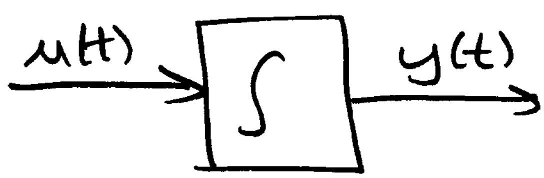
# The Superposition Principle:

The principle has 3 different aspects:

A) SISO system, no initial conditions:



We perform 2 experiments:

(1) $u(t) = u_1(t) \Rightarrow y(t) = y_1(t)$

(2) $u(t) = u_2(t) \Rightarrow y(t) = y_2(t)$

Now, we perform a third experiment:

(3) $u(t) = c_1 \cdot u_1(t) + c_2 \cdot u_2(t)$

if and only if

IFF the system is linear

$$\Rightarrow y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$

$$\forall\, u_1(t), u_2(t)$$

The system response to a mixture of the two input signals $u_1(t)$ and $u_2(t)$ is a mixture of the system responses to the individual signals.

The IF part is easy to prove:

$$y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\}$$

where: $Y_1(s) = G(s) \cdot U_1(s)$

and: $U_1(s) = \mathcal{L}\{u_1(t)\}$

Similarly:

$$y_2(t) = \mathcal{L}^{-1}\{G(s) \cdot \mathcal{L}\{u_2(t)\}\}$$

Hence:

$$u(t) = c_1 \cdot u_1(t) + c_2 \cdot u_2(t)$$

$$U(s) = c_1 \cdot U_1(s) + c_2 \cdot U_2(s)$$

$$\Rightarrow Y(s) = G(s) \cdot U(s)$$

$$\Rightarrow Y(s) = G(s) \cdot \left[ c_1 \cdot U_1(s) + c_2 \cdot U_2(s) \right]$$
$$= c_1 \cdot \left[ G(s) \cdot U_1(s) \right] + c_2 \cdot \left[ G(s) \cdot U_2(s) \right]$$
$$= c_1 \cdot Y_1(s) + c_2 \cdot Y_2(s)$$

$$\bullet\!\!-\!\!\circ$$

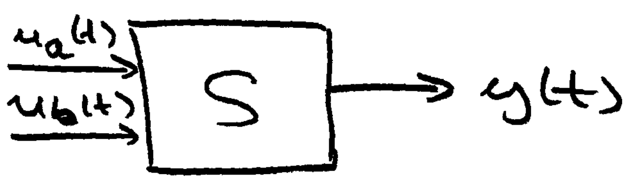
$$\underline{\underline{y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)}}$$

q.e.d.

The IFF part is not so easy to prove, and we won't even try.

## B) MISO system, no initial conditions

Given a multi-input/single-output (MISO) system with zero initial conditions, e.g.



We again perform 2 experiments:

(i) $\left| \begin{array}{l} u_a(t) = u_1(t) \\ u_b(t) = \emptyset \end{array} \right| \Rightarrow y(t) = y_a(t)$

(ii) $\left| \begin{array}{l} u_a(t) = \emptyset \\ u_b(t) = u_2(t) \end{array} \right| \Rightarrow y(t) = y_b(t)$

We now perform a third experiment:

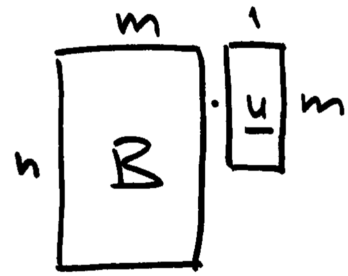
(iii) $\left| \begin{array}{l} u_a(t) = c_1 \cdot u_1(t) \\ u_b(t) = c_2 \cdot u_2(t) \end{array} \right|$

IFF the system is linear

$$\Rightarrow y(t) = c_1 \cdot y_a(t) + c_2 \cdot y_b(t)$$

The proof of the IF part is easy. It follows directly from matrix-vector calculus:

$$y(t) = \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \cdot \underbrace{B \cdot \underline{u}(\tau)}_{}\, dt$$




Let $B = [\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m]$ ; $\underline{u} = [u_1; u_2; \dots u_m]$

$$B = \left[\begin{array}{c:c:c:c} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_m \end{array}\right] \qquad \underline{u} = \begin{bmatrix} u_1 \\ \hdashline u_2 \\ \hdashline \vdots \\ \hdashline u_m \end{bmatrix}$$

$$\Rightarrow B \cdot \underline{u} = \underline{b}_1 \cdot u_1 + \underline{b}_2 \cdot u_2 + \cdots + b_m \cdot u_m$$

$$\Rightarrow y(t) = \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \cdot \left( \underline{b}_1 \cdot u_1(\tau) + \underline{b}_2 \cdot u_2(\tau) + \cdots + \underline{b}_m \cdot u_m(\tau) \right) \cdot d\tau$$

$$\equiv \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \underline{b}_1 \cdot u_1(\tau)\, d\tau + \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \cdot \underline{b}_2 \cdot u_2(\tau)\, d\tau + \cdots + \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \cdot \underline{b}_m \cdot u_m(\tau)\, d\tau$$

i.e., the response is indeed a superposition of the responses of the individual signals.

The IFF part is again not so easy to prove.

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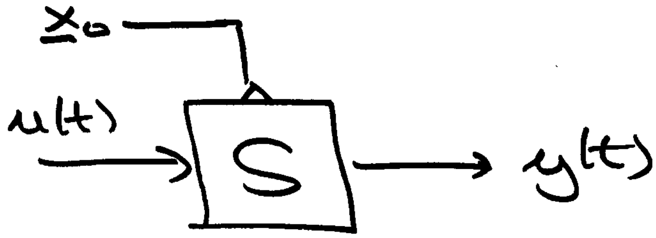
C) <u>SISO system with initial conditions</u>




We again perform 2 experiments:

(i) $\left| \begin{array}{l} u(t) = u_1(t) \\ \underline{x}_0 = \emptyset \end{array} \right| \Rightarrow y(t) = y_u(t)$

(ii) $\left| \begin{array}{l} u(t) = \emptyset \\ \underline{x}_0 = \underline{x}_{0_1} \end{array} \right| \Rightarrow y(t) = y_x(t)$

We now perform a third experiment:

(iii) $\left| \begin{array}{l} u(t) = c_1 \cdot u_1(t) \\ \underline{x}_0 = c_2 \cdot \underline{x}_{01} \end{array} \right|$

IFF the system is linear:

$$\Rightarrow y(t) = c_1 \cdot y_u(t) + c_2 \cdot y_x(t)$$

The proof of the IF portion follows directly from the convolution integral:

$$y(t) = \underline{c}' e^{At} \cdot \underline{x}_0 + g(t) * u(t)$$

Superposition

The IFF part is again much more dificult to prove.