ON THE BONDGRAPHIC POWER POSTULATE AND ITS RÔLE IN INTERPRETING THE 1905 AND 1915 RELATIVITY THEORIES

by

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE
WITH A MAJOR IN ELECTRICAL ENGINEERING

In the Graduate College

THE UNIVERSITY OF ARIZONA

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STATEMENT BY AUTHOR

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ACKNOWLEDGMENTS

I have no doubt in my mind that this work would have not been able to stand as a reality, had it not been for the will of God, who bestowed on me of his uncountabley infinite blessings what I cannot count. His true love and caring was, and will always be, the reason behind my venturing with no fear and my aspiring to loftier stratum.

The author would also like to take the opportunity to thank the following people, who played an important rôle in the process which had led to this thesis.

Although not involved directly, the company of my mother through the hard times I went through while working on this thesis, was definitely an important psychological factor that alleviated my loneliness. For her sacrifices since my birth, she can't be less than a full partner in my success.

I also cannot forget the instrumental rôle my employer, Saudi Aramco, played in achieving this goal. In particular I would like to thank my R&OED division coordinator Mr. M. A. Al-Mubarak for his confidence and support. Also Mr. E. R. Pickinpaugh, my mentor, for his wise advising through my PDP program; Dr. K. Kessinger for salvaging my nomination by securing my admission to the U of A; and finally to the Aramco Services Company's Academic Advisors for making this an unforgettable experience.

Last but not least, I would like to thank my thesis director, Dr. François E. Cellier and my relativity advisor, Dr. Lizhi Fang for their patience and illuminating remarks.

While lingering for the visitors,

Nasser M. N. Gussn
Tucson, Arizona,
September 5, 1994.

FOR THE LOVE OF HAKIMA

TABLE OF CONTENTS

TABLE OF CONTENTS — *Continued*

TABLE OF CONTENTS — *Continued*

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS AND ABBREVIATIONS

This list is intended to familiarize the reader with unfamiliar symbols or abbreviations. Please note that this is not a comprehensive list; all familiar or standard symbols or abbreviations are excused. The list is divided into three sections that sort the entries by their origin; Greek, Latin and symbols that are rather mathematical in nature or that do not belong to any of the previous categories. Where applicable, the equation number, appearing between parentheses, is indicated.

GREEK:

$\alpha, \beta, \ldots, \omega$	Indices that run from zero to three (lowercase letters only).
β	Ratio of velocity to the speed of light.(138).
$\Gamma^{\lambda}{}_{\mu\nu}$	Affine connection or the Christoffel symbol of the second kind.(40).
γ	Coefficient modulating speed in special relativity.(113a).
δ^{α}_{β}	Kronecker symbol.(18).
ε	Electric charge density.
$\varepsilon^{\mu\nu\lambda\kappa}$	Levi-Civita tensor.(29).
$\eta^{\alpha\beta}$	Minkowski metric tensor.(27).
Π	Tensorial bondgraphic power.(78) & (153).
ρ	Mass density.
$\Phi^{\alpha\beta}$	Electromagnetic field strength tensor.(142).
ϕ	Newtonian gravitational potential.
Ψ	Configuration-like variable.

LATIN:

$a,b,\ldots,z$	Indices that run from one to three (lowercase letters only).
A^{α}	Four-vector electric potential.
B	Magnetic field density.
BG	Bondgraph.
C	Capacitance or compliance (bondgraphic element).
E	Electric field intensity.
E	Energy.
e^{α}	Effort.
e	Electron charge.
E-M	Energy-Momentum.
f^{α}	Flow.
F^{α}	Four-vector force.
G	Newton's gravitational constant.
$G^{\alpha\beta}$	Einstein's tensor.(61).
$g^{\alpha\beta}$	Metric tensor.(26).
GBG	Generalized bondgraph.
GR	General relativity.
GST	General system theory.
GT	Galilean transformation.
GY	Gyrator (bondgraphic element).
H	Magnetic field intensity.

I	Inductance or inertance (bondgraphic element).
J^{α}	Four-vector electric current density.
LT	Lorentz transformation.
N or N	Dimension of abstract space or the range of the indices of a tensor.
n or n	Rank of a tensor.
MP	Multiport.
MGY	Modulated gyrator (bondgraphic element).
MTF	Modulated transformer (bondgraphic element).
ODE	Ordinary differential equation
$\wp(t)$	Non-tensorial power.(70) & (71).
P^{α}	Four-vector momentum.
p	Generalized momentum.(72b).
PDE	Partial differential equation.
Q_i	The ith state.
q	Generalized displacement.(72b).
R	Resistance (bondgraph element).
$R^{\alpha\beta}$	The Ricci tensor.(57).
$R^{\delta}_{\alpha\beta\gamma}$	Riemann-Christoffel curvature tensor.(52).
RS	Irreversible transducer (bondgraphic element).
R-C	Riemann-Christoffel.
S	Source (bondgraphic element).
SE or Se	Source of effort (bondgraphic element).

SF or Sf	Source of flow (bondgraphic element).
SGY	Symplectic gyrator (bondgraphic element).
SR	Special relativity.
$T^{\alpha\beta}$	Energy-Momentum tensor.
T	Kinetic energy.
TF	Transformer (bondgraphic element).
U^{α}	Four-vector velocity.

OTHER :

$\square 2$	d' Alembertian.(151).
∇	Del operator.
∇^2	Laplacian.
;	Covariant derivative.(38).
,	Gradient or ordinary partial derivative.
∂^{α}	Partial derivative with respect to x^{α}.(152).
$\frac{D}{D\tau}$	Covariant derivative along a curve.(164), (165) & (166).

ABSTRACT

The thesis provides a new formulation for the bondgraphic power postulate. In this formulation general tensors are used to build the efforts and flows. Examples from particle mechanics and electrodynamics were found to suggest that this tensorial power is identically equal to zero.

Commencing with modelbuilding, which is briefly introduced to stimulate the reader's interest in the formalism of the thesis, we proceed with discussions on multibondgraphs and tensor calculus; thus building the backbone of our research. Both subjects are treated only to the extent necessary for the preset objectives of this thesis.

The possibility of modeling Einstein's special relativity is also entertained. First the theory is introduced, followed by a bondgraphic model utilizing velocity modulated mass. Then another model, also based on bondgraphic concepts, that treats mass as a constant energy reservoir is compared with that of Einstein. The experimental evidence (although originally conducted for testing Einstein's energy-mass relation) is found mildly in favor of the Einstein formula.

The rôle of the tensorial power in modeling Einstein's gravitational field equations is then examined. After the theory has been introduced, the challenges facing this approach are delineated and possible solutions are provided.

"So much may be allowed to
the determinist; but his belief
that all human actions are
subservient to causal laws still
remains to be justified. If,
indeed, it is necessary that
every event should have a
cause, then the rule must apply
to human behaviour as much as
to any thing else. But why
should it be supposed that every
event must have a cause? The
contrary is not unthinkable..."
A. J. Ayer. Freedom and
necessity. [in:] Reason &
Responsibility by *Joel Feinberg*
(ed.).

1 INTRODUCTION: Objectives and Directions

This (introductory) chapter is intended to familiarize the reader with the structure of this thesis. We start by listing the objectives of the thesis, followed by a delineation of the prerequisites involved and the thesis-structure adopted. The most important fact that the reader should realize is that this thesis is highly interdisciplinary; the tools we use include sophisticated tensorial notations, differential geometry concepts, multibondgraphs, and special and general relativistic concepts. This of course does not mean that comprehending the chain of reasoning is difficult. We simply suggest that the reader will have to read and understand the introductory chapters on the subjects involved, unless he or she has already been initiated into them.

I would also like to explain how this research in relativity can help electrical engineers in particular tackle difficult problems in their research areas. First of all one should realize that relativity is very useful for analyzing many electrical engineering problems. For example, in his book on relativity and engineering, van Bladel (1984) states that:

Electrical engineers, in particular, are concerned with relativity by way of the electrodynamics of moving bodies. This discipline is of decisive importance for power engineers, who are confronted with problems such as

- *the justification of a forcing function ... in the circuit equation of a moving loop*
- *a correct formulation of Maxwell's equations in rotating coordinate systems*
- *the resolution of "sliding contact" paradoxes*
- *a theoretically satisfying analysis of magnetic levitation systems.*

The reader is also referred to the paper by Kron (1952) and to that by Hoffman (1949), which provide a sample of a direction that promotes the use of tensor calculus for establishing new theories pertaining to electrical engineering.

Unfortunately, some engineering colleges tend to deny their students such knowledge. Reasons for such negligence (by these colleges) were not researched, since such research is certain to require a considerable effort. The author was not able to reach any such studies on the subject.

Although it took one full academic year to acquire the basic knowledge to work with Einstein's special and general theories of relativity, I personally feel that the results thereof achieved are, at least, penetrating.

1 Objectives of the Thesis

The primary objective of this thesis is the extension of the bondgraphic power postulate to relativistic phenomena, including the phenomena influenced by the existence of a strong external gravitational field. By achieving this objective it is also hoped that the author will have accumulated a decent working knowledge on the art of modelbuilding. Such a knowledge is the ultimate extract that, theoretically, should enhance the author's capability of analyzing engineering systems in general, and control systems in particular. A third objective is to familiarize the author with the rudiments of differential geometry (through studying general relativity); a topic required to analyze non-linear control systems.

In order to meet these goals the following subjects will be discussed:

- In chapter **2** a cursory view of the art of modelbuilding will be provided. This should enable the reader to identify with concepts such as models, theories, physical domains and systems. Also a brief introduction to simulation and its rôle in modelbuilding will be given.
- It was found necessary to represent the power postulate in the language of tensors, which functions as the mathematical shell for the special and general theories of relativity. Hence an introduction to tensors will be given in chapter **3**.
- The bondgraph procedure for modeling systems will be presented in chapter **4**. Also in that chapter, the tensorial formulation of the power postulate of bondgraphs and the bondgraphic effort and flow variables are introduced.
- In chapter **5** we will apply the developed concepts to special relativistic particle mechanics and special relativistic electrodynamics. This, as will be shown, results in the design of a new model for the relativistic energy of a one-particle system, which will then be compared to the Einsteinian formulation.

- Finally, in chapter **6**, a modest introduction to the general theory of relativity will be provided. The author will also provide his thoughts on the meaning of Einstein's field equations as seen through a bondgraphic model.

2 Directions to the Reader

For this thesis we adopt the Harvard system for referencing. That is, we denote the reference by the last name(s) of the author(s) and the year of publication, which will appear between parentheses.

Equations are identified by Arabic numbers that are placed between parentheses. In case we need to refer to an equation from a previous section or chapter, we will refer to the equation using the above system, plus the (bold) Arabic numbers for the chapter and section respectively, separated by a colon. Thus, equation (12), **4** : **3,** refers to equation (12) in chapter four, section three.

It is also worth mentioning that, for convenience, we have coined two new words in this thesis. The first is modelbuilding, which refers to the art associated with modeling systems in general. The second word is bondgraphs. Also variants of this word such as multibondgraphs were admitted, in the belief that this approach will not impede the reader's ability of comprehending the discussed issues.

Also to be noted is that, excluding this chapter and the last one, all the chapters of the thesis furnish the reader with a selected bibliography on the subject(s) discussed in each one. Please refer to the table of contents for the location of the references.

"The success of any physical investigation depends upon the judicious selection of what is to be observed as of primary importance." *J. C. Maxwell.* [in:] Reality Rules I by *J. L. Casti*

2 THE ART OF MODELBUILDING

Modeling can justifiably be considered one of man's most distinguished activities. Although not in today's form or approach, modeling has helped humankind communicate with each other, and understand and adapt to their environment as well. Modeling can in fact be regarded as a documentation of the thinking process. Such a perspective classifies all symbolic systems invented by man, such as languages and arts as models. Even the acts of imagining and describing can be considered as modeling processes .

Thus the reader should realize how difficult it is to provide an account of the history of modeling. Still the appreciation of some of the great models man invented can be stimulated by the following exposure.

Models help understand the way processes change by explaining observations and providing predictions that can be verified. This concept was probably first applied to physics. One of the famous examples of this approach is the modeling of the solar system (Casti, part II, 1992). First there was the model for the orbits of planets in the solar system "built" by Claudius Ptolemaeus (c. 2nd. cent. AD). This model had the planets moving in orbits that were described by a collection of superimposed epicycles, the earth being at the center of the system. The model was good enough to explain the observations available at that time and produced predictions that satisfied the astronomers of the day. It was used successfully to predict eclipses, the positions of the planets, and most importantly the lunar positions influencing the flooding of the Nile. It even survived the discovery of new planets, the influence of which was accommodated using new epicycles that "updated" the predictions to agree with the new observations. Assuming that the sun is at the center of the solar system, Niklas Koppernigk (1473-1543) introduced the heliocentric model that was put in mathematical terms by Johannes Kepler (1571-1630) and Sir Isaac Newton (1642-1727). In this model the planets are pictured moving in elliptical orbits around the sun. Note that the predictions made by both models are very close. Actually the Ptolemaeus model is rather more accurate in its predictions, but the "simplicity" of the heliocentric model led to its adoption as the standard model. Abundantly, many other examples are found in science and engineering.

Although man practiced modelbuilding since probably the beginning of his existence – some times without realizing it – it never evolved to become rigidly defined. This led many scholars (see Stein and Rosenberg [1991]) to describe it as an art rather than a science (which is also the reason behind the name of this chapter). This vagueness surrounding the process of modelbuilding is rather natural especially if one realizes the vast field of applications this discipline has.

In 1947 this vagueness was considerably alleviated by the design of a standardized vehicle by which all processes can be modeled (Grinker 1967). The birth of the General Systems Theory (GST) on the hands of Ludwig von Bertalanffy (1968) provided a Global Theory for modeling processes in all branches of science. This development provided the first mover for what became known as the Systems science. (Note that although other scientists did provide earlier contributions, e.g. Köhler (1924), they did not deal with the problem in its full generality.)

Laying no claims as to the conclusiveness of this chapter, the author believes that it introduces the basic ideas of modelbuilding needed to guide the reader through the remainder of this thesis. First we clarify the connection between models and systems, and between models and theories. A classification of models and an introduction to the concept of physical domains is then provided. Also the phases of modelbuilding and the verification of the resulting model together with a simple discussion on systems will be given. Finally we present simulation as an alternative analysis tool and a natural extension to the process of modelbuilding.

1 Models, Systems and Theories

This section is devoted to providing a collection of definitions of models, systems and theories that should function as a basis for a better understanding of modelbuilding. Choosing to provide a collection (of definitions) stems from the difficulty of jotting down a universally applicable definition for any of the above concepts. The vantage of such an approach is evident in the richness one can attain by being exposed to different philosophies. We also explore the interconnections existing between these concepts in order to provide a unified picture for their rôles in modelbuilding.

Let us first consider models. In the common sense, the word model can refer both to a copy of something and to something that is to be copied. In modelbuilding, we focus on models in the first sense. To this end, Maki and Thompson (1973) state that:

When an investigator forms statements which he feels express basic principles in an area of observation and study, then it is often said that he has formed a model. The process ... is called model building. The model builder experiments and observes facts about the real world in his area of specialization. He then tries to explain and describe the phenomena that he is studying. He usually does this by proposing certain statements as the ones which are basic and most important.

McLean et al. (1978) point out that a fundamental feature of models is "that their construction and use involves a selective attitude towards information. All models are thus easier to deal with, both in mental and manipulative processes than the reality which they are designed to represent." They also state that "model-building ... seems to be the manifestation of a kind of cerebral 'least-action' principle rendering models basic to man's ability to conceptualize and deal with his environment." This selective attitude of models was also communicated by Paynter (1961) when he described modeling as "the artful act of abstracting from the totality of interactions between the elements of a physical system and the elements of its environment, and from among the various parts of the system itself, only those interactions which are relevant to the specific questions being asked ... ", and by Åström and Wittenmark (1990) who stated that "a model is a very useful and compact way to summarize the knowledge about a process."

Some of the above mentioned authors utilized concepts such as "physical system" and "process" to define models. Other authors manifestly build their definitions of models on concepts such as theory, experiment and system. For example, Casti (part I) (1992) introduced the theory of models arguing "that the concept of a model of a natural system N is a generalization of the concept of a subsystem of N, and that the essential feature of the modeling relation is the exploration of the idea that there is a set of circumstances under which the model describes the original system to a prescribed degree of accuracy. In other words, a particular facet of system behavior remains *invariant* under the replacement of the original system by a proper subsystem." Such definitions of models require a formal consideration of the concepts they introduce; what does one mean by theory?, for instance. What is a system?

James and James (1992) define theory as "the principles concerned with a certain concept, and the facts postulated and proved about it" and define system as "a set of quantities having some common property" or "a set of principles concerned with a central objective." Such set-theoretic definitions of systems were also given by Mesarovic (1968) and Blauberg et al. (1977). Maki and Thompson (1973) use the concept of an axiom system (which they define as a collection of undefined terms together with a set of axioms phrased with the use of these common undefined terms) to define the concept of theory, thus interconnecting the two concepts. According to them a theory is the collection of all theorems which can be logically deduced from an axiom system (where theorem is defined as a logical consequence of that axiom system). They also distinguish between two uses for the term theory; one in mathematical sciences and logic, which revolves around the definition they provide, and the other in social and life sciences, where a theory "is a collection of basic assumptions which is studied in an attempt to explain certain observed phenomena." Some authors associate wholeness with the definition of system. For example, Bahm (1969) states that "a system involves unity or wholeness of some sort that holds its parts together," and Bluemenfeld (1970) states that "the system interacts with the world outside as a whole." Another important feature of a system is the interaction between its elements. This feature is the basis of the Mclean et al. (1978) definition of a system as "a collection, or set, of interacting elements, the interactions between such elements giving rise to complex behavior." Other compatible definitions were also given by von Bertalanffy (1950) and Kast and Rosenzweig (1972), among others. These element-based definitions, together with the delineation (the interaction and structure) of systems can be interpreted as a modelbuilding process, leading to a model (subsystem) that can be utilized for analyzing the original process (system).

The above definitions indicate a correlation among models, theories and systems. They also indicate the ubiquitous nature of the three concepts and how difficult it is to eliminate their overlapping. Still, for our objectives in this thesis, we only need to establish a clear distinction between the concept of a model and that of a theory, and establish a fixed relation between models and systems (which is needed in order to justify our usage of the terms).

We feel that the best of all explanations on the relation between models and theories (which is the explanation we adopt throughout the remainder of this thesis) is the one given by Coombs et al. (1954):

A model is not of itself a theory; it is only an available or possible or potential theory until a segment of the real world has been mapped into it. Then the model becomes a theory about the real world. As a theory, it can be accepted or rejected on the basis of how well it works. As a model, it can be right or wrong only on logical grounds. A model must satisfy only internal criteria; a theory must satisfy external criteria as well.

(The reader should not be surprised to find that some authors, e.g. Simon and Newell (1956), use the terms "model" and "theory" interchangeably.)

On the relation between models and systems, we will adopt the views communicated by Casti (1992), thus treating models as subsystems and bestowing on them all the properties enjoyed by systems.

2 Classification of Models

This section is devoted primarily to providing a satellite view that enumerates different types of models (and systems; being models themselves.)

In literature one can find many classifications of models. Almost all of these classifications provide a dichotomy.

For example, models can be classified as either experimental or analytical (Brogan 1991). Experimental models are built by selecting mathematical relations which seem to fit observed input-output data. One basic approach to achieve this is the principle of least squares. The process of building such models and then fitting the models to data is also known as system identification. On the other hand, analytical models utilize physical laws to establish the relation between the components of the system under study.

Both types of the above models can be classified as mathematical, since they use a mathematical tool (although different for each model type) for mimicking the dynamics of the original system. Other models, usually referred to as physical models, such as the billiard ball model for the behavior of an enclosed gas, or the apparatus build by Kepler to model the orbits of planets, are more oriented towards metaphoric physical (material) representation. Sometimes it proves useful to combine both types when analyzing non-observable processes.

Maki and Thompson (1973) draw a distinction between mathematical models and real models (which they define as the approximation and idealization of the real world) while stating that "it is very difficult to decide where the real model ends and where the mathematical model begins."

Mathematical models in particular are very useful for studying the dynamics of systems. The type of mathematical tool or tools to be used depends on the process itself and on the applications intended for the model. Our interest in models in this thesis is oriented towards mathematical ones, which is natural since almost all analysis in science and engineering are based on them. (A view that is not universally accepted). Note that models based on bondgraphs are also mathematical in nature, but incorporate an important physical postulate, viz. the power postulate (cf. Ch. 4).

The following classifications are all for mathematical models.

First the dichotomy realizing deterministic and stochastic models. Deterministic models predict the future behavior of the system if provided with sufficient information at one instant in time or at one stage. This predictability is weakened in the case of stochastic models. For these models the predictions are probabilistic and no matter how much one knows about the system at a given time, it is impossible to determine with "absolute" certainty the future behavior of the system.

We also find statespace models and input-output models in literature (although more commonly referred to as systems rather than models). Both types of models are usually a subset of deterministic models, and can either be continuous or discrete. In the continuous case they are built of ODEs (statespace models are built of first order ODEs only). In the discrete case, difference equations are used. The decision to use continuous or discrete models depends on the amount of data the modeled process provides and on the nature of dynamics involved. In general, many continuous models can be converted to discrete models while successfully reproducing the dynamics of the original system (to a preset degree of accuracy). In the statespace model case, the state concept provides a complete summary of the status of the system at a particular point in time using state variables which provide information on the internal dynamics of the system. Examples of state variables are entropy in the thermodynamic domain, charge in the electric domain and displacement in the mechanical domain (more on physical domains in the next section). On the other hand input-output models are external. They provide information about the input and the output of a system without any treatment of the internal dynamics. Thus they are very similar to Ashby's black box models (1958).

Note that PDEs are also used to build models. This approach is the standard for realizing models of fields (e.g. electromagnetic or gravitational fields), or what are known as continua models (where particle models are built using ODEs).

Bondgraph models have been very successful in modeling particle based processes. They were also applied to continua phenomena via reduction to particle level. (The reader might relate better to the lumped and distributed parameter terminology referring to particle and continuum models respectively.)

Mathematical models can also be built of linear or nonlinear equations. Usually complex dynamics require nonlinear equations to describe them, although it is sometimes possible to linearize the model when focusing on small variations.

Before leaving this section we need to emphasis that it does not provide an exhaustive account of the classifications found in literature. Still the provided classifications are the most dominant ones and can easily guide the reader through the remainder of this thesis.

3 Physical Domains

According to Breedveld (1984), (section 5.2.2) "every type of scalar state variable or vector component of a directed ... state variable ... with conjugate effort and flow corresponds to a so-called 'physical domain' ." In other words, one can state that physical domains are distinguished from each other by their effort and flow pairs. Note that their is no universally accepted table for these domains, and that different identifications of effort and flow variables, produce different classifications of physical domains.

Table 2.1 provides the framework of van Dixhoorn's physical system theory for physical domains as they are identified in Paynterian bondgraphs (BGs). Another extended framework, that incorporates the magnetic and material domains, was adopted for Generalized Bondgraphs (GBGs) (Breedveld, ib. p. 50). Actually, in chapter five, we utilize the so-called electromagnetic domain. (Note that the chemical and thermodynamical domains do not have generalized momenta.)

Table 2.1. Physical Domains in Paynterian BGs

Domain	Effort	Flow	Generalized momentum	Generalized displacement
Translation	Force (N)	Velocity (m/s)	Momentum (Ns)	Displacement (m)
Rotation	Torque (Nm)	Torque velocity (rad /s)	Angular momentum (Nms)	Angle (rad)
Hydraulic	Total pressure (N/m^2)	Volume flow (m^3/s)	Pressure momentum (Ns/m^2)	Volume (m^3)
Acoustic	Pressure (N/m^2)	Volume velocity (m^3/s)	Momentum (Ns/m^2)	Volume (m^3)
Electric	Voltage (V)	Current (A)	Flux linkage (Vs)	Charge (C)
Chemical	Chemical potential (J/mol)	Molar flow (mol/s)		Molar mass (mol)
Thermo-dynamical	Temperature (K)	Entropy flow (J/sK)		Entropy (J/K)

The unifying approach of dealing with different physical domains is an important tool by which modelbuilding becomes oriented towards explaining the way different physical domains interact. The interactions are described perfectly by gyrators and transformers (see chapter four for definitions of gyrators and transformations). Thereby modelbuilders can treat all physical phenomena using a single (unifying) concept.

4 Modelbuilding phases

This section provides a delineation of building phases a model must go through before becoming a useful analysis tool. We focus our attention on mathematical tools in particular; for the reasons mentioned earlier. Note that our treatment of phases is not universal. (For example, a different outline of the steps involved is available in the book by Brogan [1991], p. 5). We adopt the work by Maki and Thompson (1973, pp. 1-7) for our presentation of the phases involved in modelbuilding.

The first phase in modelbuilding is devoted to studying the process or system to be modeled. Obviously the success of this step is related to the degree of familiarity and understanding the modelbuilder has for the to-be-modeled process. This prompts a question on the skills needed for successful modelbuilding. The author's personal view is that a good command of mathematics is the most important tool any modelbuilder needs. But mathematics alone is not going to produce real world models. As we will note later in this chapter, simulation is becoming a standard (while not the only standard) in analysis of dynamical systems. This requires a strong understanding for a number of programming languages and the ability to utilize simulation software. Also an ability to recognize patterns and general structures can be of great help for modelbuilders, which enables them to apply the results of one mathematical modeling experience to several branches of science and engineering.

The second phase involves idealizing and approximating the process; making the analysis more precise. The modelbuilder is required to make decisions on what information or features of the process are relevant to the conducted study, and then to reduce the real process to an ideal one that can, satisfactorily, mimic the involved dynamics of the parent process. This phase is usually the most difficult one due to the type of decisions that must be made.

The third phase transforms the operative processes at work in the approximated and idealized system to symbolic terms and mathematical operations. Note that such a transformation is not unique. One can design experiments to show that one mathematical model is better than others (according to Kuhn, this is impossible in practice!), but it should also be clear that it is difficult to build a model that can account for all facets of the problem under consideration.

When the mathematical model is completed, the fourth phase begins. In this phase, various analyses of the system are conducted using appropriate mathematical tools. This results in a set of predictions on the performance and dynamics of the original process.

This new information about the original process completes one modelbuilding cycle. In the next section we introduce the most important concept relating the model to its parent process.

5 Model Verification

The model evolving from the four phases already mentioned, does not reveal how rewarding our modeling is, until we can validate the data it provides. This process should be looked at as a falsification rather than a validation of the model, which is the position Karl Popper adopted. "His claim is that real scientific models are set forth in a way that spells out observations and predictions that can be tested experimentally. If the prediction fails, then the model is falsified and must be abandoned or completely re-thought. But if a model passes its crucial test, it's not validated but only 'corroborated.' In this case the process of testing must continue." (Casti II 1992).

Usually the first modeling attempt is not very successful in accounting for the observations about the dynamics of the modeled process. This requires a new cycle of the four phases of modelbuilding or more often, a partial re-iteration of that cycle. Another reason for re-building models is updating them. This is due to the measurement capabilities available. Sometimes measurements cannot attain the accuracy degree needed to judge models, a fact which usually leaves us with little choice but to learn to live with the available ones until they become obsolete as investigators acquire higher standards of measurement. In general, most models implemented in science are frequently revised; since any new information about the process(es) in question must be incorporated into the existing model(s).

It goes without saying that model verification is often not a straightforward task; the ability to design experiments that can generate the data to be compared with the analytical output from the investigated model is often hard to acquire. Some might argue that it was the data obtained from the process that enabled the modeler to produce his or her model in the first place; thus the data needed for verification should already be available. Their argument is valid. Still we need to remind ourselves of the function models should perform. Even if the model agrees with the data provided by the original process, in many cases it must also be able to successfully predict the future dynamics of that process. Often that is the part for which we design experiments.

This section should provide the reader with a dynamic picture of science; all the theories we have today are not sacred, and our minds should be able to see them constantly modified as we continue to progress. Such a mentality of change speeds up the evolution of new frontiers, enabling human beings to gain more knowledge and power than they have ever anticipated in our lifetime.

6 Hierarchical Models

In this context, hierarchy refers to any set of relations in which units are organized into more inclusive units. Note that all models (systems) are hierarchic in one way or another. All hierarchies are concerned with relations, but some are concerned more with structure, others with functional aspects of the system. Complex systems often exhibit overlapping hierarchies, so that some units are involved in more than one hierarchy, and sometimes more than one hierarchy is involved in the same function (Bowler 1981).

In order to appreciate the concept of hierarchical modeling, the reader is first required to understand the logic behind the idea of levels in modelbuilding. Take for example the model of an electric power system. Such a model must be built to meet the objectives of the conducted study. If the purpose is exploratory, simple models will suffice. On the other hand, as the study becomes more concerned with the details of the performance of the system, the modelbuilder becomes obliged to explicate more than a single level of the system. Here the levels are separated by establishing functional dependencies that allow the modeler to treat components of the studied system as black boxes when modeling at a high level (macroscopic level) and delineate the structure of the components themselves as he or she start to model the microscopic levels. Thus the influence of the components can be incorporated without the need to examine their dynamical details, while maintaining the possibility of modifying the specifications of any component as it alters the performance of the system.

The idea of the level of a model is very similar to that of the subsystem. As we start to recognize models as systems, we begin to map the levels of a model as subsystems and progress upwards, to higher modeling levels that eventually produce the total or complete system.

From the above perspective one can clearly see that such an approach becomes indispensable as the modeled process becomes complex. That is to say, in order to model any real world processes the modelbuilder will find it necessary to model the process as a system that is composed of a number of subsystems that have simpler structures or dynamical performances. This also becomes convenient as the modelbuilder engages in building models that share components or subsystems that have already been modeled.

7 The Systems Approach in Modelbuilding

Through the second half of the twentieth century, the idea of systems was able to penetrate almost all fields of science, becoming the basis for rapidly developing methods for the solution of complex problems. This approach of representing processes as systems is part of a modelbuilding procedure that is oriented towards generalizing the type of (mathematical) model used to analyze various kinds of processes under different disciplinary umbrellas. This goal is best achieved by structuring the systems approach on heterogeneous theoretical concepts. In this section we provide a cursory look into this type of models, a look that is intended to initiate the reader on this methodology without diverting our efforts away from the main stream of this thesis. For more comprehensive treatment, the author provides some of the often cited and most recent works in the bibliography section of this chapter.

One way of defining a system mathematically is the statespace approach. This approach, as mentioned earlier, uses a "system" of first order ODEs to mimic the dynamics of the parent process. When denoting the states by Q_i $(i = 1,2, \dots, N)$, the mathematical representation can be written as

1
$$\begin{cases} \dfrac{dQ_1}{dt} = f_1(Q_1, Q_2, \dots, Q_N) \\ \dfrac{dQ_2}{dt} = f_2(Q_1, Q_2, \dots, Q_N) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dQ_N}{dt} = f_N(Q_1, Q_2, \dots, Q_N) \end{cases}$$

(Bertalanffy 1968). This representation is found in many fields. For example, it was used by Skrabal (1944, 1949) to model the mass action, by Lotka (1925) to model demographic problems, and by Spiegelman (1945) for kinetics of cellular processes and the theory of competition within an organism. Werner (1947), used a similar system to model the basic law of pharmacodynamics which is used to derive the various laws of drug action.

The above representation can be used to demonstrate a variety of behavior patterns that systems may exhibit. First let us limit our study to a system with a single state. This meets our objectives without any loss of generality. The mathematical representation becomes

2 $$\frac{dQ}{dt} = f(Q).$$

Expanding the right hand side of the above equation into a Taylor series around the origin and truncating after the first term we get

3 $$\frac{dQ}{dt} = \alpha_1 Q \ ,$$

(α_1 being the Taylor series coefficient) where the zeroth term is equal to zero on the assumption of no spontaneous generation of elements. The solution of equation (3) is the well known exponential law

4 $$Q = C_1 \exp(\alpha_1 t).$$

(C_1 is the constant of integration.) This law appears in many fields. In mathematics it is often called the law of natural growth (for $\alpha_1 > 0$), and is applied to the growth of capital by compound interest. In social sciences it is known as the law of Malthus and signifies the unlimited growth of a population whose birth rate is higher than its death rate. When the constant α_1 is negative, equation (4) can be applied to radioactive decay and to the extinction of the population when the death rate is higher than the birth rate. Other applications can be found in many other disciplines, including biology and chemistry.

Another law for growth can be obtained by truncating the series after the second term. The resulting equation is

5 $$Q = \frac{\alpha_1 C \exp(\alpha_1 t)}{1 - \alpha_2 C \exp(\alpha_1 t)},$$

where C is a constant and α_1 and α_2 are the Taylor series coefficients. The growth produces the so-called logistic curve which also has applications in many fields. For example, in sociology it is called the Verhulst law and represents the growth of human populations with limited resources.

The representation of equation (1) can also be used to model the phenomenon of competition. Take for example the simplest case of two competitors. One can reduce the complexity of the model farther by assuming $\alpha_{i \neq j} = 0$. Then the resulting law is given by

6 $$Q_1 = BQ_2^{\,A},$$

where,

$$A = \alpha_1 / \alpha_2$$
$$B = C_1 / C_2^{\,A}.$$

(In biology, equation (6) is known as the allometric equation and in sociology, it is known as Pareto's law.) More complex cases (for which not all $\alpha_{i \neq j}$ are equal to zero) can be found in the works by Volterra (1931) and Spiegelman (1945).

Other behavioral patterns such as wholeness, mechanization and finality are also modeled by (1). The interested reader is referred to the masterpiece written by Bertalanffy (1968) for a detailed treatment. Note that all the above mentioned patterns are actually notions pertaining to the inner structure of the system model (Blauberg et al. 1977). This particular conceptual tool is a very powerful one in modeling processes that maintain a static structure. Such processes are dominant in engineering (practically all machines can be assumed to have static structures) and physics. Unfortunately, the structural approach faces serious challenges when applied to biology and sociology where processes tend to evolve and maintain dynamic or varying structures.

In the next section, a technique referred to as Diakoptics is introduced. It is based on the concept of tearing the structure of a complex system (in order to simplify the analyses involved).

8 Diakoptics

The etymology of the word " Diakoptics" is from the Greek "kopto" meaning "to tear" and "dia" that reinforces the word to follow and may be interpreted as "system." The word was suggested by Prof. Philip Stanly, of the Department of Philosophy at Union College, Schenectady, NY (Kron 1963). Hence the method is also known as "the method of tearing." "The method of subspaces " is an earlier name for the same method that was coined in the early 1940's (Hoffmann 1944). Kron started to tear systems in the mid 1930's, for the purpose of setting up the equations of large systems in a piecewise systematic manner. In the early 1950's he began to solve each subdivision first and then interconnect the solved equations (Kron 1956). That was the harbinger of a new vista in electrical engineering, providing a program that utilized tensor analysis as a mathematical shell.

As defined by its inventor, Diakoptics "Is a combined theory of a pair of storehouses of information, namely equations + graph, or matrices + graph, associated with a given physical or economic system. The graph of the system is also put to work to assuage the monstrous appetite of the high-speed digital or analogue computer" (Kron 1963). Diakoptics is definitely not limited to physical systems but extends to any system which has a large number of variables and is representable by a block-diagram or a graph. The system may contain negative, non-linear and other arbitrary algebraic functions.

This contribution by Kron helped expedite the development of what became known as bondgraphs. The existence of such a conceptual kinship between the two techniques was explicated by Henry Paynter (1969 & 1992), the inventor of Bondgraphs. Due to the more general structural treatment that bondgraphs supported, and to the elimination of the "awesome" tensorial notation that was adopted for Diakoptics (the author re-introduces this notation to bondgraphs, believing that it plays an important rôle in the analysis of complex systems), and also due to the use of a more general type of graphs, bondgraphs became much more established in scientific and engineering circles. An introduction that focuses on bondgraphs as a modeling tool is found in chapter four. Although not instrumental to this thesis, the reader might also want to experience the applications of Diakoptics as a modeling tool; for that purpose, a number of good texts on the subject is provided in the bibliography section.

9 Simulation and Modelbuilding

Before leaving this chapter, we will discuss briefly the process of simulation and its rôle in modelbuilding; first a definition of simulation is introduced, then we focus our attention on mathematical simulation and on the software used to simulate dynamical systems.

Shannon (1975) defines simulation as "the process of designing a computerized model of a system and conducting experiments with this model for the purpose either of understanding the behavior of the system or of evaluating various strategies for the operation of the system." Another definition given by Korn and Wait (1978) states that a simulation "is an experiment performed on a model." Although many other definitions are found in the literature; the relation between simulation and models (or modelbuilding) almost always revolves around the idea of systems. Simulation is a tool for "observing" the behavior of systems that do not exist (e.g. mathematical models), or for "observing" the unobservable (or rather the difficult-to-measure) behavior of real systems.

The application of continuous dynamic simulation (CDSS) languages on personal computers (PCs) and workstations is becoming the standard analysis tool for engineers and scientists. These languages can be discrete, continuous or combined. For discrete event simulation, the most popular languages are SIMSCRIPT, GASP, GPSS, and SIMULA (Karayanakis 1993). For more information on discrete simulation languages, the reader can consult Deo (1983) and Kreutzer (1986). Continuous languages are used in verifying analog simulation results. The most popular ones are ACSL and SIMULINK. Other languages are developed to simulate hybrid systems. GASP IV and CLASS are two of the well-known combined simulation languages. For an excellent treatment of combined simulation languages see Cellier (1979a & 1979b).

A standardization effort by the Simulation Software Committee of Simulation Councils, Inc. (SCi) resulted in developing CSSL (Continuous System Simulation Language) in 1967. CSSL is a problem-oriented language for the simulation of continuous dynamic systems that can be modeled by systems of ODEs. For more on CSSL, the reader is referred to Stephenson (1971). Some of the most popular languages based on CSSL are ACSL, RSSL, DARE, DARE-P, HYTRAN, and SL/1.

Like any other technique, modeling via simulation has its advantages and disadvantages. Some of its advantages are (Adkins and Pooch 1977):

1. Provides controlled experimentation environment; with regard to the time of the experiment, the variation of parameters, and the number of times it is carried out.
2. Permits sensitivity analysis by input variables' manipulation.
3. Permits experimentation without altering the real system.

4. Is a very effective tool for training purposes.

The salient drawbacks of simulation can be summarized as follows:

1. In terms of development time, man power and computer time, simulation may become expensive.
2. The results of simulated experiments can diverge from the behavior of the real process. This is primarily due to the selective attitude of modelbuilding.
3. Initializing the parameters involved in a model may also prove to be difficult. This involves extensive developing time in collection, analysis and interpretation.

As a final remark, the reader should realize that although simulation is widely considered an integral part of the decision-making process carried out by modelbuilding, the successful application of the technique is still an art that is heavily dependent on the modelbuilder's experience on applying simulation.

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"Starting from the concept that there exists a unique privileged observer of the cosmos, namely man himself, natural philosophy has journeyed to the opposite pole and now accepts as a fundamental principle that all observers are equivalent, in the sense that each can explain the behavior of the cosmos by application of the same set of natural laws." *D. F. Lawden*, An Introduction to Tensor Calculus, Relativity and Cosmology, 3rd edition

3 AN INTRODUCTION TO TENSOR CALCULUS

It seems that the concept of stress in mechanics is the historic origin that led to tensors (tenseur, that which exerts tension, stress [Kay 1988]). Tensor calculus was developed by Georg Friedrich Bernhard Riemann (1826-1866), Elwin Bruno Christoffel (1829-1900), Curbastro Gregorio Ricci (1853-1925), and Tullio Levi-Civita (1873-1941) as a tool for the study of n-dimensional spaces undergoing transformations of reference frames subject to some condition of invariancy. In the early 1920's, the distinguished Albert

Einstein (aided by Marcel Grossmann) championed its inclusion in physics as the mathematical shell for his relativity theory. Kron was the one to inject the compactness of such analysis into electrical engineering in 1935 (Bewley 1961).

The most important concept that tensors introduce is that of physical entities. This makes tensors the most suitable of tools on which one might successfully build a general physical systems theory. Thus there is no doubt that tensor calculus is one of the most suitable (necessary but not sufficient) bases for a unifying theory in science.

This chapter gives a cursory exposition that is intended as a guide to the subject rather than a self-contained material. The reader will find much more exhaustive treatment in the bibliography at the end of the chapter. Still the author believes that this material is quite adequate for a first encounter with tensor analysis and for the subsequent developments utilizing it in this work.

1 Generalized Spaces

The relation between analysis and geometry, as it stands in the realm of real dimensions at the end of the twentieth century, has proven to be a very powerful tool in attacking problems in science, especially sciences built on mathematical foundations (the so-called hard sciences.) The power of this method lies in the fact that it helps one "see" the relation between the variables over which the analysis is performed, by representing their variation as trajectories of points in three dimensional spaces. As the labyrinth established in the world of "systems" of more than three variables evolve, the "seeing" advantage starts to slip through our fingers (but is not exactly lost, thanks to the projection theorem). The geometric interpretation of any physical "system" can however still help our understanding in areas such as analysis and synthesis (or design) of physical "systems."

Such a generalization of three dimensional spaces allows us to think of N dimensional spaces (where the dimension, N is allowed to be finite, or infinite), with N dimensional coördinates, that are linearly independent of each other and thus form a so-called basis of the space (for more on this formalism on generalized spaces see [Dudley 1994] or [Brogan 1991]).

2 Euclidean Space

The modern meaning of a Euclidean space is that of a generalized N dimensional one. But what makes that generalized space Euclidean is the definition of distance between two points that live inside it. The square of the distance between adjacent points is given by

7 $$ds^2 = \sum_{i=1}^{N} dx^i,$$

where the x^is are rectangular Cartesian coördinates. This expression representing ds^2 is called the metric or the fundamental form or simply the square of the line element. Note that if the space allows complex numbers as values for the coördinates, the generalized space is called a unitary space and equation (7) needs to give the absolute value of the difference of the coördinates. Before introducing any other generalized spaces, a digression is in order to introduce the formalism of tensors that will facilitate writing our equations.

3 Orthogonal Transformations

Starting with rectangular Cartesian coördinates x^i and formulating the distance between two points $P_1(x^1, x^2, \dots, x^N)$and $P_2(y^1, y^2, \dots, y^N)$ in an N dimensional Euclidean space (where $x^i, y^i \in \mathbf{R}^N$), we obtain:

8 $$s^2 = \sum_{i=1}^{N}(x^i - y^i)^2.$$

Now if we carry out the linear transformation,

9 $$\bar{x}^i = \sum_{j=1}^{N} a_{ij}x^j + b^i \qquad i = 1,2,3,\ldots,N$$

and find that the coefficients of the transformation satisfy

10 $$\sum_{i=1}^{N}\left(\bar{x}^i - \bar{y}^i\right)^2 = \sum_{i=1}^{N}\left(x^i - y^i\right)^2$$

for all rectangular Cartesian coördinates $x^i, \bar{x}^i$ and $y^i, \bar{y}^i$, then the transformation is said to be orthogonal. Note that the square of the distance between points P_1 and P_2 is invariant, i.e. independent of the Cartesian Frame used.

Let $z^i = x^i - y^i$,and $\bar{z}^i = \bar{x}^i - \bar{y}^i$. Then (9) can be written as:

11 $$\bar{z}^i = \sum_{j=1}^{N} a_{ij}z^j.$$

Now in matrix notation (11) takes the form,

12 $$\bar{z} = A\,z.$$

and (10) becomes

13 $$\bar{z}^T\bar{z} = z^T z$$

taking the transpose of (12) gives,

14 $$\bar{z}^T = z^T A^T$$

If we substitute (14) and (12) in (13) we get

15 $$A^T A = I_N$$

Thus by taking the determinant of both sides we find that $|A| = \pm 1$(where the positive value belongs to "proper" rotation, while the negative value belongs to orthogonal transformations involving a reflection), which means that A is non-singular and that post multiplying (15) by A^{-1} gives (remember that we restrict our work to $\mathbf{R}^N$)

16 $$A^T = A^{-1}.$$

In components notation the above results are summarized as follows:

17 $$\sum_{i=1}^{N} a_{ij}a_{ik} = \delta^j{}_k$$
$$\sum_{i=1}^{N} a_{ji}a_{ki} = \delta^j{}_k$$

where $\delta^j{}_k$ is the Kronecker symbol defined by

18 $$\delta^i{}_j \equiv \left.\begin{matrix} 1, i = j \\ 0, i \neq j \end{matrix}\right\}$$

Note that the conditions in (17) are the necessary and sufficient conditions for the orthogonality of a linear transformation.

4 General Coördinate Transformations

The transformation in (9) can be written in the general form

19 $$\Gamma : \bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^N) \qquad (i = 1, 2, \ldots, N).$$

This form is a representation of the C^2 class (i.e. $\bar{x}^i(x^1, x^2, \ldots, x^N)$ has continuous second-partial derivatives at every point in the region) mapping in a N dimensional abstract space from x^i to $\bar{x}^i$. If this transformation is bijective (i.e. one-to-one and onto), it is called a coördinate transformation, and if x^i are rectangular Cartesian coördinates, as in (9), $\bar{x}^i$ are called affine coördinates. If the transformation is nonlinear, then $\bar{x}^i$ are called curvilinear coördinates. The most common curvilinear coördinates are the polar, cylindrical and spherical coördinates (Kay 1988).

In pursuit of our goal of a general mathematical shell for physical systems, it will become necessary to use coördinate systems that are not tied to the rectangular coördinates. Distance in those coördinates is represented through a functional such that it remains constant under all admissible transformations. Generally this is going to destroy the Euclidean properties of the spaces under consideration. At this stage its important to stress the independent character of the functional (or metric) representing the space and the coördinate system adopted to describe the space. This independence holds for all coördinate systems with the rectangular coördinates being the exception, where the coördinates are defined by the metric (Kay 1988, p.26.)

5 Range and Summation Conventions

Quoting from Synge & Schild (1949):

Range Convention:

When a small Latin suffix (superscript or subscript) occurs unrepeated in a term, it is understood to take all values 1,2,...,N, where N is the number of dimensions of the space.

Summation Convention:

When a small Latin suffix is repeated in a term, summation with respect to that suffix is understood, the range of summation being 1,2,...,N. It will be noticed that the reference is to small Latin suffixes only. Some other range (to be specified later) will be understood for small Greek suffixes, while if the suffix is a capital letter no range or summation will be understood. [in this thesis the range for Latin suffixes is from 1 to 3, where Greek suffixes run from 0 to 3]

Thus when the summation convention (due to Albert Einstein) is applied to (17) we obtain

20
$$a_{ij}a_{ik} = \delta^j{}_k$$
$$a_{ji}a_{ki} = \delta^j{}_k$$

where the summation takes place with respect to i from 1 to N. Sometimes the repeated suffixes are referred to as dummy indices and the non-repeated ones as free indices. The reason behind the nomenclature is that the repeated indices can be replaced by any other indices without changing the value of the sum, where such a change of independent (non-repeated) indices generally does not preserve the equation. Thus in tensor equations the same free indices can appear in every term only once, but a dummy index may only appear in every term twice.

6 Tensors

Einstein (1916) defines tensors as follows:

Let certain things ("tensors") be defined with respect to any system of coördinates by a number of functions of the coördinates, called the "components" of the tensor. There are then certain rules by which these components can be calculated for a new system of coördinates if the transformation connecting the two systems is known. The things hereafter called tensors are further characterized by the fact that the equations of transformation for their components are linear and homogeneous. Accordingly, all the components in the system vanish, if they all vanish in the original system.

Gabriel Kron defines "tensor" as another term for "physical entity" and defines tensor analysis as "the study of physical phenomena in terms of the physical entities themselves." He also emphasizes that a tensor is not a matrix with a definite law of transformation, but rather that the n-way matrices are the projections of the physical entity we call a "tensor" (Kron 1942). Bishop and Goldberg (1968) give the following definition:

Let V be a vector space. The scalar-valued multilinear functions with variables all in either V or V^* (the dual space of V) are called tensors over V and the vector spaces they form are called the tensor spaces over V.

Tensors have ranks (also known as valences or orders). The rank of a tensor is the number of indices of that tensor. The total number of components of a tensor is equal to N^n, where N is the range of the indices (i.e. the dimension of the space) and n is the rank of the tensor. The simplest tensors are tensors of rank cypher (scalars) which are invariant, i.e. they have the same representation in every coördinate system. First rank tensors are vectors and they transform according to one of the following rules:

Contravariant vectors (with superscripts):

21 $$\overline{V}^i = V^j \frac{\partial \overline{x}^i}{\partial x^j}$$

Covariant vectors (with subscripts):

22 $$\overline{V}_i = V_j \frac{\partial x^j}{\partial \overline{x}^i}$$

In a similar manner the higher order tensors can be defined as contra- or co- variant tensors. The following is for the most general case that involves the transformation of a (k+l) order tensor:

23 $$\overline{V}^{i_1 i_2 i_3 \ldots i_k}_{j_1 j_2 j_3 \ldots j_l} = V^{m_1 m_2 m_3 \ldots m_k}_{n_1 n_2 n_3 \ldots n_l} \frac{\partial \overline{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \overline{x}^{i_k}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{n_l}}{\partial \overline{x}^{j_l}}$$

Note that tensors with a rank of two or more can be mixed, i.e. having contra- and co-variant indices (superscripts and subscripts). An example on a mixed tensor is the Kronecker symbol:

24 $$\delta^\mu_\nu \frac{\partial \overline{x}^\rho}{\partial x^\mu} \frac{\partial x^\nu}{\partial \overline{x}^\sigma} = \frac{\partial \overline{x}^\rho}{\partial x^\mu} \frac{\partial x^\mu}{\partial \overline{x}^\sigma} = \delta^\rho_\sigma$$

Thus the Kronecker symbol is an invariant with respect to the above transformation which is the reason why it is termed the fundamental tensor of the second rank (Lawden 1975).

7 Riemannian Space

One of the most important tensors is the metric tensor which is associated with the distance concept and is associated with it via the following equation

25 $$\pm ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta .$$

With the positive sign the interval is said to be space-like, and with the negative sign it is said to be time-like. As soon as we define this squared infinitesimal distance, which is an invariant homogeneous quadratic function of the coördinate differentials, we designate the manifold a "metric space" or a "Riemannian space" (Bergmann 1976). Note that if the rectangular Cartesian coördinates in equation (7) were written in terms of curvilinear coördinates x^α, equation (7) would become similar to equation (25), except that the metric tensor will be of a different form. In Euclidean space, the metric is always positive-definite. When the metric is allowed to be indefinite, it is then associated with non-Euclidean spaces. The metric tensor itself is defined as follows

26 $$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ are the Minkowskian metric give by

27 $$\eta_{\alpha\beta} = \begin{cases} +1, & \alpha = \beta = 1,2, or 3 \\ -1, & \alpha = \beta = 0 \\ 0, & \alpha \neq \beta \end{cases}$$

This tensor will be revisited when we look into the special and general theories of relativity in more detail.

8 Tensor Densities

Tensor densities (or pseudotensors) are defined as follows

28 $$\overline{V}_{\nu}^{\mu} = \left|\frac{\partial \overline{x}}{\partial x}\right|^{W} \frac{\partial \overline{x}^{\mu}}{\partial x^{\lambda}} \frac{\partial x^{\kappa}}{\partial \overline{x}^{\nu}} V_{\kappa}^{\lambda}$$

Where W is the weight of the tensor density and $|\partial \overline{x}/\partial x|$ is the Jacobian of the transformation $x \to \overline{x}$.

The only tensor density that does not change its components in all coördinate systems, is the Levi-Civita tensor density which is defined as follows

29 $$\varepsilon^{\mu\nu\lambda\kappa} = \begin{cases} 0, & \textit{some indices are equal} \\ +1, & \textit{even permutation} \\ -1, & \textit{odd permutation} \end{cases}$$

were the odd and even permutations are with respect to a reference sequence usually taken as $1,2,\ldots,N$,where N is the dimension of the abstract space.

9 Tensor Algebra

All algebraic operations introduced in this section produce tensors, with exceptions in special cases that are mentioned below. These operations are defined only under certain conformability conditions.

Summation:

This operation is defined on tensors having the same type and order. The output is also a tensor inheriting the same type and order. This operation is commutative. For example the summation of tensors $U^{\alpha\beta}$ and $V^{\alpha\beta}$ is given by

30 $$W^{\alpha\beta} = U^{\alpha\beta} + V^{\alpha\beta}$$

Inner Product:

This is an order reducing operation. It is carried out by equating a contra- or co- variant index in one tensor to an index of the opposite type in the other tensor, then summing over the dummy indices. The output is a tensor that has a rank less than the sum of the ranks of the parent tensors by two and with a number of contra- and co- variant indices that is reduced by one for each type, as shown in (31). Note that this product will produce a scalar (an invariant) in the case of tensors of rank one, similar to the result from vector analysis. This operation is also commutative.

31 $$W_{\alpha}^{\beta} = U^{\beta\gamma} V_{\gamma\alpha}$$

Outer Product:

This operation produces a tensor that has the sum of the ranks of the parent tensors as its rank while preserving the type of the indices of the parent tensors (being either contra- or co- variant). Note that this operation is commutative (32).

32 $$W^{\alpha\beta}{}_{\gamma\delta} = U^{\alpha\beta} V_{\gamma\delta}$$

Contraction:

As we proceed in the inner product operation, we equate the indices of different types but from the same tensor (a monad operation). Thus this operation has one input and one output, that has a rank that is less than the rank the input tensor by two. In (33), if we contract the tensor by setting $\alpha = \gamma$ we get

33
$$\overline{W}^{\alpha\beta}{}_{\gamma} = W^{rs}{}_{t} \frac{\partial \overline{x}^{\alpha}}{\partial x^{r}} \frac{\partial \overline{x}^{\beta}}{\partial x^{s}} \frac{\partial x^{t}}{\partial \overline{x}^{\gamma}}$$
$$\overline{W}^{\alpha\beta}{}_{\alpha} = W^{rs}{}_{t} \frac{\partial \overline{x}^{\beta}}{\partial x^{s}} \frac{\partial x^{t}}{\partial x^{r}} = W^{rs}{}_{t} \frac{\partial \overline{x}^{\beta}}{\partial x^{s}} \delta^{t}_{r} = W^{rs}{}_{r} \frac{\partial \overline{x}^{\beta}}{\partial x^{s}}$$

Although there is no contraction operation in vector analysis, it is helpful to readers familiar with vectors to think of vector sum, inner product and outer product as special cases of the above.

It is also worth mentioning that the inverse of a second rank tensor is given by

34 $$Z_{\alpha\beta}^{-1} = Y^{\beta\alpha}$$

(which could be interpreted as impedance-admittance relation) and that the transpose of $Z_{\alpha\beta}$ is $Z_{\beta\alpha}$.

10 Tensor Symmetry

A covariant tensor of second valence is said to be symmetric if

35 $$A_{\alpha\beta} = A_{\beta\alpha}$$

and antisymmetric or skew symmetric if

36 $$A_{\alpha\beta} = -A_{\beta\alpha}$$

Note that symmetry or antisymmetry is conserved under transformation of coördinates since $A_{\alpha\beta} - A_{\beta\alpha}$ and $A_{\alpha\beta} + A_{\beta\alpha}$ are tensors and thus, if they vanish in one coördinate system, they vanish in all others as well. The same can be said about contravariant tensors. Note that these remarks do not hold in the case of mixed tensors.

The above definition can be extended to higher valence tensors by examining symmetric properties over pairs of indices provided that they are of the same type, i.e. both must be co- or contra- variant.

A result similar to the one concerning the decomposition of matrices into symmetric and antisymmetric parts applies for non-mixed tensors with valence two,

37 $$A_{\alpha\beta} = \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right) + \frac{1}{2}\left(A_{\alpha\beta} - A_{\beta\alpha}\right)$$

where the first part is the symmetric part and the second is the antisymmetric one. Obviously both parts are tensors of valence two.

11 Covariant Differentiation

Now if we try to find the derivative of a tensor of the first rank with respect to x^{α}, we will find that the derivative does not transfer as a tensor (Weinberg 1972). This prompted a definition of a covariant derivative that would restore the tensorship of the derivative. The covariant derivative for a mixed tensor of rank three is defined as follows

38 $$T^{\mu\sigma}_{\lambda;\rho} \equiv \frac{\partial}{\partial x^{\rho}} T^{\mu\sigma}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}_{\lambda} - \Gamma^{\kappa}_{\lambda\rho} T^{\mu\sigma}_{\kappa}$$

($;\rho$ indicates the covariant derivative with respect to x^{ρ}) where,

39 $$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\}$$

is known as the Christoffel symbol of the second kind or as the affine connection. It is given by

40 $$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \quad .$$

The reader is cautioned that this symbol is not a tensor (Weinberg 1972). Note that the contravariant (upstairs) indices produced the positive product terms, where the negative terms were produced by the covariant (downstairs) index. The same mechanism carries on for higher order tensors .

Finally we need to mention that the covariant derivative of the metric tensor vanishes identically. For if we differentiate (26) with respect to x^{λ} we get

41 $$\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} &= \frac{\partial^2 \xi^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 \xi^{\beta}}{\partial x^{\lambda} \partial x^{\nu}} \eta_{\alpha\beta} \\ &= \Gamma^{\rho}_{\lambda\mu} \frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} + \Gamma^{\rho}_{\lambda\nu} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\rho}} \eta_{\alpha\beta} \\ &= \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} \end{aligned}$$

and from the definition of the covariant derivative for a covariant tensor of valence two we get

42 $$g_{\mu\nu;\lambda}=\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}-\Gamma^{\rho}_{\lambda\mu}g_{\rho\nu}-\Gamma^{\rho}_{\lambda\nu}g_{\rho\mu}=0$$

This result is known as the Ricci theorem (Broisenko and Tarapov 1968).

12 Differential Operators

The simplest differential operator is the gradient of a scalar field. For a scalar it is also equal to the covariant derivative

43 $$S_{;\mu}=\frac{\partial S}{\partial x^{\mu}}$$

(Sometimes a comma [,] is used instead of a semicolon [;] to represent the gradient). Higher order tensors are defined in the same way. Note that this operation increases the tensor rank by one.

The covariant curl is also equal to the ordinary curl,

44 $$V_{\mu;\nu}-V_{\nu;\mu}=\frac{\partial V_{\mu}}{\partial x^{\nu}}-\frac{\partial V_{\nu}}{\partial x^{\mu}}$$

Note that this curl tensor is anti-symmetric in all reference frames.

Finally the divergence of a tensor is defined as its derivative with respect to one of its indices, thus lowering the rank by one:

45 $$A_{\alpha,\alpha}=\frac{\partial A_{\alpha}}{\partial x^{\alpha}}$$
$$C_{\alpha\beta,\beta}=\frac{\partial C_{\alpha\beta}}{\partial x^{\beta}}$$

13 Line, Surface, and Volume Integrals

The line integral of a tensor of any rank is equal to a contraction of one of its indices and an integration,

46 $$\int A^{\alpha}_{\dots\beta}\,dx^{\beta}=B_{\dots}^{\ \ \alpha}$$

Where the ellipse denotes other indices. The surface integral is equal to

47 $$\iint A^{\alpha}_{...\beta\gamma} dx^{\beta} dx^{\gamma} = C_{...}{}^{\alpha}$$

and the volume integral is given by

48 $$\iiint A^{\alpha}_{...\beta\gamma\delta\varepsilon} dx^{\gamma} dx^{\delta} dx^{\varepsilon} = C^{\alpha}_{...\beta}$$

In tensor analysis Stokes' theorem assumes the following form:

49 $$\int A_{\alpha} dx^{\alpha} = \iint \left(\frac{\partial A_{\alpha}}{\partial x^{\beta}} - \frac{\partial A_{\beta}}{\partial x^{\alpha}}\right) dx^{\alpha} dx^{\beta}$$

14 Index Gymnastics

Quoting from Misner et al. (1973) "index gymnastics" is defined as:

...the technique of extracting the content from geometric equations by working in component notation and rearranging indices as required...

The most exhaustive table on index gymnastics techniques can be found in Misner et al. (1973). We provide the most important entries in the following:

1. Raising an index	$S^{\alpha\beta}_{\gamma} = g^{\beta\mu} S^{\alpha}_{\mu\lambda}$
2. Lowering an index	$S^{\alpha}_{\mu\gamma} = g_{\mu\beta} S^{\alpha\beta}_{\gamma}$
3. Contracting **S** to form a new tensor **M**	$M_{\mu} = S^{\alpha}_{\mu\alpha}$
4. $\lVert\eta_{\alpha\beta}\rVert$ is the inverse of $\lVert\eta^{\alpha\beta}\rVert$	$\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^{\gamma}_{\alpha}$
5. Gradient of **N** to form a new tensor	$S^{\alpha}_{\beta\gamma} = N^{\alpha}_{\beta,\gamma}$

6. Divergence of **N** to form a new tensor **R** $R_\beta = N^\alpha_{\beta,\alpha}$

7. The contravariant index on a gradient is obtained by raising covariant index. $N^{\alpha,\gamma}_\beta \equiv N^\alpha_{\beta,\mu}\eta^{\mu\gamma}$

15 Geodesic Lines

A geodesic is the shortest distance connecting two points on a surface. In a Euclidean space, it is a straight line, but in a Riemannian space it takes forms other than straight lines. The geodesic equation is given by

50 $$\frac{d^2\xi^l}{ds^2} + \Gamma^l_{mk}\frac{d\xi^m}{ds}\frac{d\xi^k}{ds} = 0$$

Where s is the arc length. Note that the affine connection vanishes when the coördinate system is Cartesian. For a derivation of (50) the reader is referred to Bergmann (1976).

16 Curvature of Space

We found it convenient to structure this section on the chapter in the book by Synge and Schild (1949). No claim is made that this section is self contained, although we include all the elements necessary for our study of the general theory of relativity.

The concept of curvature of space is not a new one. In Euclidean geometry we speak about the curvature of a line or of a plane. This may be generalized for an N dimensional Riemannian space by considering N dimensional objects, and treating the curvature as an intrinsic property of the space (i.e. it cannot be measured by comparison of the space with another space.) A curved space can be defined as a space that does not satisfy the definition of a flat one. A space is said to be flat if it is possible to choose coördinates for which the metric form is

51 $$ds^2 = \varepsilon_1(dx^1)^2 + \varepsilon_2(dx^2)^2 + \ldots + \varepsilon_N(dx^N)^2,$$

throughout the space (not at a single point only), where the coefficients of each term are either +1 or −1. In order to examine the curvature of a space, a unique tensor, built from the first and second derivatives of the metric tensor, is used. This tensor is known as the Riemann-Christoffel curvature tensor (R-C tensor for short) and is given by

52 $$R^{\lambda}_{\mu\nu\kappa} \equiv \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} \quad ,$$

where the Γs are the affine connections (Christoffel symbols of the second kind) given by (39). The algebraic properties of the fully covariant R-C tensor (which is obtained via the metric tensor) are :

53 $$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \qquad \text{(symmetry)}$$

54 $$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu} \qquad \text{(antisymmetry)}$$

55 $$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \qquad \text{(cyclicity)}$$

A necessary condition for the flatness of any space is the vanishing of the R-C tensor at all points in the space.

The number of the independent components of the R-C tensor is given by

56 $$(1/12)N^2(N^2 - 1)$$

where N is the dimension of the space. An important form of the R-C tensor is obtained via contraction,

57 $$R_{\mu\kappa} \equiv R^{\lambda}_{\mu\lambda\kappa}$$

which is known as the Ricci tensor and is symmetric. Also of interest is the curvature scalar which is given by

58 $$R = g^{\mu\kappa}R_{\mu\kappa} \, .$$

In addition to the above algebraic identities, the R-C tensor obeys the following differential identities known as the Bianchi identities:

59 $$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 .$$

A useful form can be obtained by contracting equation (59) twice to obtain

60 $$(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{;\mu} = 0.$$

Finally we introduce the Einstein tensor defined by

61 $$G^{\mu\nu} \equiv (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R),$$

which together with equation (60) leads to the conclusion that the divergence of the Einstein tensor vanishes.

17 Special Spaces

In this section we will introduce some special types of spaces, exposing interesting properties that cannot be found in a general treatment of Riemannian spaces.

First of all we introduce constant-curvature spaces, but to do so we need to define isotropic points in a Riemannian space. An isotropic point in a Riemannian space is a point at which the Riemannian curvature satisfies

62 $$K\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) = R_{abcd}$$

Introducing $G_{abcd} \equiv \left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$ we can write

63 $$KG_{abcd} = R_{abcd}$$

Now if a Riemannian space (with a dimension of three or higher) is isotropic at each point in a region, then the Riemannian curvature is constant throughout the region. For if we take the covariant derivative of (63) with respect to x^u and permute and then sum we get

64 $$G_{abcd}K_{;u} + G_{abdu}K_{;c} + G_{abuc}K_{;d} = 0$$

Multiplying with $g^{ac}g^{bd}$ gives

65
$$\begin{aligned}
g^{ac}g^{bd}G_{abcd} &= g^{ac}g^{bd}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) = \delta^c_c\delta^d_d - \delta^c_d\delta^d_c = N^2 - N \\
g^{ac}g^{bd}G_{abdu} &= g^{ac}g^{bd}\left(g_{ad}g_{bu} - g_{au}g_{bd}\right) = \delta^c_d\delta^d_u - \delta^c_u\delta^d_d = \delta^c_u - N\delta^c_u \\
g^{ac}g^{bd}G_{abuc} &= g^{ac}g^{bd}\left(g_{au}g_{bc} - g_{ac}g_{bu}\right) = \delta^c_u\delta^d_c - \delta^c_c\delta^d_u = \delta^d_u - N\delta^d_u
\end{aligned}$$

Then summing gives

66
$$\begin{aligned}
&\left(N^2 - N\right)K_{;u} + \left(\delta^c_u - N\delta^c_u\right)K_{;c} + \left(\delta^d_u - N\delta^d_u\right)K_{;d} = 0 \\
&\left(N^2 - N\right)K_{;u} + (1 - N)K_{;u} + (1 - N)K_{;u} = 0 \\
&(N - 1)(N - 2)K_{;u} = 0
\end{aligned}$$

For $N \geq 3$, the covariant derivative of K with respect to x^u (which is arbitrary) becomes equal to its partial derivative with respect to x^u (since K is a tensor of valence cypher). Hence K must be constant (see problem 9.14 in Kay [1988]). This result is known as Schur's theorem.

Although we have already introduced flat spaces in (51), we expand our treatment by introducing some of the possible properties of a flat space of N dimensions (Synge and Schild 1949):

(1) The metric form is not always positive definite. Thus cypher distance between two points does not always indicate that the two points coincide.

(2) The number of dimensions may be greater than three. In such cases a two dimensional flat space does not divide the space into two parts.

(3) A flat space may be topologically different from Euclidean space. Although this property is interesting, for our purposes, we shall assume the space to have Euclidean topology.

Before introducing Cartesian tensors, we need to define the so-called homogenous coördinates. In (51), if the metric form can be written as

67 $$ds^2 = dy^\alpha dy^\alpha$$

where $y^\alpha \equiv \sqrt{\varepsilon_\alpha} x^\alpha$ (the summation convention being inactivated), the coördinates are said to be homogeneous and due to the simplicity of the metric form in homogeneous coördinates, they are used to study flat spaces. 3-space Cartesian coördinates are an example on homogenous coördinates. Under such coördinates, the transformations are all orthogonal (section three). Under this formalism, Cartesian tensors are defined as quantities that transform according to tensor laws when the coördinates undergo an orthogonal transformation. The analogy between homogeneous coördinates in flat space and rectangular Cartesian coördinates in a Euclidean plane prompted the name Cartesian tensors. Note that being a tensor requires more stringent conditions than being a Cartesian tensor, and that Cartesian tensors are available only if the space is flat and the coördinates are homogeneous. Also note that Cartesian tensors can not distinguish between co- and contra- variant components, and thus their transformation laws remain unchanged under raising or lowering indices. This can be seen when (9) is rewritten as

68 $$\Gamma : \bar{x}^i = A_{ij} x^j + A^i \ , \quad \Gamma^{-1} : x^i = B_{ij} \bar{x}^j + B^i \ .$$

Differentiating with respect to x^j and $\bar{x}^j$ we get

69 $$\frac{\partial \bar{x}^i}{\partial x^j} = \frac{\partial x^j}{\partial \bar{x}^i}$$ (due to the orthogonality of the transformations.)

Finally we state the fact that covariant differentiation can be introduced without introducing a metric. That is to say, without associating a length with an infinitesimal displacement or an angle with two vectors. This treatment leads to the so-called non-Riemannian spaces, which are more general than Riemannian spaces. For more on non-Riemannian spaces see Synge and Schild (1949).

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"I remain convinced that BG models will play an increasingly important role in the upcoming century..." *Henry Paynter.* Preface, ICBGM'93

4 BONDGRAPH ANALYSIS

From a mathematical perspective, a bondgraph (BG) is a member of the linear graphs family. Thus the similarities between bondgraphs and blockdiagrams, signal flow graphs, electric circuits, mechanical networks, or other linear graphs are anticipated. In fact many papers were devoted for scrutinizing such connections (e.g. [Perelson and Oster 1976], [Brown 1972a]). Still differences between bondgraphs and other linear graphs exist (Cellier 1991), and these differences are actually the source of the unique character of bondgraphs. Succinctly, a bondgraph is a modeling tool for multiport systems, which, while preserving the topological structure of the system, uses power flow (being the historic basis of the multiport concept) as a criteria for description. This criterion sometimes becomes a hurdle in the way of extending bondgraphic modeling to non-physical systems since the "power" of social and economic systems is still, mathematically speaking, a soft term. Therefore we can appreciate the physical genes imbedded in the bondgraphic representation of multiport systems.

Bondgraphs were officially born on April 24, 1959 (Paynter 1993). The classic work on BGs is "Analysis and Design of Engineering Systems", the class notes for MIT course 2.751, by their inventor Henry M. Paynter (Paynter 1961). Subsequently, BGs evolved in the hands of a second generation of bondgraphists, such as Dean C. Karnopp, Ronald C. Rosenberg, Jean Thoma, and Donald L. Margolis, who wrote numerous papers and books on BGs that led to the standardization of the BG language and the extension of its application to new physical "domains." A third generation bondgraphist, by the name of Pieter C. Breedveld championed the generalized version of bondgraphs (GBG) which was based on Thermodynamics (see below).

In this chapter we will introduce the rudiments of BG's with emphasis on the electric, magnetic and mechanical domains. This will initiate the reader to bondgraphs and enable him or her to integrate BGs into their analysis and synthesis of physical systems.

1 The Power Postulate

In this section we will focus on the power flow concept that BG's utilize to model the equations of motion of multiport systems. The basics are first covered, with emphasis on the electrical and discrete mechanical domains to demonstrate the ideas considered. Then an extension that integrates the tensor character of variables to the concept, is introduced. Power is defined as the rate at which energy is transferred, with the watt (J /sec) as its SI unit. The reader might have noticed that the formulae for power found in any introductory physics book involve the product of two time dependent variables. Take for example the formula for the instantaneous power in electric networks, $\mathcal{P}(t) = \underline{v}(t)^T \underline{i}(t)$. This is a matrix formula that gives the scalar power as a function of time, and relates it to the output of an inner product between two column vectors (hereafter the underscore is used to identify vectors), viz., the instantaneous voltage vector $\underline{v}(t)$, and the instantaneous current vector $\underline{i}(t)$. Another formula for instantaneous power is that of the discrete mechanical "domain," $\mathcal{P}(t) = \underline{f}(t)^T \underline{v}(t)$. The power in this case is the inner product of the instantaneous force vector $\underline{f}(t)$, and the instantaneous velocity vector $\underline{v}(t)$. This observation on the nature of the power formulae spans the gamut of physical systems, and in fact constitutes a basis for analogies.

Actually as we examine the types of variables in these formulae, we start to realize that they belong to two types of variables. The first type, known as the flow ($\underline{f}$), built from the time derivative of the so-called "configuration-like variable" (we will denote it by Ψ) of the system, e.g. the generalized coördinates in discrete mechanical systems (from which originated the notion of a configuration) or the electric charge in electrical networks (Toni 1977). For discrete systems we will call the configuration "generalized displacement" ($\underline{q}$) so that it agrees with the existing nomenclature. The second type, known as the effort ($\underline{e}$), (or as the source-like variable) is built from the partial derivative of the energy of the domain under consideration with respect to the configuration. Mathematically, this translates to

70 $$\mathcal{P}(t) \equiv \frac{dE}{dt} = \left(\frac{\partial E}{\partial \underline{q}}\right)\left(\frac{d\underline{q}}{dt}\right) = \underline{e}(t)^T \underline{f}(t)$$

or

71 $$\mathcal{P}(t) \equiv \frac{dE}{dt} = \left(\frac{\partial E}{\partial \underline{p}}\right)\left(\frac{d\underline{p}}{dt}\right) = \underline{f}(t)^T \underline{e}(t)$$

(where $[\underline{p}]$ is the generalized momentum. Note that each formula is associated with a specific kind of energy.) It is convenient to also define the integral of the effort with respect to time as the generalized momentum. Hence the basic quantities are defined as follows

72a
$$\underline{e} = \frac{d\underline{p}}{dt} = \left[\frac{\partial E}{\partial \underline{q}}\right]^T$$
$$\underline{f} = \frac{d\underline{q}}{dt} = \left[\frac{\partial E}{\partial \underline{p}}\right]^T$$

72b
$$\underline{p}(t) = \int_{t_0}^{t} \underline{e}(\tau)\, d\tau + \underline{p_0}$$
$$\underline{q}(t) = \int_{t_0}^{t} \underline{f}(\tau)\, d\tau + \underline{q_0}$$

The Paynter " tetrahedron of state " that depicts the relations between the variables is shown in Fig. 4.1. This generalization is a powerful tool that can provide a unified approach to the analysis and synthesis of any dynamical system. A table that summarizes the relation between the BG's fundamental variables and conventional variables for some discrete "domains" was provided by Van Dixhoorn (1982).

As we have seen in the introduction to Tensor Calculus, every vector is a tensor of valence one. Hence we can identify any vector as either a covariant or a contravariant tensor.

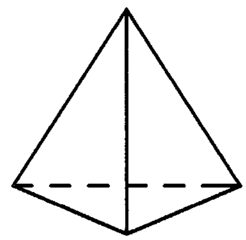

Fig. 4.1 : The Paynter tetrahedron of state

But what type of a tensor is the effort variable ? What about the flow variable ? This is really not an essential issue since we can always find a metric tensor that can lower or raise the indices. We only need to investigate the relation of power invariance and map it into the definitions of the co- and contra- variant tensors. Our starting point is the observation that most variable transformations in physical systems are power conserving. The reader is referred to Hoffmann (1957) and Karnopp (1969) for more details. Power invariance can be mathematically stated as follows

73 $$\mathcal{P}(t) = \mathcal{P}'(t)$$

where power is observed in two different coördinate systems. From (70) we can define $\mathcal{P}'(t)$ similarly to obtain a relation between the efforts and the flows of each coördinate

74 $$\underline{e}^T \underline{f} = \underline{e}'^T \underline{f}'$$

which we can delineate further by substituting for the efforts and the flows from (72a),

75 $$\left(\frac{\partial E}{\partial \underline{q}}\right)\left(\frac{d\underline{q}}{dt}\right) = \left(\frac{\partial E}{\partial \underline{q}'}\right)\left(\frac{d\underline{q}'}{dt}\right)$$

where the energy is, a fortiori, invariant. Now if we rewrite equation (22), **3** : **6**, using the chain rule, we get

76 $$V_j\left(\frac{dx^j}{dt}\right)=\overline{V_i}\left(\frac{d\overline{x}^i}{dt}\right)$$

and one can immediately realize that (76) and (75) are very similar. Still the reader is cautioned that the coördinates in (76) are not equal to the generalized coördinates in (75). This can be seen since (22), repeated as (77), does not give the correct relation between the efforts and the flows, but rather the relation between the efforts of the so-called "primitive" system and the connected real one, viz:

77 $$V_j=\left(\frac{\partial x'^i}{\partial x^j}\right)V_i'$$

The lecture notes of His (1932-33) refer to currents and voltages as contravariant and covariant vectors, respectively (Roth 1959). This was also the way Kron (1942) and Happ (1971) treated them. Although it was not universally followed (e.g. Brameller, John and Scott [1969]) we will adopt this treatment throughout the rest of this thesis, and refer to it as the His convention. This convention regards efforts as covariant tensors and flows as contravariant ones, which appears to be the natural type of both. (Again we emphasis that the covariance [contravariance] of any index can be changed to contravariance [covariance] via the metric tensor.) A generalization of the above concept will definitely maintain the scalar character of power (for conserving transformations), but will allow the effort and flow tensors to have valences greater than one. Thus power can be written as (Note that this formulation is more general than that of Fahrenthold and Wargo [1991])

78 $$\Pi=e_{\alpha\beta\gamma\ldots}f^{\alpha\beta\gamma\ldots}$$

(note that we preserve the symbol Π for tensoral power only) where it is understood that the effort and flow tensors have the same valence. In this work we will present physical phenomena that require efforts and flows with valence one only.

Before we leave this section we need to caution from the fact that Power and Energy are not invariant under all transformations. For example, under Lorentz transformations (more on these transformations when we discuss relativity in the coming chapters) energy and power are not scalars (invariants). Thus in general we cannot treat energy and power as scalars, but for many physical phenomena we will deal with transformations that leave energy and power invariant.

2 Bondgraph Anatomy

In order to model the relation between efforts and flows as it occurs in physical systems, BGs need to have multiport elements that establish relations mixing efforts and flows, and relate variables of the same type. (We will avail ourselves of the multibondgraphs formulation without introducing the 1-bond notation. Multibondgraphs were first introduced by Bonderson [1975] as vector bondgraphs and then renamed by Breedveld [1986] to prevent any ambiguity that might occur from the notion of column vectors in matrix theory and the vector concept in vector analysis. The reader is referred to Karnopp et al. [1990] for a more gradual development of the notation used in this work.) Other multiport elements should model the interaction between the environment and the system, or in other words the boundary conditions. Our criterion in classifying these multiport elements is, again, based on the concept of power (Breedveld 1984).

The first type establishes mixed relations and is power-discontinuous (where energy storage or dissipation takes place). The multiport element responsible for energy storage (hence known as energic) is denoted by **C**, which stands for capacitance (in the electric "domain") or compliance (in the mechanical "domain"). Hereafter it will be referred to as a **C**-field. It is important to realize that **C**-fields store only one type of energy, viz. potential energy (or its analogues such as the electric energy). The other familiar storage element is the inductance or inertance which is referred to as an **I**-field, which takes care of storing the mechanical energy and its analogues. In mixed energy "domains" it is more convenient to describe energy storage through mixed fields, which are referred to as **IC**-fields in BGs terminology (Karnopp et al. 1990). The other power-discontinuous element is denoted by **R**. The reader, probably, recognizes this as the resistance in the electric "domain," which actually extends to all other physical "domains" and is responsible for the dissipation of energy from the system (hence known as non-energic). We shall refer to it as an **R**-field. It is important to note that the **C** and **I** elements are reversible, i.e. energy can be stored in them and retrieved as well (after some time delay, of course), where **R** elements are irreversible; they donate the energy to the environment as entropy (Thoma 1975). If the entropy introduced is included in the model, the field becomes an irreversible transducer, denoted by **RS**, and referred to as an **RS**-field. The **S** in the mnemonic code stands for the non-linear entropy source that accounts for the liberated energy. If the entropy is not included, the assumption of an isothermal (or, in other words, linear) relation, is implied, and the **RS**-field degenerates to an **R**-field.

The multiport elements responsible for modeling the interaction between the environment and the system are also power-discontinuous, since they model power creation (power from the environment to the system) or annihilation (power from the system to the environment). Conveniently, they are called multiport sources. The mnemonic code for sources is **S**-array. The reason for the word 'array' is that, by definition, sources have no constitutive coupling between the effort and the flow variables. Thus sources produce only one type of variables. If they produce an effort they are called effort sources (if the effort produced is constant they are called Dirichlet sources), **SE**-array. The other type of sources, **SF**-array, will thus produce the other type of variables (known as Neumann sources for constant flow).

At this juncture we have already accounted for all the power-discontinuous multiport elements needed to model dissipation, storage and boundary conditions associated with a physical system. Now we need to introduce multiports (MP's for short) that will model the energy transfer within the system itself. Note that these MP elements are not required to do any function that is modeled by the power-discontinuous ones. All they need to do is to transfer energy to and fro the power-discontinuous MP elements. (Thus they must be power-continuous MP elements.) In GBG's terminology these MP elements are collectively referred to as a Birkhoff Junction Structure (after George D. Birkhoff), or as a Generalized Junction Structure (Breedveld 1984).

In electric circuit theory, we speak of wires or "ducts" through which power is allowed to flow. This means that we enforce quasi-static conditions on our model or, stated differently, power radiation is left out and the only way energy is transported is through the electric wires. In BG's we assume that there is no way of transporting energy within the system other than by means of radiation, convection and conduction. (Actually this extension was introduced for Generalized BG's.) (Nijen Twilhaar 1985.) In BG terminology, the ducts are called multibonds, hence the name bondgraphs. They are represented by a harpoon or a half arrow as shown in Fig. 4.2a (n is the valence of the effort [or flow] variable), where we follow the Thoma convention (after J.U.Thoma) which requires the half arrow to point always to the "flow side" of the multibond (Breedveld 1986). This representation is not unique (cf. Thoma [1990]), but we see it as a reasonably balanced representation between complexity and vagueness. The Thoma representation is also used when necessary. (Thoma uses a single harpoon with a ring around it to represent multiport bonds.) The complete reticulation up to this point is depicted in Fig. 4.2b.

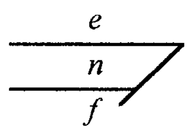

Fig. 4.2a : The multibond representation in BGs

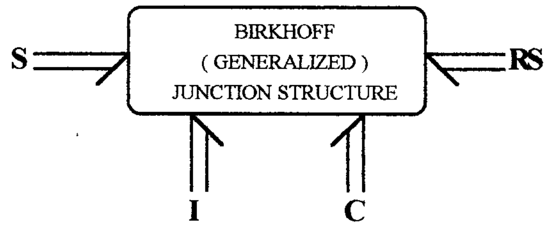

Fig. 4.2b : The Birkhoff junction structure in BGs

In order to also be able to represent the communication between the various ports, we need to augment control signals that do not carry power, but rather information. (They will have to carry a small amount that can be considered negligible for all practical purposes.) These signals are also known as active signals and are represented with full arrows as shown in Fig. 4.3 .

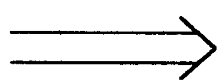

Fig. 4.3 : The representation of active signals in BGS

In a junction structure, there are four elements that model the transfer of power within the physical system. The first multiport element we focus on is the **0** (zero) junction. This junction is also known as a common effort junction and is represented by the following equations:

79

$$f^{\alpha} = f'^{\alpha} + f''^{\alpha} + \cdots$$
$$e_{\alpha} = e'_{\alpha} = e''_{\alpha} = \cdots$$

where n is the dimension of all the effort and flow vectors. Note that f^{α} is the output vector, where all the other flow vectors are input vectors. (The output is always written to the left of the equal sign, where inputs are written to the right of it.) Similarly we define the **1** (one) junction or the common flow junction as follows:

80
$$
\begin{aligned}
e_{\alpha} &= e'_{\alpha} + e''_{\alpha} + \cdots \\
f^{\alpha} &= f'^{\alpha} = f''^{\alpha} = \cdots
\end{aligned}
$$

The sign of each component is decided by the so-called computational causality (see below). The familiar parallel and series connections in electric circuit theory are special cases of the above junctions. Some authors refer to 0 (1) junctions as p (s) junctions for the same reason (Thoma 1990.)

The other two elements that are included in the Birkhoff junction structure are the MP transformer and the MP Gyrator. The MP transformer (MP **TF**) is defined by the following equations:

81
$$
\begin{aligned}
e_{\alpha} &= T_{\alpha}^{\beta} e'_{\beta} \\
f'^{\beta} &= T_{\alpha}^{\beta} f^{\alpha}
\end{aligned}
$$

where T_{α}^{β} is the transformation tensor. Similarly the MP gyrator (MP **GY**) is defined as follows

82
$$
\begin{aligned}
f^{\alpha} &= G^{\beta\alpha} e'_{\beta} \\
f'^{\beta} &= G^{\beta\alpha} e_{\alpha}
\end{aligned}
$$

where $G_{\alpha\beta}$ is the gyration tensor. Note that transformers do not change the "gender" of the effort and flow tensors, where gyrators do. Also note that **GY**s are more basic than **TF**s, for we can obtain the effect of a transformer by two consecutive gyrative operations

83a
$$f^{\alpha} = G^{\beta\alpha} e'_{\beta}$$
$$f'^{\beta} = G^{\beta\alpha} e_{\alpha}$$

83b
$$f'^{\beta} = H^{\gamma\beta} e''_{\gamma}$$
$$f''^{\gamma} = H^{\gamma\beta} e'_{\beta}$$

83c
$$f^{\alpha} = G^{\beta\alpha} e'_{\beta} = G^{\beta\alpha} K_{\beta\gamma} f''^{\gamma} = T^{\alpha}_{\gamma} f''^{\gamma}$$
$$e''_{\gamma} = K_{\beta\gamma} f'^{\beta} = K_{\beta\gamma} G^{\beta\alpha} e_{\alpha} = T^{\alpha}_{\gamma} e_{\alpha}$$

(where $K_{\beta\gamma}$ is the inverse of $H^{\gamma\beta}$) Fig. 4.4 shows the BG representation of MP **TF**'s and MP **GY**'s. Note that we can represent MP **GY**'s in two ways: as having two multiports, which is the traditional definition as in (82) or as having only one (Breedveld 1981). The new definition seems to facilitate the modeling of systems. The new 1-MP **GY** is defined as follows (no relation exists between equations [83a, b & c] and [84]):

84
$$e_{\alpha} = G_{\alpha\beta} f^{\beta}$$
$$G_{\alpha\beta} = -G_{\beta\alpha}$$

(with the other causal relation also possible.) The bondgraphic representation for the1MP **GY** is shown in Fig. 4.5.

Fig. 4.4 : The Bondgraphic representation for MP TF and GY in BG's

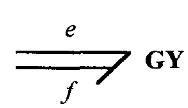

Fig. 4.5 : The 1-MP GY

If the junction structure does not include any **GY**s, the junction structure is known as a Kron (after Gabriel Kron), or a weighted junction structure. If only **0** and **1** junctions are present, the structure is known as a Kirchhoff (after Gustav Kirchhoff), or simple junction structure.

With a glance at Fig. 4.1 one notices that there is a hidden line that relates the variables p and q. The constitutive relation is called a memristor (short for memory resistor) by Chua (1971). The memristor is defined mathematically as follows

85
$$\begin{aligned} \dot{q} &= f = W(p)e \\ \dot{p} &= e = M(q)f \end{aligned}$$

where $W(p)$ is called the incremental "memductance" and $M(q)$ is called the incremental "memrisistance." Note that memristors have a meaning only for nonlinear systems, since for the linear case the constitutive relations in (85) will degenerate to the constitutive relations of an isothermal (linear) resistor.

The memristor is, like the linear resistor, causally neutral. This is because integration and differentiation are involved in viewing it in the effort-flow plain (Oster and Auslander 1972). Memristors can only be used to model non-linear, displacement modulated resistors. Oster and Auslander (ib.) provide numerous examples for the uses of memristors. Although in their paper they do distinguish memristors from modulated resistors, the author of this thesis does not "see" the difference. We wanted to present the memristor for completeness, but it will not be used in the following developments.

Some authors use the term implicit fields to refer to fields that are built from pure power-discontinuous-element fields plus power continuous elements, especially **0** and **1** junctions. In this terminology fields build from MP energy storage elements are referred to as explicit fields. For example, in the case of **C**-fields (**I**-fields) constituting the pure fields in an implicit field, the explicit field is represented by a 2-MP **C**-field (**I**-field). Fig. 4.6 is an example of an explicit **I**-field.

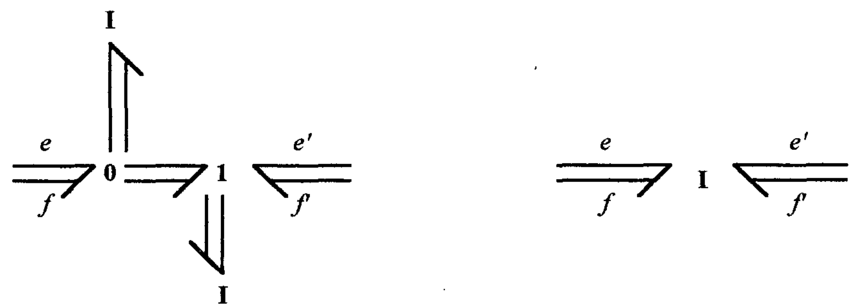

Fig. 4.6 : The explicit field (right) and its implicit representation (left)

3 Modulation of Bondgraphic MP Elements

The MP elements of BG's introduced in the previous section can be allowed to vary. This stems from our observations in many physical phenomena that can be modeled through MP elements that vary with "time" or with displacement. (In inter domain relations, we can also expect effort and/or flow modulation .) Thus we can generalize the formulation of the elements already introduced to a formulation that provides MP modulated elements.

But first let us investigate the possibility of modulation for the energy conserving elements. As storage element, **C**-fields and **I**-fields are not allowed to change the amount of energy stored, not even through dissipation, since we have to model all the dissipation through **R**-fields. Thus the very concept of storage requires the constitutive laws of energy storage elements to be modulation free with respect to time (Breedveld 1984, p.85); modulation with respect to other variables such as generalized displacement is still possible. However such extension is perfectly legitimate for non-energic power conserving MP elements such as **GY**'s and **TF**'s (Karnopp 1977). (From now on let us use elements and MP elements indiscriminately while maintaining the most general context.)

Let us now investigate the possibility of modulated **S**-arrays. Naturally physical phenomena are full of sources that are "time" variant. Take for example a.c. voltage sources. But are there any sources that are displacement modulated ? Recall that there is no constitutive relation that relates efforts and flows in **S**-arrays. Thus we have no constitutive tensor to modulate, and the possibility of displacement modulated sources is eliminated.

Modulated resistors are related to memristors, which are always displacement modulated. The constitutive relation for modulated resistors in the $e-f$ (effort-flow) plane is similar to that of memristors in the $p-q$ (momentum-displacement) plane. Solid-state electronic devices are an example of displacement modulated resistors. Such resistors are always non-linear. Are there any "time" modulated resistors? By definition, an **R**-field is an n-port, the constitutive relations of which relate the n-port efforts to the n-port flows by means of **static** (or algebraic) functions (Karnopp et al. 1990, p.251). (Note that this definition makes power-conserving elements, such as **0** and **1** junctions, **R**-fields, usually referred to as implicit **R**-fields.) Thus "time" modulation violates the basic definition of **R**-fields. (The reader might be able to think of some physical phenomena where the dissipation is "time" variant. Still, for the purposes of this thesis, we adhere to the above definition.)

Now we ask ourselves if **RS**-fields can be modulated. If we recall that **RS**-fields were introduced as an alternative to the transformer in modeling entropy generation by irreversibilities, we can argue that they can be modulated in a similar fashion to transformer modulation (Thoma 1975). In fact **RS**-fields are inherently cross coupled, with resistance being a function of temperature. Thus effort modulation is perfectly legitimate for **RS**-fields. Also from the previous two paragraphs we can conclude that displacement modulation of **RS**-fields is possible via the electric charge from the resistance side. One can also represent "time" modulation the way it was introduced for MP **TF**.

The tensors of the constitutive relations in equations (81), (82) and (83) were not explicitly referred to as constants, since we can allow for their variation and still be able to maintain the invariance of power (for a special class of transformations that are frequent in physical phenomena). These tensors, which we will refer to as the modulating tensors, are modulated by signals of active bonds that do not contribute to the power of the MP **MTF** or **MGY** (where the **M** stands for modulated), and are either functions of the displacement (configuration-like or metric) tensor or "time." Again we need to emphasize that it is possible to use other variables to modulate MP elements. We restrict the modulation on displacement and "time" since they suffice for modeling the phenomena encountered in physical systems. (We will see that in special relativity, a flow modulated transformer is of great importance.) If the modulating tensors are functions of the displacement tensor, the modulation is said to be internal and the MP **MTF** or **MGY** becomes non-linear. If they are functions of "time", the modulation is said to be external and the MP **MTF** or **MGY** becomes time-variant.

4 Structural Properties of Bondgraphic Elements

It has already been stated that sources in BG's are always decoupled. Thus we only speak of **S**-arrays and not **S**-fields. (There exists no relation between the effort and the flow of any source.) Similar restrictions are imposed on BG elements. In this section we investigate these restrictions and their influence on the structure of BG elements.

Let us start with **C**-fields. Any single-valued functional relationship between effort and displacement defines an energy conserving **C**-field. Energy of **C**-fields is given by

86 $$E=\int_{t_0}^{t} e_\alpha(t) f^\alpha(t)\, dt=\int_{t_0}^{t} e_\alpha(t)\frac{dq^\alpha(t)}{dt} dt=\int_{q_0}^{q} e_\alpha(q) dq^\alpha=E(q)$$

In general, tensors of **C**-fields are symmetric. With the assumption (over power conserving transformations) that energy is a single valued scalar function of displacement and $E \in C^2$ (i.e. assuming energy to have a smooth second derivative) we have

87 $$\frac{\partial e_\alpha}{\partial q^\beta}=\frac{\partial}{\partial q^\beta}\frac{\partial E}{\partial q^\alpha}=\frac{\partial^2 E}{\partial q^\beta \partial q^\alpha}=\frac{\partial^2 E}{\partial q^\alpha \partial q^\beta}=\frac{\partial}{\partial q^\alpha}\frac{\partial E}{\partial q^\beta}=\frac{\partial e_\beta}{\partial q^\alpha}$$
$$\therefore \frac{\partial e_\alpha}{\partial q^\beta}=\frac{\partial e_\beta}{\partial q^\alpha} \Rightarrow C_{\alpha\beta}=C_{\beta\alpha}$$

where $C_{\alpha\beta}$ is the constitutive tensor of the **C**-field for the linear case. The equations in (87) are known as Maxwell's reciprocal relations. Mutatis mutandis, similar results hold for I-fields (88), viz:

88 $$\frac{\partial f^\alpha}{\partial p_\beta}=\frac{\partial^2 E}{\partial p_\beta \partial p_\alpha}=\frac{\partial f^\beta}{\partial p_\alpha}$$

In the case of **R**-fields, if the **R**-field is implicit and composed of linear pure 1-MP **R**-fields, **0** and **1** junctions, and transformers, the implicit **R**-field's constitutive tensor (which is a covariant tensor of the second rank) is symmetric (provided it is written in terms of resistances only). It is said to be in the Onsager form in analogy to the Onsager reciprocity (Onsager 1931a & 1931b). If the implicit **R**-field contains gyrators, the reciprocity is lost. (If the constitutive tensor is written in terms of resistors and conductors, the tensor becomes skew symmetric, and is said to be in the Casimir form, again, provided that no gyrators are present [Casimir 1945].) In the general case, an **R**-field is not in the Onsager form nor in the Casimir form (Karnopp 1990).

Finally we investigate the structures of transformers and gyrators. The simplest transformer one could think of is a transformer with only two ports and a "ratio" of one. This becomes a 2-MP **TF** with its transformation tensor being the Kronecker tensor (which we always represent as a mixed tensor) of valence two. In such a case, the transformer degenerates to a multibond. Note that in order for equations 83a, b, and c to hold, we need the indices of the gyration and transformation to run over the same range, or in other words the gyration and transformation tensors must live in spaces that have the same dimension. (If a matrix representation is used the matrices must be square).

Similarly, the simplest 1-MP gyrator is build from two bonds and its matrix representation is given by

89
$$G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This type of gyrators is called symplectic. This is because the matrix in (89) is known as a symplectic matrix in differential geometry. For the multiport case the gyration tensor is represented by

90
$$G_{\alpha\beta} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

The mnemonic code for these gyrators is **SGY**. Symplectic gyrators are very useful for dualizing elements. For a comprehensive treatment of their use in dualizing, the reader is referred to Breedveld (1984). For the general case, a MP **GY** will always have a gyration tensor that is skew symmetric. If the 1-MP **GY** is used, the gyrative tensor is skew symmetric. Where if the old definition is used, the matrix representation of the gyration tensor becomes block skew symmetric. Note that it is necessary for the gyration tensor to be skew symmetric to insure that a gyrator (of any kind) conserves energy but does not store any.

5 Computational Causality

A method that declares efforts and flows as causes (independent variables) or effects (dependent variables), becomes essential when one attempts to write down the equations embedded in a BG. To achieve this goal we use a stroke (that actually resembles the palm of the hand as seen from a side view), known as the causal stroke that is placed at either end of the power bond and declares that end as the side from which the flow enters the bond. Hence it also tells us that the effort must enter from the other side, since power is a bilateral signal in BGs. Fig. 4.7 demonstrates the placement of the causal stroke on the MP elements of BGs.

Note that sources have fixed causality, where storage elements have preferable (integrable) causality for numerical evaluation reasons. Dissipation elements do not involve differentiation nor integration in their constitutive relations and hence do not have a preferable causality. The same is true for the 1-MP **GY**. On the other hand, 2-MP **TF**'s and 2-MP **GY**'s have two permissible causal assignments out of a total of four possible assignments for each. Finally the **0** and **1** junctions have restricted causality as shown in Fig. 4.8. This is due to the way they are defined in (79) and (80), that is to say that **0** junctions have only one output flow tensor, where **1** junctions have only one output effort tensor.

In augmenting a BG with causal strokes, one starts with the necessary (fixed) causalities, then preferable causalities and finally restricted ones are assigned. If all assignments turn out to be in agreement with the permissible causalities, the BG has a unique solution (i.e. the system of equations generated has a unique solution). If some necessary causalities cannot be met, there exists no solution and the BG is said to be noncausal. If some preferable causalities cannot be met, we have a degenerate BG (in other words, we have a structural singularity in the solution). Finally if assigning the causalities of the storage elements is arbitrary (can be changed without violating any of the other causal requirements), we have in our system of equations the so-called algebraic loops (Cellier 1991).

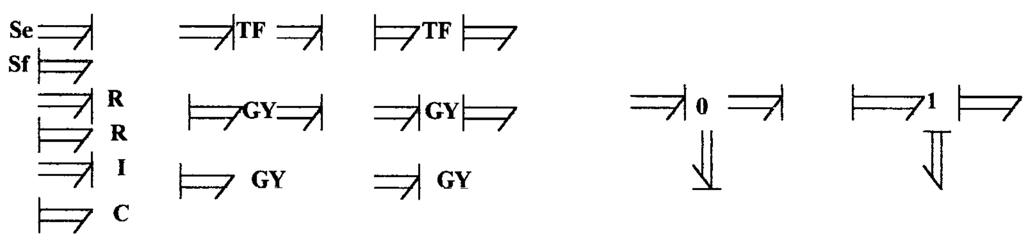

Fig. 4.7 : The causal stroke assignment for bondgraphic elements

6 Lagrangian Bondgraphs, Gyrobondgraphs and Generalized Bondgraphs

Over the past four decades, many types of BG's stemmed from mechanical (also known as dynamic or Paynterian) BG's. The evolving BG's either attempted to minimize the primitive set of elements used (either from a pure mathematical point of view or for physical considerations), or combined BG analysis with other analytical tools to produce even more powerful ones. The minutiae of such developments are not discussed here, since our treatment relies exclusively on the original Paynterian BG approach. (This is not a sequitur of an unrewarding attitude to such efforts, nor is it an underestimation of their practical exigencies. We simply judge that at this embryonic stage of development we need to maintain our analysis strictly adhering to the classical power postulate and the original nine elements used in Paynterian BG's.) We still feel obligated not to bypass these developments. A satellite view of some BG's is presented in this section.

Brown (1972b) introduced Lagrangian Bondgraphs as an analytical tool for modeling holonomic physical "systems". (A holonomic system is a system that has only holonomic constraints, i.e. constraints that can be expressed analytically via equations relating generalized coördinates and time.) His approach combines Lagranges equations and Paynterian BG's, arranged in expressions of energy and power. If the "system" is linear or mildly non-linear, the structure of such graphs contains all or almost all the information in the model. For highly non-linear systems, the graph contains little information and serves little purpose.

Six years later, a paper by R. C. Rosenberg (1978) introduced another type of BG's, Gyrobondgraphs. As he defines them, they are "a form of bond graphs containing only elements from a primitive set. " A primitive set, introduced by Paynter and Karnopp (1965), must include the dissipation element (**R**-field) plus one type of storage, junction, and source elements; and gyrators. (Since this paper was introduced before the introduction of the 1-MP **GY**, the gyrators referred to are the old 2-MP **GY**s.) Rosenberg chose to build his primitive set from inertances, **1**-junctions, and effort sources. He gave no explanation for this choice, although he pointed out that choosing other primitive sets will not change "substantially" the development of gyrobondgraphs. Duality rules provide a tool for converting Paynterian BGs to gyrobondgraphs; still such treatment of BGs damages their importance as modeling tools for physical systems by obscuring their physical interpretation of the dynamics involved and reducing them to formula manipulation tools.

This last observation prompted P. C. Breedveld (1981) to introduce a new primitive set, based on one state variable, that stemmed from the lack of an **I**-field in thermodynamics (Breedveld 1982). (The existence of inertic storage elements in other physical domains raises a serious question on the individuality of thermodynamical, and actually the chemical domains. It seems to the author that this has to do with the definition of equilibrium in thermodynamics, that requires the flow [the entropy flow] to be identically cypher, where other equilibrium definitions permit constant flow and are thus less restrictive.) To be able to provide a complete synthesis for all physical domains, a primitive set containing **C**-fields and sources of effort (which are considered to be infinitely large capacitors) was adopted. This approach was later given the name generalized bondgraphs (GBGs) to eliminate any confusion that it only applied to the thermodynamical "domain".

4 SELECTED BIBLIOGRAPHY

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"The special theory of relativity is now believed to apply to all forms of interaction except large scale gravitational phenomena. It serves as a touchstone in modern physics for the possible forms of interaction between particles. Only theories consistent with special relativity need be considered. This often severely limits the possibilities." *J. D. Jackson*, Classical Electro-dynamics, 2nd edition

5 SPECIAL RELATIVITY: Theory and Interpretation

Instead of providing a historical background on the special theory of relativity in this introduction (which the reader can find in any introductory book), I would like to impress on the reader's mind some of the controversial comments that he or she might not be able to find in the majority of the literature on the subject. First let us begin by asserting the fact that "a small but vocal tradition" of showing that Einstein did not derive the mass-energy relation in 1905 does exist. The best example on this tradition is the work by E. H. Ives (1952). (The reader should understand that this and all the other opinions in this introduction are being cited without any comments from the author!) One can also find works that are critical of the SR theory itself; two representative works being those by Luther (1966) and Irvine (1981). Also of interest is the work by Abraham (1908) in which he provides a rigid model of the electron that challenges the mass-velocity relation in SR. In fact, most of the early experimental work pertaining to this subject was conducted on the mass-velocity relation in order to decide on the model that best fits the experimental observations.

It is also important to understand the implications of the SR theory. In particular the reader should contemplate the most salient ones such as the abandonment of universal time, the frame dependence of simultaneity, the energy flow contributions to momentum, and most remarkably the relation between energy and mass.

In this chapter, we intend to develop a solid basis of the SR theory. We begin by providing a picture of the scientific scene as it appeared in the nineteenth century. In the second section, the track leading to the building of the SR theory is provided. Next, in section three, the famous Lorentz transformation is introduced. (Not being derived in this chapter, the author would like to suggest the work by Einstein [1952] for a simple derivation of the Lorentz transformation.) Over the next three sections we expose some of the kinematic and dynamic features of the SR theory together with the mathematical tool developed by Hermann Minkowski, and the energy equation of an isolated relativistic particle. Then the BG model for the relativistic energy of the one-particle system is introduced. Finally, in the last section we present the covariant form of the Maxwell equations and produce a BG interpretation for the Lorentz force equation together with an expression for the power of a particle traveling in an electromagnetic field.

1 Physics in the Nineteenth Century

Until the second half of the nineteenth century, Newtonian physics seemed to apply to all physical phenomena. Its central ideas of absolute space (i.e., the set of axes K_0 with respect to which all "true" motion should be measured) and the uniform – un-accelerated – motion of particles, that are removed from interaction with other particles (i.e., the law of inertia), provided the basis for establishing the relationship between the coördinates of rigid systems. For a rigid system K that moves with relative uniform velocity $\mathbf{w}$ with respect to another rigid system K' (see Fig. 5.1), the relation between the coördinates is given by the Galilean transformation

91
$$\mathbf{r}' = \mathbf{r} - \mathbf{w}t,$$
$$t' = t.$$

(Note the universal character of time in equation [91].) Thus if a particle obeys the equation of motion

$$\mathbf{f} = m\frac{d^2\mathbf{r}}{dt^2}$$

in system K, it will obey the equation

$$\mathbf{f}' = \mathbf{f} = m\frac{d^2\mathbf{r}'}{dt^2}$$

in system K'.

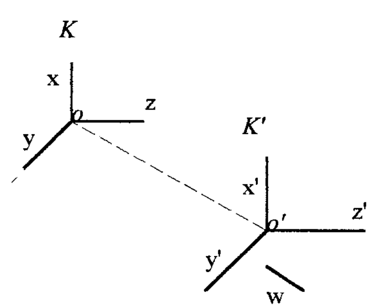

Fig. 5.1: Systems K and K' with Relative Velocity w.

Differentiating (91), the velocity addition law is found to be

92 $$\mathbf{u}' = \mathbf{u} - \mathbf{w}.$$

The above treatment shows that the first and second laws of Newton mechanics do hold under Galilean transformations. This prompted the acknowledgement of the principle of Galilean relativity (or covariance under GT's [short for Galilean transformations]) as a characteristic of all branches of physics.

After a series of experiments carried out by various physicists, it was concluded that electromagnetic radiation propagates in empty space with a uniform constant velocity which is equal to 3E8 m/s and is usually denoted by c. If we take this conclusion as a fact and use it to study the transformation properties of electromagnetics, our task becomes much easier since it can be accomplished without the need to consider the interrelation between magnetic and electric fields. If we carry out the GT in equation (91) then the speed of light in the K system of coördinates will be

93 $$\begin{aligned} u_x &= c.\cos\alpha, \\ u_y &= c.\cos\beta, \\ u_z &= c.\cos\gamma, \end{aligned}$$

where the angles α, β and γ are used to designate the direction of the light rays, and are measured from the three axes of the K system. Note that

94 $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

From equation (92), the velocity components with respect to the new system K' are

95 $$\begin{aligned} u'_x &= c.\cos\alpha - w_x, \\ u'_y &= c.\cos\beta - w_y, \\ u'_z &= c.\cos\gamma - w_z, \end{aligned}$$

and

96 $$\mathbf{u}' = (c^2 + w^2 - 2c(w_x.\cos\alpha + w_y.\cos\beta + w_z.\cos\gamma))^{1/2}.$$

Thus the velocity in the new frame will equal c only for a certain cone of directions with the vector $\mathbf{w}$ as its axis. In the direction of $\mathbf{w}$, the speed of light will be $c-\mathbf{w}$, and in the opposite direction it will be $c+\mathbf{w}$ (Bergmann 1976). Thus it is very clear that the Galilean relativity principle does not apply to electromagnetic radiation. From this difficulty, Maxwell's equations had to have a preferred frame of reference with respect to which they take their standard form!

Many efforts to establish the covariance of electromagnetism with respect to GTs are very well documented in literature. The most salient ones can be found in Bergmann (1976).

We shall rather focus on the path taken by Einstein that led to restoring the relativity principle in electromagnetics and mechanics.

2 Origin of Special Relativity

At the end of the nineteenth century, physicists were contemplating three possibilities for explaining the problems electromagnetism was facing with Galilean relativity. These possibilities are (Jackson 1975) :

1. Modifying the Maxwell equations so that invariance under GTs is established.
2. The relativity principle is not universal, and Galilean relativity applies only to classical dynamics, and that electromagnetism had a preferred reference frame where they assumed their simple form (where ether is at rest.)
3. The relativity principle is universal, but it assumes a different form than that proposed by Galilei.

The experimental evidence supporting Maxwell's equations was strong enough to eliminate the first possibility. Also the experiments of Fizeau in 1851 and 1853, and of

Michelson and Morley in 1886, made the second possibility quite implausible. Thus the third alternative was the only one left to explore.

Albert Einstein (1905a & 1905b) ventured through that alternative by building his model on the following three postulates (Jackson 1975 and Lawden 1982):

1. The Postulate of relativity: The Laws of nature and the results of all experiments performed in a given frame of reference are independent of the translational motion of the system as a whole.
2. The Postulate of the constancy of the speed of light: The speed of light is independent of the motion of its source.
3. The postulate of Euclidean Geometry: In any inertial frame, the geometry is Euclidean.

The introduction of these postulates dictated re-writing the laws of mechanics (and eventually, the laws of nature in general) for high-speed motions. They also required a series of experimental work to verify their authenticity. The theoretical work will be presented in the following sections. For more detailed treatment of the experimental work, the reader is referred to the Resource Letter on Relativity (1962).

3 Lorentz Transformations

Mathematically, the Lorentz transformation can be represented as a transformation from one system of spacetime coördinates x^α to another system x'^α ,so that

97 $$x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha$$

where $a^\alpha$and Λ^α_β are constants, restricted by the conditions

98 $$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

with

99 $$\eta_{\alpha\beta} = \begin{cases} +1, \alpha = \beta = 1, 2 or 3 \\ -1, \alpha = \beta = 0 \\ 0, \alpha \neq \beta \end{cases}$$

The above tensor (equation [99]) is known as the Minkowski tensor and represents the flat spacetime known as the Minkowski world (see chapter three.) A simple form of the LT (short for Lorentz transformation) can be found from the situation depicted in fig. 5.1, with $\mathbf{w} = wu_z$, for which the LT from K to K' takes the form

100 $$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

Where $\beta = \mathbf{w}/c$, and $\gamma = (1-\beta^2)^{-1/2}$. Note that a^α(in equation [97]) is equal to zero. This type of LT is correctly called a homogenous Lorentz transformation. On the other hand, if a^α is not equal to zero, the transformation is classified as an inhomogeneous Lorentz transformation or as a Poincaré transformation. Further, an LT, whether homogeneous or inhomogeneous, is classified as a proper transformation if Λ^α_β satisfies the additional conditions

101 $$\Lambda^0_0 \geq +1 \text{ ; } \mathrm{Det}\Lambda = +1.$$

An improper LT is then defined as a transformation that involves either space inversion ($\Lambda^0_0 \geq +1$; $\mathrm{Det}\Lambda = -1$) or time reversal ($\Lambda^0_0 \geq -1$; $\mathrm{Det}\Lambda = -1$). (Weinberg 1972.) An important subgroup of proper LT's is that consisting of rotations, and is represented mathematically with

102 $$\Lambda^i_j = R_{ij}, \qquad \Lambda^i_0 = \Lambda^0_i = 0, \text{ and } \Lambda^0_0 = +1,$$

where R_{ij} is a unimodular orthogonal matrix. Note that GT's and LT's are identical as far as spacetime translation ($x^\alpha \to x^\alpha + a^\alpha$) and rotation are concerned. Only when a boost of the coördinates is involved ($\mathbf{w} \neq \mathbf{0}$), does one see a difference between the two transformations.

A common property all LT's share is that they leave invariant the so-called proper time, which is the time one always measures in his own frame. The proper time is usually denoted by $d\tau$., and is defined as follows

103 $$d\tau^2 \equiv dx^{0^2} - d\mathbf{x}^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta .$$

Note that equation (103) is valid for both accelerated and unaccelerated bodies, but is restricted to material ones only (such as particles).

4 Special Relativistic Kinematics

In this section, we discuss some of the kinematic effects of the LT. Particularly of interest are the effects on length and time measurements in different frames of reference, as well as the relativistic addition laws of velocity.

Let us start by examining the effects on time. From equation (10), the time t' (where, $x'^0 = ct'$) is given by

104 $$t' = \gamma(t - (\beta / c)x^3).$$

Now we can write the expression for a period of time $\Delta t'$. Note that the last term in equation (104) drops out, and the relation becomes

105 $$\Delta t' = \gamma \Delta t .$$

Thus if one imagines a rigid clock located at the origin of the K frame, the time it records will be slower by a factor of $(1/\gamma)$ than that recorded by a similar clock located at the origin of the K' frame. This means that a body in motion will experience a period of time less than that experienced by a body at rest. (Note that in [105] the observer is assumed to be stationary in the K' frame; the moving frame becomes K.) Probably the reader has heard about the famous twin paradox which is based on the same relation in (105). Actually a twin-paradox experiment was carried out in the year 1966. It was conducted on muons instead of human beings, so that the journey of the muon is along a circle fourteen meters in diameter. After completing a complete circle and returning to the starting point, the muon was found to be younger that its twin particle. (Note that muons are unstable particles and that their life time is approximately two microseconds.) For more details the reader is referred to Fang and Chu (1987). The phenomenon discussed above is known as time dilatation or time dilation.

Another interesting effect of the LT is the so-called Lorentz contraction. Referring to fig. 5.1 With $\mathbf{w} = wu_z$, and if one assume that a rod is rigidly connected with K with its end points at the coördinates $x_b{}^3$ and $x_a{}^3$, one can give the rod's length as

106 $$l = x_b{}^3 - x_a{}^3$$

Now if an observer considers the length of the rod in the K' system, l', as the difference $x_b'^3 - x_a'^3$, at the same time x'^0, then the relation of the two lengths is given by

107 $$l' = \gamma^{-1}\, l.$$

Thus for a stationary observer in K', the rod will seem to be contracted by the factor γ^{-1}. A similar treatment that covers surface and volume elements can be found in Bladel (1984).

Another interesting effect that LT's produce is the relativistic law for addition of velocities. To derive the law we will assume that a point P located in the K' frame moves with a velocity $\mathbf{u}'$, then the components of this speed must be $u'^i = c\dfrac{dx'^i}{dx'^0}$ in the K' frame and $u^i = c\dfrac{dx^i}{dx^0}$ in the K frame. If we also assume that the K' frame moves with a uniform velocity of $\mathbf{w} = c\boldsymbol{\beta}$ in the positive z (x^3) direction (note that beta is a three dimensional vector in this case), then the LT gives

108
$$dx^0 = \gamma_w(dx'^0 + \beta dx'^3)$$
$$dx^1 = dx'^1$$
$$dx^2 = dx'^2$$
$$dx^3 = \gamma_w(dx'^3 + \beta dx'^0)$$

From the above, the parallel component of the velocity (with respect to $\mathbf{w}$), $u_{parallel}$ is given by

109
$$u_{parallel} = \frac{u'_{parallel} + w}{1 + \frac{\mathbf{w}.\mathbf{u}'}{c^2}} ,$$

and the perpendicular component is given by

110
$$u_{perpendicular} = \frac{u'_{perpendicular}}{\gamma_w(1 + \frac{\mathbf{w}.\mathbf{u}'}{c^2})} .$$

The above components will generate the classical laws if the term $\frac{\mathbf{w}.\mathbf{u}'}{c^2}$ is negligibly small. The total velocity is then given by

111
$$\mathbf{u} = \frac{\mathbf{u}'\gamma_w^{-1} + \mathbf{w}\left[\frac{\mathbf{w}.\mathbf{u}'}{w^2}(1 - \gamma_w^{-1}) + 1\right]}{1 + \frac{\mathbf{w}.\mathbf{u}'}{c^2}} .$$

Note that the transformation relation in equation (111) was derived for a rotation free LT.

5 The Minkowski World

As a consequence of abandoning universal time, events are now described (in a coördinate system) using four coördinates, three spatial and one temporal. Together with the postulate of the invariance of the speed of light in any inertial frame, this formalism produces the invariant quantity known as the proper time and is given by

112 $$d\tau^2 \equiv dt^2 - d\mathbf{x}^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta$$

(Note that we are now using natural units for which $c = 1$). In (112) the time t is considered a fourth coördinate (namely $x^0 = ct$) that together with the other three conventional coördinates constitutes a four dimensional generalized space known as the Minkowski world or the Minkowski spacetime. The above equation can be re-written to obtain a relation between the proper time and time as follows:

113 $$\begin{aligned} d\tau &\equiv (dt^2 - d\mathbf{x}^2)^{1/2} \\ &= (1-\mathbf{u}^2)^{1/2} dt = \gamma^{-1} dt \end{aligned}$$

where

113a $$\begin{aligned} \mathbf{u} &\equiv \frac{d\mathbf{x}}{dt} \\ \gamma &\equiv (1-\mathbf{u}^2)^{-1/2} \end{aligned}$$.

(Keep in mind that we are using natural units for which $c = 1$.)

Note that $\mathbf{u}$ is a vector relative to stationary rectangular axes only. In order to have a vector relative to Lorentz transformations in spacetime, we define

114 $$U^\alpha \equiv \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt}\frac{dt}{d\tau} = \gamma \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix}$$

which is known as the four-velocity of the system. Note that

115 $$\begin{aligned} U^2 &= U^\alpha U_\alpha = \eta_{\alpha\beta} U^\alpha U^\beta \\ &= -\gamma^2 (1-\mathbf{u}^2) = -1 \end{aligned}$$

In a similar manner the four-momentum of a particle whose rest mass is m_0 and whose four-velocity is U^α is given by

116 $$\begin{aligned} P^\alpha &= m_0 U^\alpha = m_0 \gamma \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} m_0\gamma \\ m_0\gamma\mathbf{u} \end{bmatrix} = \begin{bmatrix} m \\ m\mathbf{u} \end{bmatrix} \end{aligned}$$

where m ($\equiv m_0\gamma$) is known as the relativistic mass.

6 Special Relativistic Dynamics and Einstein's Energy Equation

Continuing with the previous one-particle system we can proceed to define a four-force given by

117 $$F^\alpha \equiv \frac{dP^\alpha}{d\tau} = m_0 \frac{dU^\alpha}{d\tau} = m_0 \frac{dU^\alpha}{dt}\frac{dt}{d\tau}$$

$$= m_0\gamma \begin{bmatrix} \dot{\gamma} \\ \dfrac{d(\gamma \mathbf{u})}{dt} \end{bmatrix} = \begin{bmatrix} \gamma\dot{\gamma}m_0 \\ \gamma f_1 \\ \gamma f_2 \\ \gamma f_3 \end{bmatrix} = \begin{bmatrix} F^0 \\ F^1 \\ F^2 \\ F^3 \end{bmatrix}$$

where

$$\mathbf{f} = \frac{d(\gamma m_0 \mathbf{u})}{dt}$$

Now differentiating (115) with respect to τ (and using direct notation), we get

118 $$\mathbf{U}.\frac{d\mathbf{U}}{d\tau} = 0, \qquad \text{i.e.} \quad \mathbf{U}.\mathbf{F} = 0$$

Thus the four-velocity and four-force are, in a sense, orthogonal. Substituting for $\mathbf{U}$ and $\mathbf{F}$ in (118), we get

119 $$\gamma^2(\mathbf{u}.\mathbf{f} - \dot{\gamma}m_0) = \gamma^2(\mathbf{u}.\mathbf{f} - \dot{m}) = 0$$
$$\therefore (\mathbf{u}.\mathbf{f} - \dot{m}) = 0$$

(Note that we have divided by γ^2 .) And since, by definition, the first term on the left hand side of (119) is the rate of work, we can obtain the kinetic energy, T as

$$T = \int \dot{m}\,dt = m + \text{constant}.$$

Now since at rest ($u = 0$), $T = 0$, we get

120 $$T = m - m_0 = E - m_0$$

where

121 $$E = \gamma m_0$$

is the Einstein energy equation for the system, and E is the total energy associated with it (Lawden 1982).

7 A Model Based on BGs

In this section we revisit what is known in bondgraphs as the power postulate (see chapter four. For an introduction to bondgraphs, the reader is referred to Breedveld [1984], Cellier [1991], and Karnopp et. al. [1990].) This postulate gives the power for any system as follows

122 $$\Pi = \frac{dE}{d\tau} = e_\alpha f^\alpha$$

where e_α is the effort tensor (e.g. the four-vector force for a one-particle system in classical mechanics) and f^α is the flow tensor (e.g. the four-vector velocity of the one-particle system.) Note that since the right-hand side of equation (122) is a tensor of valence zero, the left-hand side must also be a tensor of valence zero. Equation (122) will still hold if we use efforts and flows that do not transfer as tensors under orthogonal transformations. In this case the power will not be a tensor as well.

Applied to our previous example, assuming non-relativistic flow (velocity) we can use Newton's second law giving

123 $$\frac{dE}{dt} = m_0 \dot{\mathbf{u}} . \mathbf{u}$$

Thus the energy is given by

124 $$E = T = \frac{1}{2} m_0 \mathbf{u} . \mathbf{u} = \frac{1}{2} m_0 u^2$$

This formalism spans the gamut of classical physics and successfully produces a unifying approach for modeling physical phenomena. In order to extend this formalism to SRT we need first to observe the following discrepancies between SR mechanics and classical mechanics. In the SR case we model the relativistic effects through a mass modulation, thus defining the relativistic mass m. In classical mechanics such effects would rather be obtained by attributing the modulation to the flow (velocity) and maintaining the parameter character of mass. That is to say we can model the relativistic effects in the one-particle system as shown in Fig. 5.2.

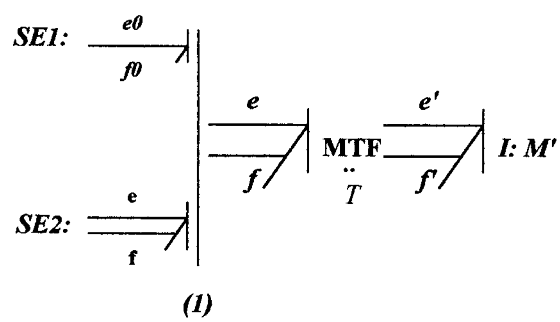

Fig. 5.2: BG Representation for a Relativistic Particle.

Where

$$M' = m_0 I_4 \ ,$$

(note that I_n is an $n \times n$ identity matrix)

and

$$T = \gamma,$$
$$\boldsymbol{e} = e_\alpha = F_\alpha = \gamma \dot{P}_\alpha,$$
$$\boldsymbol{e}' = e'_\alpha = \boldsymbol{e} / \gamma = \gamma^{-1} F_\alpha = \dot{P}_\alpha,$$
$$\boldsymbol{f}' = f'^{\alpha} = U^\alpha,$$
$$\boldsymbol{f} = f^\alpha = \boldsymbol{f}' / \gamma = \gamma^{-1} U^\alpha.$$

Thus substituting (114) and (117) in (122) gives

125 $$\Pi = e'_\alpha f'^{\alpha} = F_\alpha \gamma^{-1} U^\alpha$$

$$= \gamma^{-1}\eta_{\alpha\beta}F^{\beta}U^{\alpha}$$
$$= -\gamma\dot{\gamma}m_0 + \gamma\mathbf{u}.\mathbf{f} = 0$$

(compare with equation [119]) which can be re-written as

126
$$e_i'f'^i = \gamma^{-1}U^1F^1 + \gamma^{-1}U^2F^2 + \gamma^{-1}U^3F^3$$
$$= \frac{dE}{dt} = m_0\gamma\dot{\gamma}$$

(Note that the left-hand side of equation [126] is not a tensor with respect to orthogonal [Lorentz] transformations. Thus the energy is not a tensor either. Fortunately, the power postulate still holds. One can also consider equation [126] as a tensor equation applicable only to rectangular, stationary axes in spacetime.)

Since

127
$$\dot{\gamma} = (-1/2)(1-u^2)^{-3/2}(-2\mathbf{u}).\dot{\mathbf{u}}$$
$$= \gamma^3\mathbf{u}.\dot{\mathbf{u}}$$

we have

128
$$\frac{dE}{dt} = m_0\gamma^4\mathbf{u}.\dot{\mathbf{u}}$$

Integrating (128) with respect to time, we get

129
$$E = (1/2)\gamma^2 m_0 + E_0$$

(compare with [121]) where E_0 is the constant of integration. For any system with $m_0 = 0$, the total energy E must be equal to zero (unless $|\mathbf{u}| = 1$). Thus the constant of integration must be zero for zero mass. Still, in order to align this model with the Einsteinian one, we can use the rest energy formula as follows.

For zero velocity, equation (129) becomes

129a
$$E_{rest} = (1/2)m_0 + E_0 \,.$$

Now, by using Einstein's formula

$$E_{rest} = m_0 \quad ,$$

we can take the constant of integration to be equal to

$$E_0 = (1/2)m_0 \ .$$

The total energy can now be written as

130
$$\begin{aligned} E &= (1/2)\gamma^2 m_0 + (1/2)m_0 \\ &= m_0 + (1/2)m_0\gamma^2 u^2 \end{aligned}$$

(In the more familiar un-normalized units, equation (130) becomes

131
$$\begin{aligned} E &= (1/2)m_0\gamma^2 c^2 + (1/2)m_0 c^2 \\ &= m_0 c^2 + (1/2)m_0\gamma^2 u^2 \\ &= E_{rest} + T \end{aligned}$$

where the first term is the rest energy and the second is the relativistic kinetic energy.) We will call (130) the Special Relativistic Quadratic (SRQ) energy equation to distinguish it from (121). Note that under this formalism, the equation for kinetic energy, in both classic and relativistic mechanics, can be written as

132
$$T = (1/2)m_0(flow)^2$$

where the flow is given by (γu) in the relativistic case and by (u) in the classical one. (When $u \ll c$, the factor modulating the flow can be set equal to one.)

The concept of flow can be interpreted as the effect of spacetime on matter. Since the general theory of relativity forsakes the fixity of spacetime and shows that it is curved due to the existence of matter (Fang and Li 1989), one can regard the flow as the effect of spacetime on matter. (An established fact in general relativity is that spacetime acts on matter, telling it how to move and matter re-acts back on spacetime telling it how to curve). Thus the simple relation of change of position (in spacetime) to the change of time (i.e. velocity), is lost when matter (particles) travels with speeds commensurate to the speed of light.

To compare the existing formulae for kinetic energy, we can express the kinetic energy T (not to be confused with the above modulus) as a function of γ as shown below:

133 $$T_{Ein} = m_0 c^2 (\gamma - 1)$$

which is the formula derived from equation (121), and

134 $$T_{SRQ} = m_0 c^2 (\frac{\gamma^2 - 1}{2})$$

which is the formula derived from equation (131). Now let us define the quantity Δ as follows:

135 $$\Delta \equiv \frac{dT_{SRQ}}{d\gamma} - \frac{dT_{Ein}}{d\gamma}$$

which can be easily shown to equal T_{Ein} . By defining α as $T/(m_e c^2)$, where m_e is the rest mass of the electron, Einstein's energy equation then gives the following formula

136 $$(u/c)^2 = 1 - (1+\alpha)^{-2}.$$

On the other hand, the SRQ formula gives

137 $$(u/c)^2 = 1 - (1+2\alpha)^{-1}.$$

It is easy to show that (136) and (137) are equal to the first order, which is actually the order used in classical mechanics. To support (137) we need to compare experimental results with analytical results from both equations. Using the results obtained by W. Bertozzi (1964) shown in Table 5.1 (the values for $(u/c)_{obs}$), we can see (Fig. 5.3) that the Einstein formula is the closer to the data. However, neither formula can be excluded on this basis due to the fact that the data was provided to test equation (136) only, thus not focusing on the maximum difference region between the two equations.

We can also calculate the maximum difference between the two formulae by defining β as follows

138 $$\beta \equiv (u/c).$$

Then we can calculate the maximum difference by setting $\frac{d(\beta_{SRQ} - \beta_{Ein})}{d\alpha}$ equal to zero. This yields a sixth order equation with two complex roots, a second order root at zero and two real roots (one being negative). Thus the only values that α can take are 0 and 1.3276, from which the maximum difference is found to be equal to 5.07053E-02 c (where c is 3E8 m/s.).

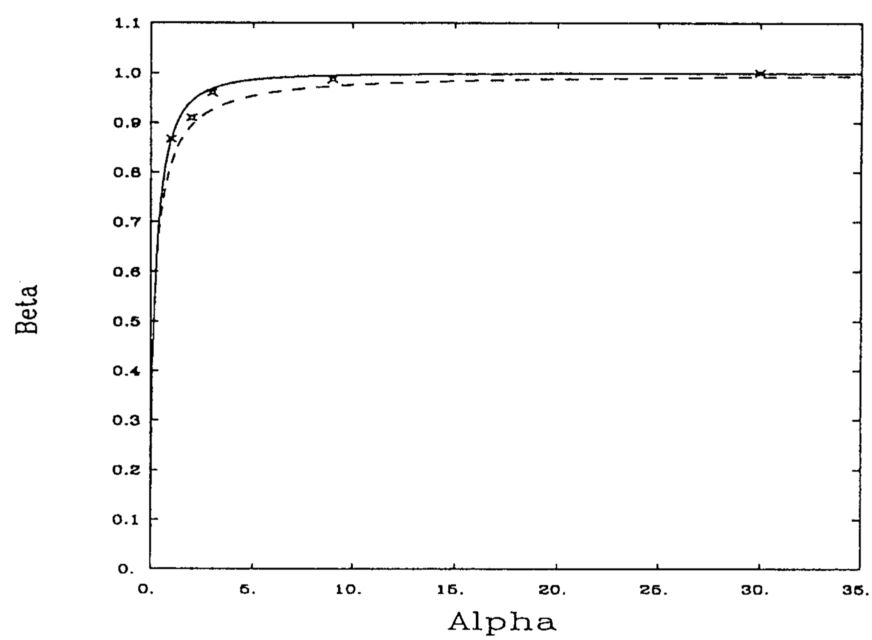

Fig. 5.3 : Beta vs. Alpha. The solid line corresponds to equation (136) where the dashed one corresponds to equation (137). The diagonal crosses correspond to the experimental data from table 5.1.

T (MeV)	α	$(u/c)_{obs}$	$(u/c)^2_{obs}$	$(u/c)^2_{Ein}$	$(u/c)^2_{SRQ}$
0.5	1	0.867	0.752	0.750	0.667
1.0	2	0.910	0.828	0.889	0.800
1.5	3	0.960	0.922	0.937	0.857
4.5	9	0.987	0.974	0.990	0.947
15.0	30	1.0	1.0	0.999	0.984

Table 5.1: The results from the Bertozzi experiment. The table also provides the analytical data from equations (136) and (137).

One can also obtain the difference of the two formulae as a percentage. Defining the difference as a percentage of β_{Ein} , we get

139 $$P.D. = ((\beta_{Ein} - \beta_{SRQ}) / \beta_{Ein}) \times 100 \quad ,$$

from which the maximum difference is found to be 5.615 % of the value of β_{Ein} (see Fig. 5.4).

Other experimental results that can be shown to support the new model are abundant. For example, the experiment by Perry and Chaffee at Harvard University (although used as an evidence for the contribution of the kinetic energy to the inertia; a dependence that is also supported by the new model) can also be used to explicate the close relation between the experimental data and the proposed model (Copeland and Bennett 1961). Other promising results are those of the Guye, Ratnowsky and Lavanchy (1921) experiment. For more on the experimental work on SRT, the reader is referred to the Resource letter on special relativity (1962).

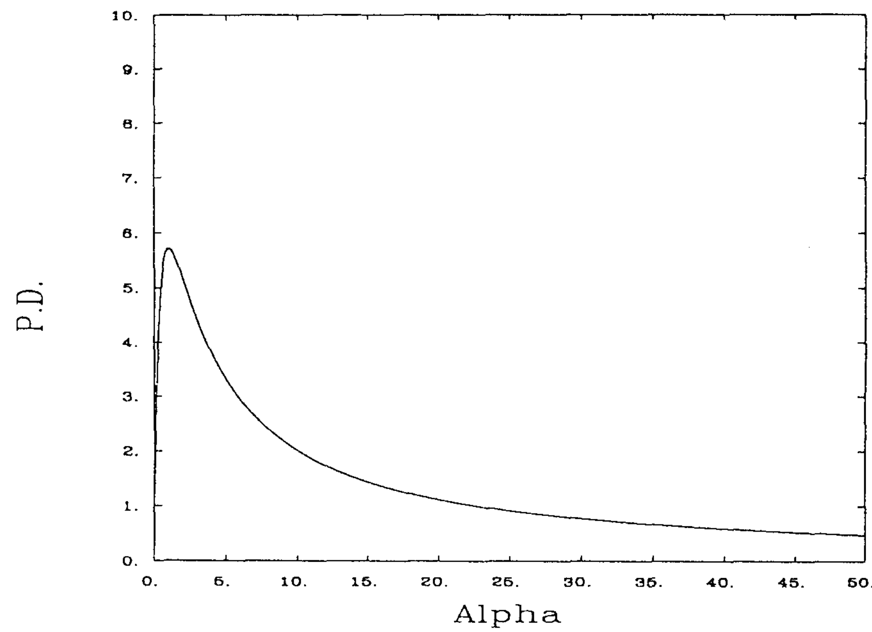

Fig. 5.4 : The difference between equations (136) and (137) expressed as a percentage of β_{Ein}.

At this juncture we need to remind ourselves of the objective of this investigation. According to Karl Popper "...science is not in the business of validating models at all, but rather should be trying to falsify them." (Casti 1992). The work by Parker (1972) seems to falsify the new model in favor of the Einsteinian one; still he declares his data as circular at the end of his paper. The best approach is probably the direct measurement of the time-of-flight of electrons within the range $\alpha < 5$, and the comparison of the experimental results to the analytical ones. Such an experiment is yet to be conducted (note that although the experiment by Bertozzi is structured as proposed, it lacks focusing on the suggested range where the difference between the competing models is maximum.)

It should also be noted that BG's are perfectly capable of reproducing the Einstein formula, if we proceed as follows. Using $\mathbf{f} = \frac{d(\gamma\mathbf{u})}{dt}$, which is the normalized form of $\mathbf{f} = \frac{d(m_0\gamma\mathbf{u})}{dt}$, we can write

$$\mathbf{f} = (\dot{\gamma}\mathbf{u} + \gamma\dot{\mathbf{u}}),$$

and substituting for $\dot{\gamma}$ from equation (127) we get

$$\mathbf{f} = \gamma^3\dot{u}.$$

Now utilizing a transformer (see fig. 4.4) to obtain the involved equations, with the integral causality for the **I** element, and setting $\mathbf{f} = T[d(Tu)/dt]$, one can easily find that the modulus required, the so-called Lorentz modulus, is equal to

140 $$T = \left[(2u^{-2})(\gamma-1)\right]^{1/2}.$$

Finally it's worth mentioning that our effort of building a model that competes with the Einsteinian one for special relativity is not an unprecedented one. Abraham (1908) proposed a rigid model for the electron that produced a competing relation between mass and velocity. Actually most of the early experiments where carried out to support the validity of either the Abraham or Einstein mass-velocity equation (or rather to falsify one of them.) The prevailing of the Einstein equation over Abraham's should not deter inquiring minds from building new models (equations) that might bring about a more unified scientific structure.

8 A BG Interpretation of the Field-Strength Tensor in Electrodynamics

This section is designed in a very simple fashion that introduces the basic invariant form of the Maxwell equations and the Lorentz force equation. It also introduces a BG interpretation for the Lorentz force equation that leads to an interesting generalized statement concerning the power composed of tensorial efforts and flows. Let our first step be the derivation of the invariant form of the Maxwell equations and the Lorentz force.

For the electric and magnetic fields **E** and **B**, produced by charge density ε and current density **J**, the Maxwell equations are

141a $$\nabla.\mathbf{E} = \varepsilon$$

141b $$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$$

141c $$\nabla.\mathbf{B} = 0$$

141d $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Now let us define the following tensor

142 $$\Phi^{\alpha\beta} \equiv \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

This then allows us to write (141a) and (141b) as follows

143 $$\frac{\partial}{\partial x^{\alpha}} \Phi^{\alpha\beta} = -J^{\beta} ,$$

and (141c) and (141d) become

144 $$\varepsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^{\beta}} \Phi_{\gamma\delta} = 0,$$

which can be rewritten as

145 $$\Phi_{\beta\gamma,\alpha} + \Phi_{\gamma\alpha,\beta} + \Phi_{\alpha\beta,\gamma} = 0 ,$$

where $\Phi_{\beta\gamma,\alpha} \equiv \frac{\partial \Phi_{\beta\gamma}}{\partial x^{\alpha}}$. Note that $\Phi_{\gamma\delta} \equiv \eta_{\gamma\alpha}\eta_{\delta\beta}\Phi^{\alpha\beta}$, and since J^{α} is a four-vector, $\Phi^{\alpha\beta}$ is also a tensor, which is usually referred to as the (electromagnetic) field strength tensor, that satisfies

146 $$\Phi'^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\Phi^{\gamma\delta}.$$

Hence we can provide the invariant form of the Lorentz force equation (giving the electromagnetic force on a charged particle)

147 $$F^{\alpha} = e\eta_{\beta\gamma}\Phi^{\alpha\beta}U^{\gamma}.$$

Now equation (144) allows us to represent $\Phi_{\alpha\beta}$ as a curl of a four-vector A_{α} (=(ϕ,**A**), where ϕ is the scalar potential and **A** is the vector potential):

148 $$\Phi_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha}}A_{\beta} - \frac{\partial}{\partial x^{\beta}}A_{\alpha}.$$

Hence the Maxwell equations can be written as follows

149 $$\Box^2 A^{\alpha} = -J_{\alpha}$$

150 $$\partial^{\alpha} A_{\alpha} = 0$$

Where $\Box^2$ is the d'Alembertian defined by

151 $$\Box^2 \equiv \eta^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}} = \nabla^2 - \frac{\partial^2}{\partial t^2},$$

and

152 $$\partial^{\alpha} \equiv \frac{\partial}{\partial x_{\alpha}} = (\frac{\partial}{\partial x^{0}}, -\nabla).$$

Recalling the power postulate presented in chapter three (section one), we can formulate the power for a particle traveling in the electromagnetic field as follows

153 $$\begin{aligned}\Pi &= e_{\alpha}f^{\alpha} = F_{\alpha}U^{\alpha}\\ &= \eta_{\alpha\beta}F^{\beta}U^{\alpha}\\ &= \eta_{\alpha\beta}e\eta_{\delta\gamma}\Phi^{\beta\delta}U^{\gamma}U^{\alpha}\end{aligned}$$

Note that the (tensorial relativistic) power is identically equal to zero. The reader might have already noticed the resemblance between this result and the one in equation (125). Actually one can even postulate that this is a general result for any tensorial (relativistic) power built from four-vector efforts and flows, since by definition the length of a four-vector is unchanged under rotation of axes (that is by a Lorentz transformation) (Rosser 1967, p. 206), or in other words one can postulate that equation (78), **4 : 1** can be re-written as follows

154 $$\Pi = e_\alpha f^\alpha = 0.$$

Of course a rigorous study of other physical domains is first necessary before one can claim such a generalization. The reason this postulate was presented is to impress the fact that it was obtain for the two covered cases.

Note also that the result in (153) was discussed in chapter four. The reader can easily see the gyrative character of the tensor $\Phi^{\alpha\beta}$– from the antisymmetric nature of the matrix in equation (142) – and since gyrators are non-energic elements, the result in (153) becomes natural (see fig. 4.5, **4 : 2** for the bondgraphic representation.) In a similar treatment to the one carried out for equation (125), the power of the particle in the electromagnetic field can be given as

155 $$\frac{dE}{d\tau} = F_i U^i = eE_i U^i = \gamma e\mathbf{u}.\mathbf{E},$$

or recalling the classical form of the Lorentz force equation, one can write

156 $$\begin{aligned}\frac{dE}{dt} &= \mathbf{F}.\mathbf{u} = e(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}).\mathbf{u} \\ &= e\mathbf{u}.\mathbf{E} + cypher\end{aligned}$$

Thus one can see that the effect of the magnetic field is not shown in the formula obtained from the power postulate. This of course is explained by noting that the function of the magnetic field is to influence the direction of the particle rather than influencing its transverse motion (this can be related to the Larmor theory which explains the rôle of the magnetic field as generating angular velocity.)

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"Of the general theory of
relativity you will be convinced,
once you have studied it.
Therefore I am not going to
defend it with a single word."
Albert Einstein, in a postcard to
A. Sommerfeld, February 8, 1916

6 ON MODELING GRAVITATION

Almost ten years elapsed, after his special version, before Einstein published his general theory of relativity. This theory models the effect known as gravitation utilizing the so-called Riemannian geometry, and provides the gravitational field equations even for strong gravitational fields. From another perspective, it can be stated that the theory investigates the changes suffered by the laws of nature as powerful gravitational fields curve spacetime. Thus, in general relativity the effects of gravity are injected to flat spacetime laws to produce the same laws but in curved spacetime.

This chapter will provide the reader with the rudiments of the theory. The author reckons that the first concept to be discussed must be the so-called energy-momentum tensor; After providing the definition, we explain how this tensor acts under powerful gravitational fields. Also the so-called principle of equivalence is introduced. This principle clears the way for the transition from special to general relativity to take place. From there we establish a link between the results from chapter five and their counterparts under strong gravitational fields.

The crux of the chapter is the discussions in the last two sections. The penultimate one provides Einstein's field equations, where the last section is devoted to discussing the rôle tensorial power can play in the modeling and understanding of the equations.

I also feel obliged to indicate that, similar to the case of SR theory, Einstein's general theory of relativity (GR theory for short) is also challenged by other theories. For example, the so-called relativistic theory of gravitation that was invented by Logunov and Mestvirishvili (1989) preaches a gravitational field which is rather physical and identifies with the Faraday-Maxwell-Hertz type of fields. In other words it provides the field with locally defined energy. The interested reader might find other theories in the literature that also compete towards explaining the dynamics of gravitation. This chapter however will mainly be restricted to Einstein's work since his approach is highly regarded as the standard one.

1 The Energy-Momentum Tensor

The energy-momentum tensor can be thought of as a tool that determines the amount of mass-energy in a unit volume (Misner et al. 1973). In a similar fashion to the treatment in section eight of chapter four, the density and current of the energy-momentum four-vector can be defined so that (see Weinberg [1972], p. 43f.):

157 $T^{\alpha\beta}{}_{,\beta} = G^{\alpha}.$

(Note that the relation in [157] holds only in the absence of gravitation.) In (125), $T^{\alpha\beta}$ is the energy-momentum tensor, G^{α} is the density of the external four-vector force F^{α}, and $T^{\alpha\beta}{}_{,\beta} \equiv \frac{\partial T^{\alpha\beta}}{\partial x^{\beta}}$. The tensor G^{α} can be easily identified with the effort concept in bondgraphs. Thus one can think of the density of the external four-vector force as the effort tensor of the matter or material domain.

Note that for a system of free particles, the four-vector momentum of the system is constant and the E-M tensor (short for Energy-Momentum) is conserved. Mathematically this is stated as follows

158 $T^{\alpha\beta}{}_{,\beta} = 0$.

(i.e. the bondgraphic effort is identically equal to cypher!) Equation (158) will also hold for a system of particles that interacts only during space-localized collisions. Now if one introduces forces to this system of particles, the conservation of the E-M tensor is lost. For example the E-M tensor for a gas of charged particles is given by

159 $T^{\alpha\beta}(x)_{,\beta} = \Phi^{\alpha}{}_{\gamma}(x) J^{\gamma}(x).$

Actually one can restore the conservation law to this particular setting, if the term

160 $T_{em}{}^{\alpha\beta} \equiv \Phi^{\alpha}{}_{\gamma}\Phi^{\beta\gamma} - (1/4)\,\eta^{\alpha\beta}\Phi_{\gamma\delta}\Phi^{\gamma\delta}$,

(i.e. $T_{em}{}^{00} = (1/2)(\mathbf{E}^2 + \mathbf{B}^2)$, $T_{em}{}^{i0} = (\mathbf{E}\times\mathbf{B})_i$) is added to the existing $T^{\alpha\beta} T^{\alpha\beta}$, thus creating a new (conserved) E-M tensor

161 $T^{\alpha\beta}{}_{conserved} \equiv T^{\alpha\beta} + T_{em}{}^{\alpha\beta}$.

Finally please note that the E-M tensor is always a symmetric tensor (even the one in equation [161]).

The following sections will manifest the importance of the E-M tensor especially as a building block for the gravitational field equations.

2 The Bridge between SR and GR

Due to the efforts of great scientists such as Galileo Galilei (1564-1642), Christiaan Huygens (1629-1695), Isaac Newton (1642-1727), Friedrich W. Bessel (1784-1846), Roland Eötvös (1848-1919), and Robert Dicke (1916-) the nucleus of the idea of the equality between the gravitational and inertial mass was created. This effort led to a more successful idea that was able to constitute a link between the SR and GR theories; viz., Einstein's equivalence principle, especially in its weak form (Weinberg 1972). This principle has two versions, strong and weak. The strong version of the principle can be stated as follows (ib. page 68f.):

[A]t every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coördinate system" such that, within a sufficiently small region [,small enough so that the gravitational field is sensibly constant throughout it,] of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coördinate systems in the absence of gravitation [i.e. the form given to the laws of nature by special relativity].

On the other hand, the weak version of the principle is restricted only to the laws of motion of freely falling particles instead of the more general laws of nature. Actually, one can even distinguish yet another version of the principle. One can talk about a very strong version which applies to all phenomena, and a strong version that excludes only gravitation. This simply has to do with the experimental evidence and its limitations to particular phenomena.

In its strongest form, the principle can be stated as follows (Misner et al. 1973):

[I]n any and every Lorentz frame, anywhere and anytime in the universe, all the (nongravitational) laws of physics must take on their familiar special-relativistic forms.

As was also pointed out by Weinberg (ib.), the equivalence principle, especially in its strong form, and the Gauss axiom are very close to each other. This directly leads us to conclude that gravitation can be modeled by the derivatives $\frac{\partial \xi^\alpha}{\partial x^\mu}$ of the transformation functions $\xi^\alpha(x)$ that transform the observer from the laboratory coördinates x^μ to the local inertial ones ξ^α.

In the next section, we will explore, added by the principle of equivalence, the effects of gravitation on systems we have already covered in the previous chapter, viz., the electrodynamic and (particle) mechanical systems.

3 The Consequences of Gravitation

As an extension to earlier treatment of special relativistic dynamics given in **5** : **6**, and the treatment of electrodynamics given in **5** : **8**, we introduce, in this section, the effects of gravitational fields on these domains. Most importantly, the author explicitly states the formalism by which he extends the bondgraphic concepts to cover the effects of gravitation. Also at the end of this section a discussion on the effects of gravitation on E-M tensor is provided. The reader interested in a more detailed treatment of the presented concepts in advised to consult the book by Weinberg (1972).

Before discussing the effects of gravitation on particle mechanics and electrodynamics we need to introduce a mathematical tool that is used to build the equations involved.

Let us first focus on the character of the momentum, a fortiori, the force tensor, and the velocity tensors. It should be obvious to the reader that these tensors "live" or function on curves, say the curve $x^\mu(\tau)$. In other words they are defined only over a curve. Hence our using the covariant differentiation introduced in **3** : **11**, which is designed for tensor fields which "live" within the spacetime as a whole, is quite inappropriate. Thus we need to define a covariant derivative with respect to the curve, or rather with respect to the invariant τ that parameterizes the curve x^{μ}.

Now let us introduce a contravariant vector $V^{\mu}(\tau)$ that transforms as follows:

162 $$\bar{V}^{\mu}(\tau) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} V^{\nu}(\tau),$$

where the derivative $\partial \bar{x}^{\mu} / \partial x^{\nu}$ is to be evaluated at $x^{\nu} = x^{\nu}(\tau)$. Differentiating with respect to τ, we get

163 $$\frac{d\bar{V}^{\mu}(\tau)}{d\tau} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} \frac{dV^{\nu}(\tau)}{d\tau} + \frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\nu} \partial x^{\lambda}} \frac{dx^{\lambda}}{d\tau} V^{\nu}(\tau).$$

Comparing with the definition of the affine provided by equation (40), **3** : **11**, we can define the covariant derivative along the curve $x^{\nu}(\tau)$ as follows:

164 $$\frac{DV^{\mu}}{D\tau} \equiv \frac{dV^{\mu}}{d\tau} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\lambda}}{d\tau} V^{\nu}.$$

For a covariant vector, the above equation becomes

165 $$\frac{DV_{\mu}}{D\tau} \equiv \frac{dV_{\mu}}{d\tau} - \Gamma^{\lambda}_{\nu\mu} \frac{dx^{\nu}}{d\tau} V_{\lambda}.$$

Note that the covariant derivative defined in equations (164) ([165]) is a contra- (co-) variant tensor, of valence one, that transforms according to the law given in equation (21) ([22]), **3** : **9**. In general the covariant derivative along a curve $x^{\nu}(\tau)$of a tensor, of any valence, is defined by adding, to the ordinary derivative with respect to τ, a term such as that in equation (163) for each upper index, and such as the one in equation (164) for every lower index (Weinberg 1972).

The covariant derivative along a curve for a tensor field can be found by utilizing the ordinary covariant derivative. For example, the covariant derivative along $x^{\nu}(\tau)$ for the tensor field $T^{\mu}{}_{\nu}$ is equal to

166 $$\frac{DT^{\mu}{}_{\nu}}{D\tau} = T^{\mu}{}_{\nu;\lambda}\frac{dx^{\lambda}}{d\tau} .$$

Thus for the one-particle system we have introduced in **5** : **6**, the influence of an external gravitational field will alter equation (117), of **5** : **6**, as follows:

167 $$F^{\mu} \equiv \frac{DP^{\mu}}{D\tau} = m_0 \frac{DU^{\mu}}{D\tau} .$$

Equation (167) is definitely interpretable as an inertance. The difference between this element and its counterpart is the covariant derivative along the curve over which the tensors involved are defined. It should be obvious that this element reduces to the classical one when the gravitational force is absent. Note that the development of the energy equation(s) in chapter **5**, does not extend to this setting, which involves gravitational forces. The author restricts his interpretation only to bondgraphic elements which survive the existence of gravitational forces. (In the following this point is going to be discussed in more detail.) Still, one can easily see that the tensorial power for this one-particle system, under the influence of a gravitational force is given by

168 $$\Pi = e_{\mu}f^{\mu} = mU_{\mu}\frac{DU^{\mu}}{D\tau}.$$

Now since $U^{\alpha}U_{\alpha} = -1$, holds in the absence of gravitation, and applying the equivalence principle, we should have the same result in the presence of gravitation. Thus if we take the covariant derivative, along the curve we get

169 $$\begin{aligned} \frac{D}{D\tau}(U^{\mu}U_{\mu}) &= U^{\mu}\frac{DU_{\mu}}{D\tau} + U_{\mu}\frac{DU^{\mu}}{D\tau} \\ &= U^{\mu}\frac{DU_{\mu}}{D\tau} + g_{\mu\nu}U^{\nu}\frac{D(g^{\mu\nu}U_{\nu})}{D\tau} \\ &= U^{\mu}\frac{DU_{\mu}}{D\tau} + g_{\mu\nu}U^{\nu}\left(g^{\mu\nu}\frac{DU_{\nu}}{D\tau} + U_{\nu}\frac{Dg^{\mu\nu}}{D\tau} \right) \end{aligned}$$

Now we need to evaluate the term $\frac{Dg^{\mu\nu}}{D\tau}$. For that we utilize equation (166). Thus we get

170 $$\frac{Dg^{\mu\nu}}{D\tau} = g^{\mu\nu}{}_{;\lambda}\frac{dx^{\lambda}}{d\tau}.$$

From our previous result concerning the covariant derivative of the metric tensor, found in **3** : **11**, we realize that the expression in (170) is equal to zero. Hence we can write

171 $$\frac{D}{D\tau}(U^{\mu}U_{\mu}) = U^{\mu}\frac{DU_{\mu}}{D\tau} + g_{\mu\nu}g^{\mu\nu}U^{\nu}\frac{DU_{\nu}}{D\tau}.$$

Thus the expression in (171) vanishes and we get

172 $$\Pi = mU_{\mu}\frac{DU^{\mu}}{D\tau} = 0 .$$

Similarly, the treatment of electrodynamics is modified to indicate the existence of the gravitational forces. Equations (143) and (145), **5** : **8**, are modified by the covariant derivative to read

173 $$\Phi^{\mu\nu}{}_{;\mu} = -J^{\nu},$$

174 $$\Phi_{\beta\gamma;\,\alpha} + \Phi_{\gamma\alpha;\,\beta} + \Phi_{\alpha\beta;\,\gamma} = 0 ,$$

(Note that the raising or lowering of the indices is carried out via the metric $g^{\mu\nu}$ instead of $\eta^{\alpha\beta}$.) and equation (147), also of **5** : **8**, becomes

175 $$F^{\mu} = eg_{\nu\lambda}\Phi^{\mu\lambda}V^{\nu}.$$

Equation (157), of this chapter, is also modified by the existence of gravitational fields to read

176 $$T^{\mu\nu}{}_{;\mu} = G^{\nu}.$$

Thus the transfer from flat spacetime to curved spacetime, or from Lorentz to Non-Lorentz frames, changes the writing of the laws of physics by changing the commas to semicolons.

4 Einstein's Gravitational Field Equations

In this section we will follow one of the approaches given by Weinberg (1972) for deriving Einstein's field equations. First we build the equations for weak fields and then utilize the equivalence principle to arrive at the equations for the strong ones.

For a particle moving slowly in a weak stationary gravitational field, the equations of motion can be written as follows (see **3** : **15**):

177 $$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{00}\left(\frac{dt}{d\tau}\right)^2=0$$

where $d\mathbf{x}/dt$ was neglected with respect to $dt/d\tau$ (on the assumption of slow motion). Also the value of the affine connection is simplified considerably to give

178 $$\Gamma^\alpha{}_{00}=-\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta}$$ (see equation (39), **3** : **11**).

Another simplifying step can be achieved due to the weak nature of the field, since such fields can be described by a semi-Cartesian coördinate system that is given by

179 $$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}\qquad\left|h_{\alpha\beta}\right|<<1\,.$$

Thus, truncating after the first term, the affine connection can be written as follows

180 $$\Gamma^\alpha{}_{00}=-\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta}\,.$$

From which we can arrive at the following equation:

181 $$\frac{d^2\mathbf{x}}{dt^2}=\frac{1}{2}\nabla h_{00}$$

which corresponds to the Newtonian result

182 $$\frac{d^2\mathbf{x}}{dt^2}=-\nabla\phi\,.$$

In (182), ϕ is the gravitational potential given by Poisson's equation

183 $$\nabla^2 \phi = 4\pi G\rho ,$$

where G is Newton's constant, equal to 6.670 E-8 in c.g.s. units, and ρ is the nonrelativistic mass density. Comparing (181) and (182), we can write

184 $$h_{00} = -2\phi + C,$$

where C is a constant. Now examining the values of h_{00} and ϕ at infinity, which logically must vanish, one finds that the constant C must be equal to cypher. Thus we arrive at

185 $$g_{00} = -(1+2\phi).$$

Now let us divert our attention to extending the above results to (arbitrarily) strong gravitational fields. We can achieve this by focusing first on a point in this field where we can erect a locally inertial coördinate system for which the following holds true:

186 $$g_{\alpha\beta}(X) = \eta_{\alpha\beta}$$

$$\left(\frac{\partial g_{\alpha\beta}(x)}{\partial x^{\gamma}} \right)_{x=X} = 0.$$

Thus in the neighborhood of X, we can treat the strong field as a weak one that is described by linear differential equations.

On the other hand, since the energy density T_{00} for nonrelativistic matter is equal to the mass density ρ, and utilizing equation (185), one can rewrite Poisson's equation (183) as follows:

187 $$\nabla^2 g_{00} = -8\pi G T_{00} \ ,$$

For the general distribution of the E-M tensor one can anticipate

188 $$G_{\alpha\beta} = -8\pi G T_{\alpha\beta} \ ,$$

which can be taken one step further, via the equivalence principle, to give the equations for an (arbitrarily) strong gravitational field:

189 $$G_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

(where $G_{\mu\nu}$ reduces to $G_{\alpha\beta}$ for weak fields.) Thus recapitulating, one can see that $G_{\mu\nu}$ is a symmetric, conserved tensor (properties necessary to salvage the equality in [189]). We shall also assume that $G_{\mu\nu}$ contains only terms that are linear in the second derivative, or quadratic in the first derivative, of the metric. This assumption is justifiable since $G_{\mu\nu}$ must be of the second rank. If higher, or lower, order terms are to be allowed, they will have to be multiplied by constants having dimensions that reduce, or increase, the overall rank of the term to two. Thus reducing the term's contribution for sufficiently large or small fields.

Another important fact we need to consider, is that contracting the R-C tensor is the most general way of constructing a second rank tensor field. Since such contractions can yield only two tensors, viz., the Ricci tensor and the curvature scalar, we can build the $G_{\mu\nu}$ as follows

190 $$G_{\mu\nu} = C' R_{\mu\nu} + C'' g_{\mu\nu} R.$$

where C' and C'' are constants. For the evaluation of these constants, and for an alternative derivation, the reader is referred to Weinberg (1972). The final result will actually give the Einstein tensor already introduced in equation (61), **3** : **16**. Thus the final shape of the Einstein field equations becomes

191 $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}.$$

Although providing some of the solutions of the Einstein field equations might add some flavor to the discussion, we believe such treatment to be void of any nutritious value that benefits the objectives of the chapter. The interested reader can still find such solutions in Hoenselaers and Dietz (1984).

5 The BG Interpretation of GR Theory

In this section we reflect on the previous discussions and their implications on achieving our objectives, viz., on modeling the dynamics of gravitational fields via bondgraphs.

First let us interpret the character of the effort in GR. Our earlier treatment of the E-M tensor revealed its conservation; which means that its covariant derivative, which we interpreted as the effort tensor, is identically equal to cypher. Hence, again, we find that the tensorial power vanishes, although this time its vanishing is due to the conservation of the effort rather than the "generalized orthogonality" between the effort and the flow we experienced in other situations. This of course reduces our chances of modeling the power (energy) transactions for gravitational fields. To this end Hilbert writes (1917):

I declare that ... for the general theory of relativity, that is, in the case of general invariance of the Hamiltonian function, there are generally no energy equations that ... correspond to energy equations in orthogonal-invariant theories. I could even note this fact as being a characteristic feature of the theory.

Another inherent problem in GR is the local energy density. Quoting from Wald (1984):

The issue of energy in general relativity is a rather delicate one. In general relativity there is no known meaningful notation of local energy density of the gravitational field. The basic reason for this is closely related to the fact that the spacetime metric, g_{ab}, describes both the background spacetime structure and the dynamical aspects of the gravitational field, but no natural way is known to decompose it into its "background" and "dynamical" parts.

More recently, Logunov and Mestvirishvili (1989), championed a new theory that built the gravitational field in the spirit of the Faraday-Maxwell fields, i.e., built the field "as a material substratum that can never be destroyed by the choice of reference frame." By doing so they eliminated the difficulty of splitting the metric tensor, since "... to retain the concept of a gravitational field as being a Faraday-Maxwell physical field we must completely renounce its identity with the metric tensor." (ib.)

In spite of all the problems, someone with some background in bondgraphs can easily see that equation (190) represents a transformer of power between two domains; the material and gravitational ones. Unfortunately, upon taking the covariant derivative of both sides, one arrives at a trivial equality. Borrowing the Karnoppic expression, our treatment is definitely "penetrating but Olympian", in the sense that it does identify the problems involved but stands short from overcoming them.

The author is still confident that bondgraphs can become a unifying method for modeling dynamics, including those of gravitational fields. It is also believed that the hurdles preventing us from achieving this goal result from the structure of the GR theory rather than from the geometrodynamics involved. It is quite possible that other alternative theories can produce bondgraphic models that can help understand gravitation better. Unfortunately, such claims will have to be kept on hold until a second stage of this research, towards a Ph.D. hopefully, is conducted to that end.

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"He spoke, but Meneláus overheard;
and when he answered, these were his winged words:
'Dear sons, no mortal man can vie with Zeus:
his halls are everlasting — and his goods.
As for my wealth, there may be other men
to match me, or there may, by chance, be none.
But it has cost me many wanderings
and many griefs to bring these treasures here,
stowed in my ships; for more than seven years
I traveled till at last I reached my home…'"
Book IV, The Odyssey of Homer.
A new verse translation by A. Mandelbaum.

7 CONCLUSIONS

This "summary" chapter provides an account of the results achieved in this thesis. The sequence they are listed in here, does not reflect their ordering in the thesis. The author rather felt that the results should be arranged according to their importance and the degree they affect future research on the subject. Please note that this chapter does not exhaust all the work presented in the thesis and that it is intended to highlight the salient results only.

1 On Whether Bondgraphs are the Appropriate Tool for Modeling SR and GR

Although the bondgraphic treatment of SR particle mechanics via modulating the flow by γ, resulted in a divergence from the standard model, its educational value is still preserved when the flow is modulated by $T=\left[(2u^{-2})(\gamma-1)\right]^{1/2}$, thus admitting velocity modulated inertance. In SR electrodynamics, bondgraphs were successful in modeling the electromagnetic field tensor in congruence with the existing literature.

As far as the GR theory is concerned, the tensorial power postulate was penetrating but Olympian. Again the insight provided is of considerable importance.

Thus although bondgraphs provide penetrating problem analysis, they, in some cases introduce models that diverge from the standard ones. We still endorse their use, at least as a parallel analyzing tool.

2 The Charm of Tensorial Power

A promising result obtained in this thesis is the tensorial bondgraphic power. This formulation worked beautifully in the examples provided in chapters five and six. It was also postulated that this power always vanishes, although such conclusions are not more than educated guesses outside the work we conducted. More exhaustive treatment and testing of the postulate is necessary before a rigid belief of the results is established.

3 The Impact of Parametrizing Mass on SR Theory

In case of SR particle mechanics, BG's were found to produce a new energy-mass formula. Upon comparing the new formula with the standard 1905 one, it was found that the experimental data , from one of the more accurate experiments, agreed better with the standard formula. This result must be viewed while keeping in mind the fact that the data used were actually conducted to test the standard formula only. Although we highly respect the Einstein energy-mass relation, we maintain that conducting new experiments, designed specifically to comparing the two models, will definitely be the only decisive factor on adopting a new formula or renewing our faith in the existing one. Other factors, such as the conservation of energy, must definitely be taken into consideration in the proposed experiments.

4 On Reconciling GR with Bondgraphics

Other than identifying the effort tensor in GR, the research on this topic was hindered by the lack of a definition of local energy for "Einsteinian" gravitational fields. A second phase of the research can be conducted to focus on other gravitation theories which support local energy density.

The Einstein field equations provided some insight on the power transfer between the material and gravitational fields, despite the collapse of the constitutive relations of the bondgraphic transformer.

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