d\tau + d \cdot u(t)$$

$$Y(s) = \underline{c}' \left[sI^{(n)} - A \right]^{-1} \cdot \underline{x}_0 + \underline{c}' \left[sI^{(n)} - A \right]^{-1} \underline{b} \cdot U(s) + d \cdot U(s)$$

$$Y(s) = \underbrace{G(s)}_{\text{transfer function}} \cdot U(s) + \underbrace{\Gamma(s)}_{\text{state transition function}} \cdot \underline{x}_0$$

$$\Rightarrow \boxed{\begin{array}{l} G(s) = \underline{c}' \left(sI^{(n)} - A \right)^{-1} \cdot \underline{b} + d \\ \Gamma(s) = \underline{c}' \left(sI^{(n)} - A \right)^{-1} \end{array}}$$

The transfer function $G(s)$ only accounts for the <u>input</u>, not for the <u>initial condition</u>.

<u>Let</u>: $\underline{x}_0 = \emptyset$

$$\Rightarrow \quad Y(s) = G(s) \cdot U(s)$$

<u>Let</u>: $u(t) = \delta(t)$

$$\Rightarrow U(s) = 1$$

$$\Rightarrow \quad Y(s) = G(s) \cdot 1 = G(s)$$

$$y(t) = \underline{c}' \int_{0^-}^{t} e^{A(t-\tau)} \cdot \underline{b} \cdot \delta(\tau)\, d\tau + d \cdot \delta(t)$$

$$= \underline{c}' e^{At} \cdot \underline{b} + d \cdot \delta(t)$$

because of the sifting property of the Dirac distribution.

$y(t) = \mathcal{L}^{-1}\{G(s)\} = g(t)$ is the <u>impulse response</u>

$$\underline{\underline{g(t) = \underline{c}' e^{At} \cdot \underline{b} + d \cdot \delta(t)}}$$

Let $G(s)$ be <u>strictly proper</u>:

$$G(s) = \frac{N(s)}{D(s)} \; ; \; \operatorname{ord}(N(s)) < \operatorname{ord}(D(s))$$

$$\Longleftrightarrow d = \emptyset$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = \underline{c}' e^{At} \cdot \underline{b}$$

We can write:

$$g(t-\tau) = \underline{c}' \cdot e^{A(t-\tau)} \cdot \underline{b}$$

and therefore:

$$y(t) = \underline{c}' \cdot \int_{0-}^{t} e^{A(t-\tau)} \underline{b} \cdot u(\tau) \, d\tau$$

$$= \int_{0-}^{t} \underline{c}' \cdot e^{A(t-\tau)} \underline{b} \cdot u(\tau) \, d\tau$$

$$= \int_{0-}^{t} g(t-\tau) \cdot u(\tau) \, d\tau$$

To summarize:

Given a strictly proper SISO system with zero initial conditions:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}\,u \\ y = \underline{c}'\underline{x} \end{array} \right| ; \quad \begin{array}{l} \underline{x}(t = 0_+) = 0 \\ u(t) = 0, \ \forall t < 0 \end{array}$$

$$\Rightarrow y(t) = \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \cdot \underline{b} \cdot u(\tau)\, d\tau$$

$$\equiv \int_{0-}^{t} g(t-\tau) \cdot u(\tau)\, d\tau$$

where: $\quad g(t) = \mathcal{L}^{-1}\{G(s)\}$

and: $\quad G(s) = \dfrac{Y(s)}{U(s)}$

and: $\quad U(s) = \mathcal{L}\{u(t)\}$

$\quad Y(s) = \mathcal{L}\{y(t)\}$

We define:

$$Y(s) = G(s) \cdot U(s)$$

$$y(t) = g(t) * u(t)$$

convolution operator

$$g(t) * u(t) := \int_{0-}^{t} g(t-\tau) \cdot u(\tau)\, d\tau$$

is defined as

Variable transformation:

$$\sigma = t - \tau$$
$$\Rightarrow \tau = t - \sigma$$
$$d\tau = -d\sigma$$

τ	σ
$0-$	t
t	$0-$

$$\Rightarrow g(t) * u(t) = \int_{0-}^{t} g(t-\tau) \cdot u(\tau)\, d\tau$$

$$= \int_{t}^{0-} g(\sigma) \cdot u(t-\sigma)(-d\sigma) = \int_{0-}^{t} u(t-\sigma) \cdot g(\sigma)\, d\sigma$$

$$= u(t) * g(t)$$

$$\Rightarrow \quad \underline{\underline{g(t) * u(t) \equiv u(t) * g(t)}}$$

The (scalar) convolution operator is <u>commutative.</u>

The convolution also works for systems that are not strictly proper, because:

$$y(t) = g(t) * u(t) = \int_{0-}^{t} g(t-\tau)\, u(\tau)\, d\tau$$

$$= \int_{0-}^{t} \left[\underline{c}' e^{A(t-\tau)} \underline{b} + d \cdot \delta(t-\tau) \right] u(\tau)\, d\tau$$

$$= \int_{0-}^{t} \underline{c}' e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau + \int_{0-}^{t} d\, \delta(t-\tau)\, u(\tau)\, d\tau$$

$$= \underline{\underline{\underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau + d \cdot u(t)}}$$