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Bootstrap percolation with individual thresholds

Bachelor Thesis

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Abstract

Bootstrap percolation is a random process that models the spread of activation on a linked structure. Starting with an initial set of active vertices, in each round all inactive vertices with at least r active neighbors become and remain active until there are no inactive vertices with enough active neighbors left.

Due to its numerous applications in modeling of physical processes, bootstrap percolation has been intensively studied on various (deterministic and random) graph models. The thereby recurringly encountered sharp threshold phenomenon states that there exists a percolation threshold such that we typically either have percolation or no percolation, depending on whether the active starting set lies above or below this value. That is, with high probability either the activation spreads to almost the whole structure, or the process ceases with almost no additionally activated vertices.

The main goal of this thesis is to extend the current results for r-neighbor bootstrap percolation to a model where each vertex i draws its individual activation threshold r_i from a given distribution whose support is assumed to be independent from the number of vertices. This enhancement is mainly motivated by real-world applications, as for example neural networks and viral marketing.

We prove the existence of a sharp threshold for an underlying Erdős-Rényi random graph model and provide estimates for the final set size and the time until percolation. Moreover, we investigate the bootstrap percolation process on a directed random graph model with arbitrary degree distribution. Under certain assumptions about the distribution of the out-degrees we can show a similar sharp threshold result. Namely, starting below a certain percolation threshold results with high probability in no percolation, whereas starting above typically leads to a final active set of linear size in the number of vertices.

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Contents

Contents **iii**

1 Introduction **1**

2 Basic concepts and notation **5**
2.1 Basic notation 5
2.2 Tail estimates 7
2.2.1 Markov's inequality 7
2.2.2 Chebyshev's Inequality 7
2.2.3 Chernoff bounds 7
2.3 Poisson approximation 8
2.3.1 Binomial distribution 8
2.3.2 Sum of Bernoulli distributions 9
2.3.3 Multinomial distribution 9
2.4 Stochastic processes 9
2.4.1 Martingale 9
2.4.2 Galton-Watson branching process 10
2.4.3 Inhomogeneous random walk 10
2.5 Miscellaneous 11

3 Model and overview **13**
3.1 Two formulations 13
3.1.1 Step model 14
3.1.2 Round model 15
3.2 Proof overview 16

4 Erdős-Rényi random graph model **19**
4.1 Model 19
4.2 Approximation of $S(t)$ by its mean 21
4.3 Approximation of the mean 23

4.3.1 Approximation at the percolation threshold 24
4.3.2 Approximation at the end 25
4.4 Approximation of the probability of activation in a certain step 27
4.4.1 Approximation at the beginning 27
4.4.2 Approximation at the end 28

5 Sharp threshold **29**
5.1 Subcritical case 30
5.2 Supercritical case 31

6 Probability threshold **37**

7 Number of rounds **39**
7.1 Starting phase 40
7.2 Explosion phase 40
7.3 End phase 44

8 Boundary cases **51**
8.1 Sparse case 51
8.2 Dense case 57

9 Other random graph models **61**
9.1 Model 61
9.2 Assumptions 62
9.3 Approximations 63
9.3.1 Restriction to a subset of all possible degrees 64
9.3.2 Approximation of the mean 65
9.4 Sharp threshold result 67
9.4.1 Subcritical case 67
9.4.2 Supercritical case 67

10 Conclusion **69**

A Declaration of originality **71**

Bibliography **73**

Chapter 1

Introduction

Bootstrap percolation is a random process that models the spread of activation on a linked structure. Starting with an initial set $\mathcal{A}(0)$ of active vertices, the process evolves in rounds. In r-neighbor bootstrap percolation, in each round all inactive vertices with at least r active neighbors become and remain active. The process continues until there are no inactive vertices with enough active neighbors left. Based on this local update rule, one analyzes the typical global behavior of such a process, where the main interest lies in the size A^* of the final active set.

Broadbent and Hammersley introduced this problem in 1956 for modeling the spread of fluid through a medium [18]. Due to its numerous applications in modeling of physical processes (for an overview see [3]), bootstrap percolation has been intensively studied on various (deterministic and random) graph models, for example on infinite trees [11], Galton-Watson trees [15], random geometric graphs [17], and random regular graphs [12]. Vallier adapted the bootstrap percolation process to an Erdős-Rényi random graph model [46], for which Janson et al. provided a more thorough analysis [30]. We refer to this paper and the references there for a more detailed discussion of related works.

It is observed for all these underlying linked structures that the bootstrap percolation process exhibits a sharp threshold behavior, a phenomenon recurringly encountered in graph theory. Namely, for a graph with n vertices there exists a percolation threshold $a_c = a_c(n)$ such that typically either the process ceases with almost no additionally activated vertices or the activation spreads to almost the whole graph, depending on whether the size a of the starting set lies below or above a_c. More precisely, if $a \sim \alpha a_c$ for some $\alpha < 1$, then we have $A^* < 2a$ with probability $1 - o(1)$, and if $\frac{a}{a_c} \geq 1 + \delta$ for some $\delta > 0$, then we have $A^* = n - o(n)$ with probability $1 - o(1)$. In the former subcritical case, we say that the process does not percolate, and in the latter supercritical case, we say that the process percolates.

Most of the existing literature in this area examines the bootstrap percolation process under assumption of the same fixed activation threshold r for every vertex. Some attempts in the direction of variable thresholds have already been made. Majority percolation, that is, setting the threshold to half of the degree, has been studied on grids [37] and hypercubes [13]. The regularity of these graphs, however, still leads to a nearly constant activation threshold. Moreover, Amini studied bootstrap percolation on random graphs with fixed degree sequences and a fixed threshold function dependent on the degree [6].

The main goal of this thesis is to extend the current results for r-neighbor bootstrap percolation to a model where each vertex i draws its individual activation threshold r_i from a given distribution. We assume the support $\mathcal{R}$ of this distribution to be independent from n and suppose that the minimum possible activation threshold $r_{\min}$ is at least 2. The introduction of individual activation thresholds seems to be a natural extension and has, besides its intrinsic relevance, the purpose to enhance current bootstrap percolation models in order to better adjust to real-world applications. In the following, we provide two such examples.

One of the main motivations to analyze bootstrap percolation stems from neuroscience. The suggestion of Abeles to model neural circuits as feed-forward networks [1, 2] gives rise to use bootstrap percolation as a simplified model for propagation of activity in a neural network [43, 45]. We refer to [44] for a thorough discussion of the biological background. In simplified terms, if the sum of input signals into a neuron surpasses a certain threshold, a signal is transmitted to all neighboring neurons. Numerous factors, as for example the diameter of a neuron, can influence this threshold [16, 42]. Therefore, it is reasonable to model neurons with individual activation thresholds.

Likewise, bootstrap percolation can be used to model the spread of information in a (social or real-life) network [25], with applications mainly in viral marketing [7, 8]. Starting from an initial set, messages are broadcast to all neighbors. In this case, r-neighbor bootstrap percolation implements a policy which is analogous to "What I tell you three times is true", as suggested by Carroll [21]. This rule implies that each vertex in the network has to be equally gullible or diffident. However, it might be desirable to enable more credulous and more doubting vertices by allowing individual activation thresholds, as proposed by Lelarge [32].

In Chapter 3, we provide a general model for bootstrap percolation and sketch the proof idea for the sharp threshold result. Then, we refine the bootstrap percolation process model to adapt to the Erdős-Rényi random graph $G_{n,p}$ with individual activation thresholds r_i in Chapter 4. For $n^{-1} \ll p \ll n^{-\frac{1}{r_{\min}}}$ we show a sharp threshold result in Chapter 5, also differenti-

ating between almost and complete percolation. The percolation threshold can also be studied in a different way, by fixing a starting size $a = a(n)$ and determining the percolation threshold $p_c = p_c(n)$ for the density of the graph. Precise estimates for p_c are provided in Chapter 6. For the supercritical case, that is, when starting above the percolation threshold, we analyze the number of rounds τ the process takes until completion in Chapter 7. We then loosen the assumption on the relation between n and p, thus look at more sparse and more dense graphs in Chapter 8. When $p \ll n^{-1}$, we show that typically almost no additional activation occurs, as the graph is too sparse. In the moderately sparse case $p = \Theta\left(n^{-1}\right)$, for a sufficiently large starting set $a = \Theta(n)$ we observe a jump between a small and a large linear fraction of n as final active set size, depending on an additionally introduced percolation threshold $\theta_c n$. In the moderately dense case $p = \Theta\left(n^{-\frac{1}{r_{\min}}}\right)$, we either have no percolation or complete percolation, both arising with constant probability. For an even denser graph, that is, for $p \gg n^{-\frac{1}{r_{\min}}}$, we have complete percolation as long as the starting set is not smaller than the minimal activation threshold.

The terminology and the proof structure of these results are strongly based on the work of Janson et al. [30], wherefore we will adopt the proofs verbatim and only introduce minor changes to adapt to the new model with individual activation thresholds. In particular, in order to analyze the bootstrap percolation process, approximations to the core quantities, namely to the probability $\overline{\pi}(t)$ of a vertex being activated up to the time step t and to the probability $\Pr[Y_i = k]$ of a vertex becoming active in one particular step k, have to be supplied. The idea is to provide two different approximations for these probabilities, one at the beginning and one at the end of the process. In the former case, for $\overline{\pi}(t)$ an estimate similar to the one for $\pi(t)$ in [30, (8.1)], replacing r by $r_{\min}$, can be found in (4.21). In the latter case, the approximation of $1 - \overline{\pi}(t)$ in (4.28) is basically the same as the one of $1 - \pi(t)$, using $r_{\max} := \max \mathcal{R}$ instead of r. Besides the slightly changed error bounds, only an additional constant factor $p(r_{\min})$ (or $p(r_{\max})$) emerges. Similarly, $\Pr[Y_i = k]$ can be approximated by (4.38) in the former and by (4.39) in the latter case, arising from replacing the constant activation threshold r in [30, (2.7)] by $r_{\min}$ and $r_{\max}$, respectively. As these estimates, provided in Section 4.3, are asymptotically very similar to the respective expressions in [30], the proofs in Chapters 5–8 can be adopted, by letting the minimum value $r_{\min}$ or the maximum value $r_{\max}$ play the role of r, depending on whether we perform the analysis of the process at the beginning or at the end.

Even though the Erdős-Rényi graph model is very popular, it does not fit well many realistic network structures [4, 5], as for example it does neither exhibit clustering nor scaling. In contrast to the limiting Poisson degree distribution arising from the Erdős-Rényi graph, naturally evolved graphs

often reveal some heavy-tailed degree distribution [27, 34]. It is found that the internet and many other networks can be modeled as a graph whose degree distribution follows a power law $\Pr[\deg = d] \sim d^{-\beta}$ [19, 24], yielding a scale-free network. Amini et al. examined the r-neighbor bootstrap percolation process on an undirected power-law random graph with exponent $\beta \in (2, 3)$, assuming an upper bound on the possible degrees. They proved the existence of a weak threshold between almost no activation and a linear final active set size [9, 10].

In Chapter 9, we investigate the bootstrap percolation process with individual activation thresholds on a directed random graph model with arbitrary degree distribution. That is, we assume that the out-degree of each vertex follows an arbitrary distribution and that the target vertices are chosen uniformly at random. The introduction of directed edges captures the fundamental asymmetry in the bootstrap percolation process, as for instance there is an inherent difference between *influencing* and *being influenced*.

Under certain assumptions about the degree distribution we can show a sharp threshold result similar to the above-mentioned by Amini et al., namely that starting below the percolation threshold leads with high probability to no percolation whereas starting above typically results in a final active set of linear size in n. The approach to prove this result again is to find an approximation for the probability $\widehat{\pi}(t)$ of a vertex being active at a certain time step t. For $t \approx t_c$, the estimate provided in (9.12) is very similar to [30, (8.1)], wherefore the proof works along the same lines.

Chapter 2

Basic concepts and notation

In this chapter, we introduce some of the fundamental concepts and notation. Basic knowledge in probability theory and calculus is assumed. For the sake of convenience, we often present simplified results and only refer to more general versions.

2.1 Basic notation

This section aims to introduce some of the basic notation used for probability theory, calculus, and asymptotic analysis of sequences of numbers and random variables.

- We define $\log_+ x := \max\{0, \log x\}$.
- For a natural number $n \in \mathbb{N} := \{1, 2, \dots\}$ we let $[n]$ denote the set $\{i \in \mathbb{N} \colon i \leq n\}$.
- For a (random) predicate P we use $[P]$ to denote a (random) indicator variable for the event P, thus $[P]$ is 1 if P is true and 0, otherwise.
- We use the abbreviation i.i.d. for independent and identically distributed.
- For two random variables X and Y, the notation $\mathrm{d_{TV}}(X, Y)$ stands for the total variation distance between X and Y.
- Landau notation

  We adopt the commonly used Landau notation [38]. The terminology for asymptotic concepts is introduced in detail in [28] and [29]. All unspecified limits are as $n \to \infty$.

  Let $a(n)$ and $b(n)$ be sequences of numbers for $n \in \mathbb{N}$.

  We write $b(n) = \mathcal{O}(a(n))$ if $\limsup_{n\to\infty} \left|\frac{b(n)}{a(n)}\right| < \infty$ and use $b(n) = o(a(n))$ to denote that $\lim_{n\to\infty} \frac{b(n)}{a(n)} = 0$. If $\limsup_{n\to\infty} \left|\frac{b(n)}{a(n)}\right| > 0$, we

say that $b(n) = \Omega(a(n))$, whereas $\limsup_{n\to\infty} \left|\frac{b(n)}{a(n)}\right| = \infty$ is denoted by $b(n) = \omega(a(n))$. We have $b(n) = \Theta(a(n))$ if $b_n = \mathcal{O}(a(n))$ and $a_n = \mathcal{O}(b(n))$.

Moreover, $b(n) \ll a(n)$ if $b(n) = o(a(n))$, and therefore $b(n) \gg a(n)$ if $a(n) = o(b(n))$.

Further, we say $b(n) \sim a(n)$ if $\lim_{n\to\infty} \frac{b(n)}{a(n)} = 1$, and thus $b(n) = a(n)(1 + o(1))$.

- Common probability distributions

  We use $\mathrm{Be}(p)$ to denote the Bernoulli distribution with success probability p and write $\mathrm{Bin}(n, p)$ for the corresponding binomial distribution with n trials. Moreover, $\mathrm{Po}(\lambda)$ stands for a Poisson distribution with mean λ and $\mathrm{Mul}\left(n, (p_j)_{j=1}^k\right)$ for a multinomial distribution with n trials and success probabilities p_j for $j \in [k]$.

- Convergence of random variables

  Let $X(n)$ for $n \in \mathbb{N}$ and X be random variables with the cumulative distribution functions F_n and F, respectively.

  The sequence $X(n)$ convergences to X in distribution, denoted by $X(n) \xrightarrow{d} X$, if $\lim_{n\to\infty} F_n(x) = F(x)$ for every $x \in \mathbb{R}$ at which F is continuous.

  If $\lim_{n\to\infty} \Pr[|X(n) - X| \geq \varepsilon] = 0$ for all $\varepsilon > 0$, then $X(n) \xrightarrow{p} X$, that is, $X(n)$ converges to X in probability.

  We have almost sure convergence of $X(n)$ to X, denoted by $X(n) \xrightarrow{\text{a.s.}} X$, if $\Pr[\lim_{n\to\infty} X(n) = X] = 1$.

  If an event occurs with probability 1, we say that it happens almost surely (abbreviated as a.s.), if its probability converges to 1 it is said to occur asymptotically almost surely (abbreviated as a.a.s.). Note that $\xrightarrow{a.s.}$ means the same as *a.a.s.*

- Probability asymptotics

  Let $a(n)$ be a sequence of numbers and $X(n)$ a sequence of random variables for $n \in \mathbb{N}$. A probabilistic version of $\mathcal{O}$ is given by $\mathcal{O}_p$, where $X(n) = \mathcal{O}_p(a(n))$ means that for every $\varepsilon > 0$ there exist constants C_ε and n_ε such that for every $n \geq n_\varepsilon$ we have $\Pr[|X(n)| \leq C_\varepsilon |a(n)|] > 1 - \varepsilon$. Analogously, the probabilistic version o_p of o can be defined as follows. We say that $X(n) = o_p(a(n))$ if $\frac{X(n)}{a(n)} \xrightarrow{p} 0$.

We use the abbreviation w.h.p. to denote that the probability of an event tends to 1 as $n \to \infty$. Note that $o(1)$ *w.h.p.* is equivalent to $o_p(1)$ and $\xrightarrow{p} 0$.

The following lemma reveals some more equivalences.

Lemma 2.1 *[cf. [28, Lemma 3]] The following statements are equivalent.*

(i) $X(n) = \mathcal{O}_p(a(n))$.

(ii) For every function $\omega(n) \to \infty$, $|X(n)| \leq \omega(n)|a(n)|$ *w.h.p.*

(iii) For every function $\omega(n) \to \infty$, $\frac{|X(n)|}{\omega(n)|a(n)|} \xrightarrow{p} 0$.

2.2 Tail estimates

This section introduces some upper bounds for the probability of a random variable taking values far away from its mean. For a detailed discussion, we refer to [41].

2.2.1 Markov's inequality

The following theorem gives an upper bound on the probability that a non-negative random variable exceeds some value, depending on its expected value.

Theorem 2.2 (Markov's inequality) *Let X be a non-negative random variable. For all $x > 0$ we have*

$$Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}.$$

2.2.2 Chebyshev's Inequality

In contrast to the preceding result, the next one provides the probability for a two-sided interval around the mean, depending on the variance.

Theorem 2.3 (Chebyshev's inequality) *Let X be a random variable. We have*

$$Pr[|X - \mathbb{E}[X]| \geq x] \leq \frac{Var[X]}{x^2}$$

for all $x > 0$.

2.2.3 Chernoff bounds

For a sum of independent Bernoulli random variables one can supply even better, exponentially decreasing bounds.

Theorem 2.4 (Chernoff bounds) *Let $X_1, \dots, X_n$ be independent random variables with $X_i \in Be(p_i)$ for $i \in [n]$. Then for $X := \sum_{i=1}^n X_i$ and $\mu := \mathbb{E}[X] = \sum_{i=1}^n p_i$ we have for all $\delta \in (0, 1]$*

$$Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\mu\delta^2}{3}}$$

and

$$Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\mu\delta^2}{2}}.$$

2.3 Poisson approximation

As some expressions for probability distributions are too involved to compute directly, one often resorts to simpler estimates. The approximation by a Poisson distribution can be applied in various settings. For a thorough discussion of Poisson approximations we refer to [14].

2.3.1 Binomial distribution

In the following, we present some basic results for the approximation of $\mathrm{Bin}(n, p)$ by $\mathrm{Po}(np)$.

If $np \to \lambda \in (0, \infty)$, then we have $\mathrm{Bin}(n, p) \xrightarrow{d} \mathrm{Po}(np)$.

The relative error bound

$$\Pr[\mathrm{Bin}(n, p) = k] = \frac{(np)^k}{k!} e^{-np} \left(1 + \mathcal{O}\left(np^2 + \frac{k^2}{n-1}\right)\right),$$

provided in [14, (1.1)], can be simplified to

$$\Pr[\mathrm{Bin}(n, p) = k] = \frac{(np)^k}{k!} e^{-np} (1 + o(1)) \tag{2.1}$$

for $np \ll 1$, $n \gg 1$, and $k = \mathcal{O}(1)$.

For the total variation distance we have, see [39],

$$\mathrm{d_{TV}}(\mathrm{Bin}(n, p), \mathrm{Po}(np)) \leq np^2,$$

and thus

$$\mathrm{d_{TV}}(\mathrm{Bin}(n, p), \mathrm{Po}(np)) < p \tag{2.2}$$

for $np < 1$.

2.3.2 Sum of Bernoulli distributions

For a sum of independent but not identically distributed Bernoulli random variables one can find a similar result.

Theorem 2.5 (Le Cam) *[cf. [20]] Let $X_1, \ldots, X_n$ be independent random variables with Bernoulli distribution $X_i \in Be(p_i)$ for $i \in [n]$. Then*

$$\sum_{j=0}^{\infty} \left| Pr\left[\sum_{i=1}^{n} X_i = j\right] - \frac{\lambda_n^j e^{-\lambda_n}}{j!} \right| < 2 \sum_{i=1}^{n} p_i^2,$$

where $\lambda_n := \sum_{i=1}^{n} p_i$.

2.3.3 Multinomial distribution

Let $\left(X_j\right)_{j=1}^{k}(n) \in \mathbb{R}^k$ for $n \geq 1$ be a sequence of random vectors with a multinomial distribution $\mathrm{Mul}\left(n, \left(p_j\right)_{j=1}^{k}\right)$ with $p_j \to y_j$ for $j \in [k]$. Then, see [33], it follows that $\left(X_j\right)_{j=1}^{k}(n)$ have a joint Poisson limit,

$$\left(X_j\right)_{j=1}^{k}(n) \xrightarrow{d} \left(Y_j\right)_{j=1}^{k},$$

where $Y_j \in \mathrm{Po}\left(y_j\right)$.

2.4 Stochastic processes

In this section, we present several frequently used stochastic processes.

2.4.1 Martingale

We formally introduce the notion of a martingale and then provide one important result applicable to such processes.

Definition 2.6 *[cf. [31, Proposition 7.16]] A discrete-time martingale is a discrete-time stochastic process $\{M(t)\}_{t\in\mathbb{N}}$ that satisfies for any t*

$$\mathbb{E}\left[|M(t)|\right] < \infty$$

and

$$\mathbb{E}\left[M(t+1) \mid M(1), \ldots, M(t)\right] = M(t).$$

A discrete-time reverse martingale is a discrete-time stochastic process $\{M(t)\}_{t\in\mathbb{N}}$ that satisfies for any t

$$\mathbb{E}\left[|M(t)|\right] < \infty$$

and

$$\mathbb{E}\left[M(t) \mid M(t+1), \ldots\right] = M(t+1).$$

Note that, by linearity of conditional expectation, a sum of independent martingales is a martingale, too.

The following theorem provides an upper bound on the values a martingale can take over a given time interval T.

Theorem 2.7 (Doob's norm inequality) *[cf. [31, Proposition 7.16]] Let M be a martingale on an index set $T \subseteq \mathbb{N}$. Then we have for any $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $t, t' \in T$*

$$\|\sup_{t \leq t'} |M(t)|\|_p \leq q \|M(t')\|_p.$$

2.4.2 Galton-Watson branching process

The notion of a Galton-Watson process arises from the behavior of extinction of family names [47].

Definition 2.8 *A Galton-Watson process with offspring distribution ξ is a discrete-time stochastic process $\{X(t)\}_{t \geq 0}$ which evolves according to the recurrence formula*

$$X(0) = 1$$

and

$$X(t+1) = \sum_{j=1}^{X(t)} \xi_j,$$

where $\{\xi_j\}_{j \geq 0}$ is a sequence of i.i.d. natural number-valued random variables.

A branching process may either become extinct, that is, $X(t) = 0$ for some t, or survive forever. The extinction probability is given by

$$\lim_{t \to \infty} Pr\left[X(t) = 0\right].$$

We provide a basic result on the extinction probability of a Galton-Watson process.

Theorem 2.9 *Suppose that not $X(t) = 1$ for all t. Then, if $\mathbb{E}\left[\xi_j\right] \leq 1$, the probability of final extinction is 1, and if $\mathbb{E}\left[\xi_j\right] > 1$, the extinction probability is strictly smaller than 1.*

For a more detailed discussion we refer to [26, 22, 35].

2.4.3 Inhomogeneous random walk

A random walk process models a path consisting of independent random steps.

Definition 2.10 *An inhomogeneous random walk is a discrete-time stochastic process* $\{X(t)\}_{t\in\mathbb{N}}$ *with* $X(t) = \sum_{j=1}^{t} \xi_j$*, where* ξ_j *for* $j \in \mathbb{N}$ *is a sequence of independent random variables.*

The hitting time

$$\widetilde{T}_P := \min\{t \in \mathbb{N} \colon P(X(t))\}$$

for a predicate P *is defined as the minimum time for which this condition* P *is satisfied.*

A thorough analysis of random walks can be found in [40].

2.5 Miscellaneous

In this section, we list remaining assorted results.

The next theorem shows the uniform convergence of the empirical to the cumulative distribution function.

Theorem 2.11 (Glivenko-Cantelli) *[cf. [31, Proposition 4.24]] Let* $X(n)$ *for* $n \in \mathbb{N}$ *be i.i.d. real random variables, where* $X(n) \in [0,1]$*, with the common cumulative distribution function* F *and empirical distribution functions* $\widehat{F}_n$ *given by*

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} [X(i) \leq x].$$

Then,

$$\lim_{n\to\infty} \sup_x \left|\widehat{F}_n(x) - F(x)\right| = 0 \text{ a.s.}$$

The following basic lemma is repeatedly used in the course of the thesis.

Lemma 2.12 *[cf. [30, Lemma 8.4]] For* $r \geq 2$*,*

$$\min_{x\geq 0}\left(\frac{x^r}{r} - r\right) = \frac{1}{r} - 1,$$

attained at $x = 1$ *only.*

Chapter 3

Model and overview

In this chapter, we introduce the formal model used to analyze the bootstrap percolation process, closely following [30, Section 1–2]. We consider the percolation process on a general random graph $G = (V_n, E)$ with a random starting set $\mathcal{A}(0)$ of fixed size a, where $0 < a < n$. For simplicity, we refer to a vertex by its index $i \in [n]$, using $[n]$ and V_n interchangeably, and assume that the vertices in the starting set have indices from $n - a + 1$ to n.

Each vertex i draws its activation threshold r_i independently from a given discrete distribution with probability density function $p(r)$ and finite support $\mathcal{R}$. We assume that this distribution is identical for each i and independent from n, thus, in particular, the minimum value $r_{\min} := \min \mathcal{R}$ and the maximum value $r_{\max} := \max \mathcal{R}$ are constant. Additionally, we suppose that $r_{\min} \geq 2$. Each vertex i becomes and remains active if at least r_i of its neighbors are active. We are mainly interested in the size A^* of the final active set, that is, in the number of active vertices at the end of the process, when no more inactive vertices have enough active neighbors. The process percolates completely if every vertex is active at the end, thus if $A^* = n$. We further say that the process almost percolates if $A^* = n - o(n)$, thus if almost all vertices are active at the end. However, if $A^* \neq n - o(n)$, the process does not percolate.

3.1 Two formulations

There are mainly two ways to look at the percolation process. Either, one proceeds in *rounds*, beginning with the starting set as active vertices in round 0, and adding, in each consecutive round, the vertices that have enough neighbors in the preceding rounds. Or one proceeds in *steps*, where in each step an active vertex gives a mark to all its neighbors and a vertex i becomes active if it has received at least r_i marks. The former formulation is more intuitive, similar to how one would observe and explain such a process, the

latter seems to be more artificial, but is easier to work with. For that reason, we will mainly focus on the step model, only resorting to the round model in Chapter 7 to study the time the process takes until completion. The two formulations lead to the same results, merely the time scale is chosen differently, as one round subsumes multiple steps.

3.1.1 Step model

We assume some initial ordering of the vertices in the starting set $\mathcal{A}(0)$ and think of newly activated vertices as added in, for example, first-in-first-out manner. Then in every step the first vertex according to this ordering gives a mark to each of its neighbors, up to a step T when there is no active vertex left. For $t \in \{0, \dots, T\}$, we let $\mathcal{A}(t)$ denote the set of active vertices at time t and $A(t) := |\mathcal{A}(t)|$ its size. Note that thus $A(0) = a$. Further, let $\mathcal{Z}(t)$ be the set of used vertices, that is, the vertices that have already given marks to its neighbors. We start with $\mathcal{Z}(0) := \emptyset$ and successively add v_t, the vertex used in step t. Hence, $\mathcal{Z}(t) = \{v_s \colon 1 \leq s \leq t\}$ for $t \in [T]$. Since in each step exactly one active vertex is used, we have $\mathcal{Z}(t) \subseteq \mathcal{A}(t)$ and $Z(t) := |\mathcal{Z}(t)| = t$. Consequently, $A(t) \geq Z(t) = t$. At the end of the process, all active vertices are used, that is, $\mathcal{Z}(t) = \mathcal{A}(t)$, and therefore $A(t) = t$. Thus, the stopping time T of the percolation process can be defined as

$$T := \min \{t \geq 0 \colon A(t) - t = 0\} = \min \{t \geq 0 \colon A(t) \leq t\}. \tag{3.1}$$

Obviously, we have $T \leq n$. The size of the final active set is given by

$$A^* := A(T) = Z(T) = T.$$

Let

$$I_i(s) := [(v_s, i) \in E]$$

be the indicator variable for the presence of an edge from the currently used vertex v_s to a certain (inactive) vertex i for $s \in [T]$ and $i \in [n]$. We further define

$$M_i(t) := \sum_{s=1}^{t} I_i(s)$$

as the number of marks a vertex i has received until and including time step t. Recall that a vertex i becomes active if it has received at least r_i marks. Thus, either $i \in A(0)$, or else $i \in \mathcal{A}(t)$ if and only if $M_i(t) \geq r_i$.

We define $\Delta\mathcal{A}(t) := \{i \in V_n \setminus \mathcal{A}(t) \colon M_i(t) \geq r_i\}$ as the set of inactive vertices with enough marks to become active, and thus $\mathcal{A}(t) = \mathcal{A}(t-1) \cup \Delta\mathcal{A}(t)$. Note that $i \in V_n \setminus \Delta\mathcal{A}(t)$ for $t < r_i$, since each vertex can receive at most one mark per time step. We let

$$Y_i := \min \{t \colon M_i(t) \geq r_i\}$$

be the time (measured in steps) of activation of a vertex i for $i \in V_n \setminus \mathcal{A}(0)$. $Y_i = t$ for $t \leq T$ means that v_t is the vertex that has given i the r_i-th mark, after which i became active. If $Y_i > T$, then i will remain inactive.

The set of activated vertices up to time t is $\mathcal{S}(t) := \{i \in V_n \setminus \mathcal{A}(0) \colon Y_i \leq t\}$, and thus

$$S(t) := |\mathcal{S}(t)| = \sum_{i=1}^{n-a} [Y_i \leq t] \tag{3.2}$$

denotes its size. Hence,

$$A(t) = A(0) + S(t) = S(t) + a. \tag{3.3}$$

It will be useful to consider the probability that a vertex i has been activated up to time t, denoted as

$$\pi_i(t) := \Pr[Y_i \leq t] = \Pr[M_i(t) \geq r_i]. \tag{3.4}$$

For simplicity, we define $\pi_i(t) := \pi_i(\lfloor t \rfloor)$ and $S(t) := S(\lfloor t \rfloor)$ for real $t \geq 0$.

3.1.2 Round model

We use the notion of generations $\mathcal{G}_k$ to denote the vertices that become active in a certain round k. Generation 0 contains all the initially active vertices, thus $\mathcal{G}_0 = \mathcal{A}(0)$. In round k, all inactive vertices i with at least r_i active neighbors in generations $0, \ldots, k-1$ become active and thus are part of generation k. The process stops after a round without further activations, thus if an empty generation occurs. Let τ denote the number of non-empty generations, hence

$$\tau := \max\{k \geq 0 \colon \mathcal{G}_k \neq \emptyset\} = \min\{k \geq 1 \colon \mathcal{G}_k = \emptyset\} - 1. \tag{3.5}$$

In the following, we relate the two models. We inductively define

$$T_0 := 0, \quad T_{j+1} := A(T_j) \quad \text{for } j \geq 0.$$

Thus, we have $A(T_0) = A(0) = |\mathcal{A}(0)| = |\mathcal{G}_0|$, $\mathcal{Z}(T_1) = \mathcal{Z}(A(0)) = \mathcal{A}(0) = \mathcal{G}_0$, and $\mathcal{Z}(T_2) = \mathcal{Z}(A(T_1)) = \mathcal{A}(T_1) = \mathcal{G}_0 \cup \mathcal{G}_1$. All vertices in $\mathcal{G}_k$ have been found and activated at time T_k, and they have been used at time $T_{k+1} = A(T_k)$.

Hence,

$$\bigcup_{j=0}^{k} \mathcal{G}_j = \mathcal{A}(T_k) = \mathcal{Z}(T_{k+1}), \quad \text{for } k \geq 0.$$

In particular, the size of generation k is

$$|\mathcal{G}_k| = |\mathcal{Z}(T_{k+1}) \setminus \mathcal{Z}(T_k)| = T_{k+1} - T_k.$$

Therefore, (3.5) can be rewritten as

$$\tau = \max\{k \geq 0 \colon T_{k+1} > T_k\} = \min\{k \geq 1 \colon T_{k+1} = T_k\} - 1.$$

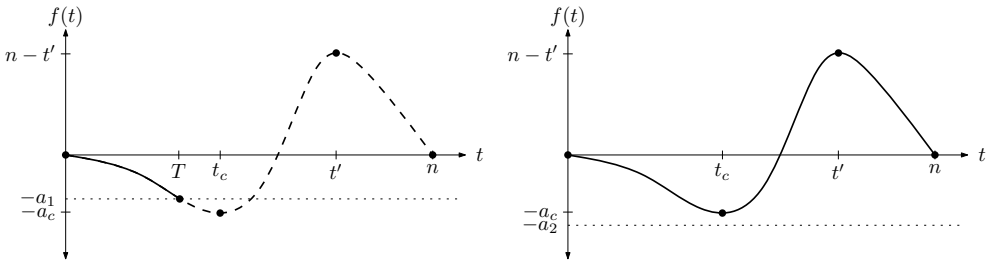

Figure 3.1: At the end of the starting phase, the process faces a bottleneck at $t \approx t_c$. In the subcritical case, shown as $a = a_1$, the process dies out at $t = T$. In the supercritical case, depicted as $a = a_2$, the process survives this critical stage and grows exponentially up to $t \approx t'$. The last few vertices are activated in the end phase.

3.2 Proof overview

In this section, also see [30, Section 6], we briefly sketch the main idea for the analysis of such a process, based on the step model.

Note that for $u \in [n]$ the percolation process needs at least u steps, that is, $T \geq u$ and hence $A^* \geq u$, if and only if the stopping criteria has not been satisfied in all previous steps, thus if and only if

$$\min\{t < u \colon A(t) - t\} = a + \min\{t < u \colon S(t) - t\} > 0.$$

Consequently, the process percolates completely if and only if for every $t < n$ the process has not stopped yet, thus if and only if $\min\{t < n \colon A(t) - t\} > 0$, or, equivalently, $\min\{t < n \colon S(t) - t\} > -a$. Hence, the discrete-time stochastic process $\{\min\{t < u \colon S(t) - t\}\}_{u \in \mathbb{N}}$ completely determines the final size A^*. In particular, $A^* = n$ if and only if $\min\{t < n \colon S(t) - t\} > -a$.

As this stochastic process is too difficult to analyze directly, we introduce some asymptotic approximations. To obtain a deterministic behavior, we replace the random variable $S(t)$ by its mean $\mathbb{E}[S(t)]$ and look for minima in the resulting function $f(t) := \mathbb{E}[S(t)] - t$, whose behavior is qualitatively sketched in Figure 3.1. After starting at $f(0) = 0$, f decreases to a minimum at $t \approx t_c$, where t_c is defined by (4.22). Then f increases until $\approx n$ vertices are active, that is, until a time step $\approx t'$ where $\mathbb{E}[S(t')] \approx n$, and thus $f(t') \approx n - t'$. Then no more vertices can become active, and thus $f(t) \approx n - t$ holds until $t = n$, so f decreases again in this range to a final value $f(n) = \mathbb{E}[S(n)] - n \approx 0$. Hence, there are two possible minima of f, either at $t \approx t_c$ or at $t \approx n$. Whether the process (almost) percolates or not thus depends on whether $S(t_c) - t_c > -a$. As $f(t_c) = \mathbb{E}[S(t_c)] - t_c \approx a_c$, where a_c is defined by (4.24), the percolation threshold for a is $\approx a_c$.

At $t \approx t_c$, there are two possibilities. Either, the activation dies out with $A^* \approx t_c$, or the process overcomes this bottleneck and grows fast until $A^* \approx$

n. Given, that the percolation process does not cease (due to a too small starting set), the function value $\approx f(n)$ distinguishes complete and almost percolation, thus determines whether there remain some inactive vertices at the end.

The bootstrap percolation process can be divided into three different phases, a starting phase up to and including the bottleneck at $t \approx t_c$, an explosion phase of double exponential growth after the bottleneck, and finally an end phase, where the last few vertices may be activated.

We call a process starting with an initial active set size below the threshold, that is, with $a \sim \alpha a_c$ for some $\alpha < 1$, subcritical, and a process starting above, hence with $a \geq (1+\delta)a_c$ for some $\delta > 0$, supercritical.

Chapter 4

Erdős-Rényi random graph model

This chapter provides approximations for the analysis of a bootstrap percolation process in the case of an Erdős-Rényi random graph model with individual activation thresholds. These results are used the in subsequent chapters to prove the sharp threshold phenomenon and additional statements, analogously to the proofs in [30] for a fixed activation threshold r.

4.1 Model

We refine the step model for an Erdős-Rényi graph $G_{n,p} = (V_n, E)$, an undirected graph with n vertices where every edge independently is present in the graph with the same probability p. We assume that $p = p(n)$ depends on n such that

$$n^{-1} \ll p \ll n^{-\frac{1}{r_{\min}}}. \tag{4.1}$$

Based on this specific graph model, we can give more concrete expressions for some of the quantities introduced in Section 3.1.1.

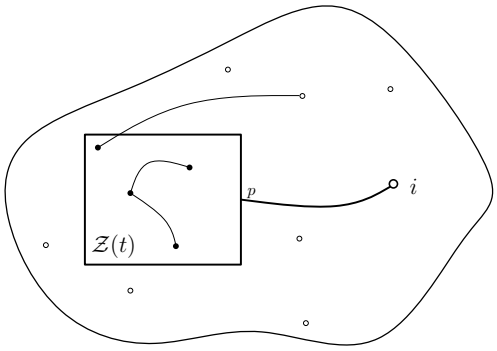

Figure 4.1: The unused vertices are drawn as circles, and the used ones are depicted as disks, summarized in the box labeled with $\mathcal{Z}(t)$. From the view of vertex i, each of these vertices in the box is connected to it with the same probability p.

We consider the set $\mathcal{Z}(t)$ of all active and used vertices at time t, where by definition, cf. Section 3.1.1, there are exactly t active vertices at time t. This setting is schematically depicted in Figure 4.1. For one distinguished inactive vertex i we count the number of edges that lead from $\mathcal{Z}(t)$ to this vertex. Each of these edges is present with probability p. Moreover, the indicators $I_i(s)$ for $s \in [T]$ are independent and identically $\mathrm{Be}(p)$-distributed. Consequently, $M_i(t) \in \mathrm{Bin}(t,p)$ for $t \leq T$. Using (3.4), we have

$$\pi_i(t) = \Pr\left[\mathrm{Bin}(t,p) \geq r_i\right] = \sum_{j=r_i}^{t} \binom{t}{j} p^j (1-p)^{t-j}. \tag{4.2}$$

For the sake of brevity, we define

$$\pi^x(t) := \Pr\left[\mathrm{Bin}(t,p) \geq x\right]$$

as well as

$$\overline{\pi}(t) := \mathbb{E}\left[\pi^{r_i}(t)\right]. \tag{4.3}$$

Note that $\pi^x(t)$ is a non-negative, monotonically decreasing function in x, that is, for every t

$$\pi^x(t) \geq \pi^y(t) \geq 0, \quad \text{for } x < y. \tag{4.4}$$

Further, see [30, (8.1)], $\pi^x(t)$ has the following asymptotic behavior

$$\pi^x(t) = \binom{t}{x} p^x \left(1 + \mathcal{O}(pt)\right) = \frac{(pt)^x}{x!}\left(1 + \mathcal{O}(pt + t^{-1})\right) \tag{4.5}$$

for fixed x and for $pt \leq 1$.

Since the r_i are independent and identically distributed, we have, recalling (3.4) and (4.3),

$$\begin{aligned} \pi_i(t) &= \sum_{r \in \mathcal{R}} p(r) \Pr\left[\mathrm{Bin}(t,p) \geq r\right] \\ &= \mathbb{E}\left[\Pr\left[\mathrm{Bin}(t,p) \geq r_i\right]\right] = \overline{\pi}(t). \end{aligned} \tag{4.6}$$

Therefore,

$$S(t) \in \mathrm{Bin}\left(n-a, \overline{\pi}(t)\right), \tag{4.7}$$

what yields

$$\mathbb{E}\left[S(t)\right] = (n-a)\overline{\pi}(t) \tag{4.8}$$

and

$$\mathrm{Var}\left[S(t)\right] = (n-a)\overline{\pi}(t)\left(1 - \overline{\pi}(t)\right). \tag{4.9}$$

To simplify calculations, we will sometimes consider

$$S_n(t) := \sum_{i=1}^{n} \left[Y_i \leq t\right],$$

a binomial random variable with parameters n and $\overline{\pi}(t)$. Therefore,

$$\mathbb{E}\left[S_n(t)\right] = n\overline{\pi}(t)$$

and

$$\mathrm{Var}\left[S_n(t)\right] = n\overline{\pi}(t)\left(1-\overline{\pi}(t)\right).$$

We further have, see [30, (2.7)],

$$\Pr\left[Y_i = k\right] = \sum_{r\in\mathcal{R}} p(r)\binom{k-1}{r-1}p^r(1-p)^{k-r}. \tag{4.10}$$

4.2 Approximation of $S(t)$ by its mean

As mentioned in Section 3.2, it is our goal to approximate $S(t)$ by its mean $\mathbb{E}\left[S(t)\right]$ in order to get rid of the randomness of the stochastic process. Closely following [30, Section 7], we provide different bounds on the error of this approximation.

First, we give a uniform error bound, based on an argument for empirical distribution functions.

Lemma 4.1 *[cf. [30, Lemma 7.1]] We have*

$$\sup_{t\geq 0}|S(t) - \mathbb{E}\left[S(t)\right]| = o_p(n).$$

Proof For $n-a < \sqrt{n}$ the result is trivial, since $0 \leq S(t) \leq n-a$. Note that $(n-a)^{-1}S(t) = (n-a)^{-1}\sum_{i=1}^{n-a}\left[Y_i \leq t\right]$ is the empirical distribution function of $\{Y_i\}_{i=1}^{n-a}$. The Glivenko-Cantelli theorem, see Theorem 2.11, implies that $\sup_{t\geq 0}\left|(n-a)^{-1}S(t) - \overline{\pi}(t)\right| = o_p(1)$, and thus for $n-a \geq \sqrt{n}$ the claim follows. □

For small values of t, however, this uniform bound leads to rather poor results. For that reason, we improve the bound by a martingale argument.

The following lemma shows that the bootstrap percolation process can be modeled as a martingale.

Lemma 4.2 *[cf. [30, Lemma 7.2]] The stochastic process*

$$\xi(t) := \frac{S(t) - \mathbb{E}\left[S(t)\right]}{1-\overline{\pi}(t)}, \quad t \geq 0 \tag{4.11}$$

is a martingale, and the stochastic process

$$\widetilde{\xi}(t) := \frac{S(t) - \mathbb{E}\left[S(t)\right]}{\overline{\pi}(t)}, \quad t \geq r_{min} \tag{4.12}$$

is a reverse martingale.

Proof Note that, by (4.6) and (4.2), we have that $\Pr[Y_i \leq t] = \overline{\pi}(t)$ is independent from i. Thus,

$$\xi(t) = \sum_{i=1}^{n-a} X_i(t),$$

where

$$X_i(t) := \frac{[Y_i \leq t] - \Pr[Y_i \leq t]}{1 - \Pr[Y_i \leq t]} = 1 - \frac{[Y_i > t]}{\Pr[Y_i > t]} \tag{4.13}$$

and

$$\widetilde{\xi}(t) = \sum_{i=1}^{n-a} \widetilde{X}_i(t),$$

with

$$\widetilde{X}_i(t) := \frac{[Y_i \leq t] - \Pr[Y_i \leq t]}{\Pr[Y_i \leq t]}. \tag{4.14}$$

Since $\xi(t)$ is a sum of $n-a$ i.i.d. processes $X_i(t)$, it suffices to treat each of these separately, that is, for the first part to show that $X_i(t)$ is a martingale for each i.

If $Y_i \leq t$, then $X_i(t) = X_i(t+1) = 1$. However, if $Y_i > t$, then $X_i(t) = \frac{-\overline{\pi}(t)}{1-\overline{\pi}(t)}$ either jumps to $X_i(t+1) = 1$ or decreases to $X_i(t+1) = \frac{-\overline{\pi}(t+1)}{1-\overline{\pi}(t+1)}$. It follows from

$$\Pr[Y_i = t+1 \mid Y_i > t] = \frac{\overline{\pi}(t+1) - \overline{\pi}(t)}{1 - \overline{\pi}(t)}$$

as well as

$$\Pr[Y_i > t+1 \mid Y_i > t] = \frac{1 - \overline{\pi}(t+1)}{1 - \overline{\pi}(t)}$$

that

$$\mathbb{E}[X_i(t+1) \mid Y_i > t] = \frac{-\overline{\pi}(t)}{1 - \overline{\pi}(t)} = X_i(t).$$

For the second part we similarly find that $\mathbb{E}\left[\widetilde{X}_i(t) \;\middle|\; \widetilde{X}_i(t+1), \dots\right] = \widetilde{X}_i(t+1)$, what proves the second claim. □

The martingale property allows us to obtain improved error bounds, based on Doob's norm inequality.

Lemma 4.3 *[cf. [30, Lemma 7.3]] For any t_0, we have*

$$\mathbb{E}\left[\left(\sup_{t \leq t_0} |S(t) - \mathbb{E}[S(t)]|\right)^2\right] \leq 16 n \overline{\pi}(t_0) \tag{4.15}$$

and

$$\mathbb{E}\left[\left(\sup_{t \geq t_0} |S(t) - \mathbb{E}[S(t)]|\right)^2\right] \leq 16 n \left(1 - \overline{\pi}(t_0)\right). \tag{4.16}$$

Proof First, assume $\overline{\pi}(t_0) \leq \frac{1}{2}$. Lemma 4.2 implies that $\xi(t)$ is a martingale, and Theorem 2.7 yields

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{t\leq t_0}|S(t)-\mathbb{E}\left[S(t)\right]|\right)^2\right] &\leq \mathbb{E}\left[\left(\sup_{t\leq t_0}|\xi(t)|\right)^2\right] \\ &\leq 4\mathbb{E}\left[|\xi(t_0)|^2\right] = 4\frac{\mathrm{Var}\left[S(t_0)\right]}{\left(1-\overline{\pi}(t_0)\right)^2} \\ &\leq 8n\overline{\pi}(t_0),\end{aligned} \tag{4.17}$$

which proves (4.15) in this case.

Similarly, if $\overline{\pi}(t_0) \geq \frac{1}{2}$, then we obtain, using the reverse martingale (4.12),

$$\mathbb{E}\left[\left(\sup_{t\geq t_0}|S(t)-\mathbb{E}\left[S(t)\right]|\right)^2\right] \leq 4\frac{\mathrm{Var}\left[S(t_0)\right]}{\overline{\pi}(t_0)^2} \leq 8n\left(1-\overline{\pi}(t_0)\right), \tag{4.18}$$

what shows (4.16) for $\overline{\pi}(t_0) \geq \frac{1}{2}$.

Let t_1 be the largest integer such that $\overline{\pi}(t_1) \leq \frac{1}{2}$. We can apply (4.17) with $t_0 = t_1$ and (4.18) with $t_0 = t_1 + 1$, and then, combining these results, we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{t\geq 0}|S(t)-\mathbb{E}\left[S(t)\right]|\right)^2\right] &\leq \mathbb{E}\left[\left(\sup_{t\leq t_1}|S(t)-\mathbb{E}\left[S(t)\right]|\right)^2\right] \\ &\quad + \mathbb{E}\left[\left(\sup_{t\geq t_1+1}|S(t)-\mathbb{E}\left[S(t)\right]|\right)^2\right] \\ &\leq 8n\overline{\pi}(t_1) + 8n\left(1-\overline{\pi}(t_1+1)\right) \leq 8n.\end{aligned}$$

This immediately implies (4.15) for $\overline{\pi}(t_0) > \frac{1}{2}$ and (4.16) for $\overline{\pi}(t_0) < \frac{1}{2}$. □

4.3 Approximation of the mean

Replacing $S(t)$ by its expected value $\mathbb{E}\left[S(t)\right]$ indeed derandomizes the process. Nevertheless, we lack a simple closed-form expression for $f(t) := \mathbb{E}\left[S(t)\right] - t$. For that reason, we first provide an approximation for $\overline{\pi}(t)$, and then, based on this, supply a simplified expression for $\mathbb{E}\left[S(t)\right]$, and hence for $f(t)$.

Note that, as briefly explained in Section 3.2, we are mainly interested in the size of the active set at two different time steps, namely for $t \approx t_c$ around the percolation threshold and for $t \approx n$ at the end of the process. As we will see later, this implies that at the first point $pt \ll 1$ and $t \gg 1$, and at the second point, by (4.1), $pt \approx pn \gg 1$. Since the behavior of the process at these two points is substantially different, we need to analyze each case separately.

4.3.1 Approximation at the percolation threshold

For $tp \ll 1$ and $t \gg 1$ we approximate $\overline{\pi}(t)$, recalling (4.6), by its first summand $p(r_{\min})(r_{\min})\pi^{r_{\min}}(t)$. In the following, we present several error bounds.

Applying (4.4) and (4.5), we can find an upper bound

$$\begin{aligned}
\overline{\pi}(t) &\leq p(r_{\min})\pi^{r_{\min}}(t) + \sum_{r \in \mathcal{R}\setminus\{r_{\min}\}} p(r)\pi^{r_{\min}+1}(t) \\
&= p(r_{\min})\pi^{r_{\min}}(t) + (1 - p(r_{\min}))\,\pi^{r_{\min}+1}(t) \\
&= p(r_{\min})\pi^{r_{\min}}(t)\left(1 + \frac{1-p(r_{\min})}{p(r_{\min})}\frac{\pi^{r_{\min}+1}(t)}{\pi^{r_{\min}}(t)}\right) \\
&= p(r_{\min})\pi^{r_{\min}}(t)\left(1 + \frac{1-p(r_{\min})}{p(r_{\min})}\frac{\frac{(pt)^{r_{\min}+1}}{(r_{\min}+1)!}\left(1+\mathcal{O}\left(pt+t^{-1}\right)\right)}{\frac{(pt)^{r_{\min}}}{(r_{\min})!}\left(1+\mathcal{O}\left(pt+t^{-1}\right)\right)}\right) \\
&= p(r_{\min})\pi^{r_{\min}}(t)\left(1+\mathcal{O}(pt+(pt)^2+p)\right) \\
&= p(r_{\min})\pi^{r_{\min}}(t)\left(1+\mathcal{O}(pt+t^{-1})\right).
\end{aligned} \tag{4.19}$$

Using the non-negativity of probabilities, we can derive a trivial lower bound

$$\overline{\pi}(t) \geq p(r_{\min})\pi^{r_{\min}}(t). \tag{4.20}$$

Combining the upper and lower bounds, (4.19) and (4.20), yields, again using (4.5),

$$\begin{aligned}
\overline{\pi}(t) &= p(r_{\min})\pi^{r_{\min}}(t)\left(1+\mathcal{O}\left(pt+t^{-1}\right)\right) \\
&= p(r_{\min})\frac{(pt)^{r_{\min}}}{r_{\min}!}\left(1+\mathcal{O}\left(pt+t^{-1}\right)\right).
\end{aligned} \tag{4.21}$$

For $pt \ll 1$ and $t \gg 1$, it is thus reasonable to approximate $\mathbb{E}[S(t)]$ by $np(r_{\min})\frac{(pt)^{r_{\min}}}{r_{\min}!}$, and hence

$$f(t) := \mathbb{E}[S(t)] - t$$

by

$$\overline{f}(t) := np(r_{\min})\frac{(pt)^{r_{\min}}}{r_{\min}!} - t.$$

The function $\overline{f}$ has a unique global minimum on $[0,\infty)$ at, see [30, (3.1)],

$$t_c := \left(\frac{(r_{\min}-1)!}{np(r_{\min})p^{r_{\min}}}\right)^{\frac{1}{r_{\min}-1}}. \tag{4.22}$$

It is important to understand that, analogously to [30, (3.4)],

$$np(r_{\min})\frac{(pt_c)^{r_{\min}}}{r_{\min}!} = \frac{t_c}{r_{\min}}. \tag{4.23}$$

With this relation we easily can find the minimum value $\overline{f}(t_c) = -a_c$, where, see [30, (3.2)],

$$a_c := \left(1 - \frac{1}{r_{\min}}\right) t_c. \tag{4.24}$$

Assumption (4.1) implies that, compare [30, (3.5)],

$$t_c \to \infty, \quad pt_c \to 0, \quad \frac{t_c}{n} \to 0, \quad a_c \to \infty, \quad \frac{a_c}{n} \to 0. \tag{4.25}$$

Note that for $t \approx t_c$, we have indeed $pt \ll 1$ and $t \gg 1$.

The approximation of $\mathbb{E}[S(t)] - t$ by $\overline{f}(t)$ yields, as in [30, (8.2)],

$$\mathbb{E}[S(t)] - np(r_{\min})\frac{(pt)^{r_{\min}}}{r_{\min}!} = \mathcal{O}\left(n(pt)^{r_{\min}}\left(pt + t^{-1} + \frac{a}{n}\right)\right). \tag{4.26}$$

4.3.2 Approximation at the end

For $pt \gg 1$, we approximate $1 - \overline{\pi}(t) = \sum_{r\in\mathcal{R}} p(r)\left(1 - \pi^r(t)\right)$ by its last term $p(r_{\max})\Pr[\mathrm{Bin}(t,p) < r_{\max}]$. We supply a lower and upper bound on the error.

As both, $1 - \pi^x(t)$ in x, see (4.4), and $\binom{t}{j}p^j(1-p)^{t-j}$ in j for $pt \geq j$, are monotonically increasing, it follows, at least for large n, that

$$\begin{aligned}
1 - \overline{\pi}(t) &= \sum_{r\in\mathcal{R}} p(r)\Pr[\mathrm{Bin}(t,p) < r] \\
&\leq p(r_{\max})\Pr[\mathrm{Bin}(t,p) < r_{\max}] \\
&\quad + (1 - p(r_{\max}))\Pr[\mathrm{Bin}(t,p) < r_{\max} - 1] \\
&\leq p(r_{\max})\Pr[\mathrm{Bin}(t,p) < r_{\max}] \\
&\quad + (1 - p(r_{\max}))\,(r_{\max} - 1)\Pr[\mathrm{Bin}(t,p) = r_{\max} - 2] \\
&\leq p(r_{\max})\Pr[\mathrm{Bin}(t,p) < r_{\max}]\left(1 + \right. \\
&\quad \left.\frac{1 - p(r_{\max})}{p(r_{\max})}\,(r_{\max} - 1)\,\frac{\Pr[\mathrm{Bin}(t,p) = r_{\max} - 2]}{\Pr[\mathrm{Bin}(t,p) = r_{\max} - 1]}\right) \\
&\leq p(r_{\max})\Pr[\mathrm{Bin}(t,p) < r_{\max}]\left(1 + \right. \\
&\quad \left.\frac{1 - p(r_{\max})}{p(r_{\max})}\,(r_{\max} - 1)\,\frac{(r_{\max} - 1)\,(1-p)}{(t - r_{\max} + 2)p}\right) \\
&= p(r_{\max})\Pr[\mathrm{Bin}(t,p) < r_{\max}]\,(1 + o(1))\,.
\end{aligned} \tag{4.27}$$

The non-negativity property of probabilities yields a trivial lower bound

$$\begin{aligned} 1-\overline{\pi}(t) &= \sum_{r\in\mathcal{R}} p(r)\Pr\left[\mathrm{Bin}(t,p)<r\right] \\ &\geq p(r_{\max})\Pr\left[\mathrm{Bin}(t,p)<r_{\max}\right]. \end{aligned}$$

Combining these two bounds, we obtain

$$1-\overline{\pi}(t) = p(r_{\min})\Pr\left[\mathrm{Bin}(t,p)<r_{\max}\right](1+o(1)). \tag{4.28}$$

Thus, analogously to [30, (3.7)], we have

$$n-\mathbb{E}\left[S_n(n)\right] = n\left(1-\overline{\pi}(n)\right) \sim b_c', \tag{4.29}$$

with

$$b_c' := np(r_{\max})\frac{(np)^{r_{\max}-1}}{(r_{\max}-1)!}(1-p)^n. \tag{4.30}$$

We define, as in [30, (3.3)],

$$b_c := np(r_{\max})\frac{(np)^{r_{\max}-1}}{(r_{\max}-1)!}e^{-np}. \tag{4.31}$$

For $p \ll n^{-\frac{1}{2}}$ we have $(1-p)^n \sim e^{-np}$, and thus, see [30, (3.8)],

$$n-\mathbb{E}\left[S_n(n)\right] = n\left(1-\overline{\pi}(n)\right) \sim b_c' \sim b_c.$$

For $p=\Omega\left(n^{-\frac{1}{2}}\right)$ both b_c' and b_c decrease to 0 very fast. Consequently, in all cases

$$n-\mathbb{E}\left[S_n(n)\right] = n\left(1-\overline{\pi}(n)\right) = b_c + o\left(b_c+1\right), \tag{4.32}$$

as in [30, (3.9)].

Note that (4.1) implies that, also compare [30, (3.5)],

$$\frac{b_c}{n}\to 0, \quad pb_c\to 0. \tag{4.33}$$

Further, by (4.31), for any $\beta\in(-\infty,\infty)$ and $p\geq n^{-1}$, we have, analogously to [30, (3.10)],

$$b_c \to \begin{cases} \infty \\ \frac{e^{-\beta}}{(r_{\max}-1)!} \\ 0 \end{cases} \iff np-\left(\log(np(r_{\max}))+(r_{\max}-1)\log\log n\right) \to \begin{cases} -\infty \\ \beta \\ \infty \end{cases}$$

as well as

$$b_c \to \begin{cases} \infty \\ \frac{e^{-\beta}}{(r_{\max}-1)!} \\ 0 \end{cases} \quad \Longleftrightarrow np - (\log(np(r_{\max})) + (r_{\max}-1)\log\log n) \to \begin{cases} -\infty \\ \beta \\ \infty. \end{cases} \tag{4.34}$$

We let

$$b^* := b_c \omega(n), \tag{4.35}$$

where $\omega(n) \to \infty$ slowly but otherwise is arbitrary.

4.4 Approximation of the probability of activation in a certain step

Y_i denotes the time of activation of a vertex i for $i \leq n - a$. Hence, the probability of activation in a certain step k is indicated by $\Pr[Y_i = k]$. We are interested in this quantity for $k = \mathcal{O}(1)$ and $k = \Theta(n)$, to determine the probability of a vertex becoming active at the beginning and at the end of the process, respectively. Recalling (4.10), we thus want to approximate

$$\Pr[Y_i = k] = \sum_{r \in \mathcal{R}} p(r)\chi(k,r),$$

where

$$\chi(k,r) := \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

Note that

$$\frac{\chi(k,r+1)}{\chi(k,r)} = \frac{p}{1-p}\frac{k-r}{r},$$

and thus

$$\chi(k,x) \geq \chi(k,y), \quad \text{for } x < y, \text{ if } k < \frac{r}{p} \tag{4.36}$$

and

$$\chi(k,x) \leq \chi(k,y), \quad \text{for } x < y, \text{ if } k > \frac{r}{p}. \tag{4.37}$$

4.4.1 Approximation at the beginning

For any $k = \mathcal{O}(1)$, we have, using (4.36) and $\frac{p}{1-p} \to 0$, at least for large n,

$$\begin{aligned} \Pr[Y_i = k] &\leq p(r_{\min})\chi(k,r_{\min}) + (1 - p(r_{\min}))\,\chi(k,r_{\min}+1) \\ &= p(r_{\min})\chi(k,r_{\min})\left(1 + \frac{1-p(r_{\min})}{p(r_{\min})}\frac{\chi(k,r_{\min}+1)}{\chi(k,r_{\min})}\right) \\ &= p(r_{\min})\chi(k,r_{\min})\left(1 + \frac{1-p(r_{\min})}{p(r_{\min})}\frac{p}{1-p}\frac{k-r_{\min}}{r_{\min}}\right) \\ &= p(r_{\min})\chi(k,r_{\min})\,(1+o(1))\,, \end{aligned}$$

and therefore, using the trivial lower bound

$$\begin{aligned} \Pr[Y_i = k] &\geq p(r_{\min})\chi(k, r_{\min}+1), \\ \Pr[Y_i = k] &\sim p(r_{\min})\binom{k-1}{r_{\min}-1}p^{r_{\min}}(1-p)^{k-r_{\min}} \\ &\sim p(r_{\min})\binom{k-1}{r_{\min}-1}p^{r_{\min}}. \end{aligned} \tag{4.38}$$

4.4.2 Approximation at the end

For $k = \Theta(n)$, and thus $kp \gg 1$, we have, applying (4.37), at least for large n,

$$\begin{aligned} \Pr[Y_i = k] &\leq p(r_{\max})\chi(k, r_{\max}) + (1 - p(r_{\max}))\,\chi(k, r_{\max}-1) \\ &= p(r_{\max})\chi(k, r_{\max})\left(1 + \frac{1-p(r_{\max})}{p(r_{\max})}\frac{\chi(k, r_{\max}-1)}{\chi(k, r_{\max})}\right) \\ &= p(r_{\max})\chi(k, r_{\max})\left(1 + \frac{1-p(r_{\max})}{p(r_{\max})}\frac{1-p}{p}\frac{r_{\max}-1}{k-r_{\max}+1}\right) \\ &= p(r_{\max})\chi(k, r_{\max})\,(1+o(1))\,, \end{aligned}$$

and hence, again using the trivial lower bound,

$$\begin{aligned} \Pr[Y_i = k] &\sim p(r_{\max})\binom{k-1}{r_{\max}-1}p^{r_{\max}}(1-p)^{k-r_{\max}} \\ &\sim p(r_{\max})\binom{k-1}{r_{\max}-1}p^{r_{\max}}. \end{aligned} \tag{4.39}$$

Since the error converges to 0 for any $k \geq \varepsilon n$ for some $\varepsilon > 0$ and decreases with increasing k, we can additionally state that the convergence is uniform in such k.

Chapter 5

Sharp threshold

In this chapter, we prove the sharp threshold result for the Erdős-Rényi random graph with individual activation thresholds, closely following [30, Section 8]. More precisely, we show that the active set w.h.p. at most doubles, that is, $A^* < 2a$, if it is smaller than the percolation threshold at the beginning, whereas starting with more initially active vertices implies w.h.p. percolation, thus $A^* = n - o(n)$.

We first give an error estimate for the approximation of $S(t) - t$ by $\overline{f}(t)$ and then discriminate between the subcritical and the supercritical case to prove the respective results.

Lemma 5.1 *[cf. [30, Lemma 8.1]] Suppose that $a = o(n)$. Then*

$$\sup_{0 \le x \le 10 r_{min}} \left| S(x t_c) - \frac{1}{r_{min}} x^{r_{min}} t_c \right| = o_p(t_c).$$

Proof First, (4.26) and (4.23) yield, recalling (4.25),

$$\begin{aligned} \mathbb{E}\left[S(x t_c)\right] &= (n-a)\,\overline{\pi}\,(x t_c) = n \overline{\pi}\,(x t_c)\,(1 + o(1)) \\ &= n p (r_{\min}) x^{r_{\min}} \frac{(p t_c)^{r_{\min}}}{r_{\min}!} \left(1 + o\left(\frac{1}{x}\right)\right) \\ &= x^{r_{\min}} \frac{t_c}{r_{\min}} \left(1 + o\left(\frac{1}{x}\right)\right) = \frac{x^{r_{\min}}}{r_{\min}} t_c + o(t_c), \end{aligned}$$

uniformly for $t_c^{-1} \le x \le 10 r_{\min}$.

Further, Lemma 4.3 implies, by (4.21) and (4.23),

$$\begin{aligned} \sup_{0 \le x \le 10 r_{\min}} |S(x t_c) - \mathbb{E}[S(x t_c)]|^2 &= \mathcal{O}_p\left(n \overline{\pi}(10 r_{\min} t_c)\right) \\ &= \mathcal{O}_p\left(n (p t_c)^{r_{\min}}\right) = \mathcal{O}_p(t_c) = o_p(t_c^2), \end{aligned}$$

and the result follows by taking the square root on both sides. □

5.1 Subcritical case

In the subcritical case, that is, for $a \sim \alpha a_c$ with $\alpha < 1$, the bootstrap percolation process typically ceases before the threshold value t_c is reached. In order to prove this, we use Lemma 5.1 to find a time step $t < t_c$ for which w.h.p. $A(t) \leq t$, and thus the process stops. We further provide an asymptotic estimate for the final set size.

Theorem 5.2 *[cf. [30, Theorem 3.1(i)]] If* $\frac{a}{a_c} \to \alpha < 1$, *then*

$$A^* = \big(\varphi(\alpha) + o_p(1)\big)\, t_c,$$

where $\varphi(\alpha)$ *is the unique root in* $[0,1]$ *of*

$$r_{min}\varphi(\alpha) - \varphi(\alpha)^{r_{min}} = (r_{min} - 1)\alpha. \tag{5.1}$$

Further, $\frac{A^*}{a} \xrightarrow{p} \varphi_1(\alpha)$, *where*

$$\varphi_1(\alpha) := \frac{r_{min}}{r_{min} - 1}\frac{\varphi(\alpha)}{\alpha}, \quad \varphi_1(0) := 1. \tag{5.2}$$

Proof The assumption on a may be written by (4.24) as

$$a = (\alpha + o(1))\, a_c = \left(\alpha\left(1 - r_{\min}^{-1}\right) + o(1)\right) t_c. \tag{5.3}$$

Hence, (3.3) and Lemma 5.1, taking $x = 1$, yield

$$\begin{aligned} A(t_c) - t_c = S(t_c) + a - t_c &= \frac{t_c}{r_{\min}} + o_p(t_c) + a - t_c \\ &= t_c\left(r_{\min}^{-1} + \alpha\left(1 - r_{\min}^{-1}\right) - 1 + o_p(1)\right). \end{aligned}$$

Since $\alpha\left(1 - r_{\min}^{-1}\right) < 1 - r_{\min}^{-1}$, w.h.p. $A(t_c) - t_c < 0$. And thus, recalling the definition of T given in (3.1), we have $T < t_c$.

Applying Lemma 5.1 with $x = \frac{T}{t_c}$ yields $S(T) = \left(\frac{T}{t_c}\right)^{r_{\min}} \frac{t_c}{r_{\min}} + o_p(t_c)$. Since $S(T) = A(T) - a = T - a$, we find, using (5.3), that

$$T - \alpha(1 - r_{\min}^{-1})t_c = S(T) + o(t_c) = \left(\frac{T}{t_c}\right)^{r_{\min}} \frac{t_c}{r_{\min}} + o_p(t_c),$$

and thus

$$r_{\min}\frac{T}{t_c} - (r_{\min} - 1)\alpha = \left(\frac{T}{t_c}\right)^{r_{\min}} + o_p(1). \tag{5.4}$$

Since the function $h(x) := r_{\min}x - x^{r_{\min}}$ is strictly increasing from 0 to $r_{\min} - 1$ on $[0,1]$, (5.4) implies, using $\frac{T}{t_c} < 1$ w.h.p., that $\frac{T}{t_c} \xrightarrow{p} y$, where $y = \varphi(\alpha)$. This proves the first assertion, and if $\alpha > 0$, the second follows.

If $\alpha = 0$, then $a = o(t_c)$, and (4.26) implies that $\mathbb{E}[S(\lambda a)] = \mathcal{O}(n(ap)^{r_{\min}}) = o(a)$ for every fixed $\lambda > 0$. Hence, for every fixed $\lambda > 1$ we have $A(\lambda a) = S(\lambda a) + a = a + o_p(a)$, so w.h.p. $A(\lambda a) < \lambda a$, and thus $a \le T < \lambda a$. Consequently, when $\alpha = 0$, then $\frac{T}{a} \xrightarrow{p} 1$. □

The precise estimate for the final set size can be simplified to provide a more intuitive upper bound on the number of active vertices at the end of the process.

Corollary 5.3 *Under the same assumption as in Theorem 5.2, we have w.h.p. $A^* < 2a$.*

Proof As φ_1 is a continuous, strictly increasing function $[0,1] \to \left[1, \frac{r_{\min}}{r_{\min}-1}\right]$, we have w.h.p. $A^* < \frac{r_{\min}}{r_{\min}-1} a \le 2a$. □

5.2 Supercritical case

In the supercritical case, that is, when $\frac{a}{a_c} \ge 1 + \delta$ for some $\delta > 0$, we prove that w.h.p. either almost or complete percolation occurs. In a second step, we profoundly investigate the difference between these two possible outcomes.

We talk about surviving the bottleneck if there are at least $3t_c$ active vertices and say that the process reached a point of (almost) percolation if there are at least $n - b^*$ active vertices, where b^* is given by Equation (4.35). Note that other distinguished values could be chosen as well for these definitions.

The following lemma shows that if the bottleneck can be surmounted, the process continues until (almost) percolation.

Lemma 5.4 *[cf. [30, Lemma 8.2]] For any a we have w.h.p. $A(t) > t$ for all $t \in [3t_c, n - b^*]$.*

Proof By (3.3), $A(t) = S(t) + a \ge S_n(t)$, so it suffices to show that $S_n(t) > t$. We proceed by differentiating between several ranges of t.

Case 1: $t \in [3t_c, 8r_{\min}t_c]$. By Lemma 5.1, we have w.h.p. for all such t

$$S_n(t) \ge \frac{1}{r_{\min}} \left(\frac{t}{t_c}\right)^{r_{\min}} t_c - t_c \ge \frac{3^{r_{\min}-1}}{r_{\min}} t - t_c \ge \frac{3}{2} t - t_c > t.$$

Case 2: $t \in \left[8r_{\min}t_c, p^{-1}\right]$. Let $t_j := 2^j r_{\min} t_c$ for $j \ge 1$, and define $J := \min\left\{j \ge 1 \colon pt_j \ge 1\right\}$. For $t_c \le t \le p^{-1}$, using (4.20), (2.1) and (4.23),

$$\begin{aligned}
\overline{\pi}(t) &\ge p(r_{\min})\pi^{r_{\min}}(t) \ge p(r_{\min})\Pr\left[\mathrm{Bin}(t,p) = r_{\min}\right] \\
&= p(r_{\min}) \frac{(pt)^{r_{\min}}}{r_{\min}!} e^{-pt}\left(1 + o(1)\right) \\
&\ge \frac{p(r_{\min})}{3} \frac{(pt)^{r_{\min}}}{r_{\min}!} = \frac{t}{3r_{\min}n} \left(\frac{t}{t_c}\right)^{r_{\min}-1}.
\end{aligned}$$

Hence, for $3 \le j \le J-1$

$$\mathbb{E}\left[S_n(t_j)\right] = n\overline{\pi}(t_j) \ge \frac{2^j}{3}t_j \ge \frac{8}{3}t_j,$$

and thus, using Chebyshev's inequality and (4.9), we get

$$\begin{aligned}\Pr\left[S_n(t_j) \le 2t_j\right] &\le \Pr\left[S_n(t_j) \le \frac{3}{4}\mathbb{E}\left[S_n(t_j)\right]\right] \\ &\le \Pr\left[\left|S_n(t_j) - \mathbb{E}\left[S_n(t_j)\right]\right| \ge \frac{1}{4}\mathbb{E}\left[S_n(t_j)\right]\right] \\ &\le \frac{\mathrm{Var}\left[S_n(t_j)\right]}{\left(\frac{1}{4}\mathbb{E}\left[S_n(t_j)\right]\right)^2} \le \frac{16}{n\overline{\pi}(t_j)} \le \frac{6}{t_j}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Pr\left[S_n(t) \le t \text{ for some } t \in [8r_{\min}t_c, t_J]\right] &\le \sum_{j=3}^{J-1}\Pr\left[S_n(t_j) \le 2t_j\right] \\ &\le \sum_{j=3}^{J-1}\frac{6}{t_j} < \frac{12}{t_3} < \frac{2}{r_{\min}t_c} \\ &= o(1).\end{aligned}$$

Case 3: $t \in \left[p^{-1}, c_1 n\right]$ for a suitable small $c_1 > 0$. Let $t_1' := \lceil p^{-1} \rceil$. Then, recalling (4.20), (8.3), and (8.5),

$$\begin{aligned}\overline{\pi}(t_1') &\ge p(r_{\min})\Pr\left[\mathrm{Bin}(t_1', p) \ge r_{\min}\right] \\ &= p(r_{\min})\Pr\left[\mathrm{Po}(t_1'p) \ge r_{\min}\right] + \mathcal{O}(p) \ge 2c_1\end{aligned}$$

for some small c_1. Hence, w.h.p. $S_n(t_1') > c_1 n$, and consequently $S_n(t) \ge S_n(t_1') > c_1 n \ge t$.

Case 4: $t \in \left[c_1 n, n - p^{-1}\right]$. Let $t_2' := \lfloor c_1 n \rfloor$ and $t_3' := n - p^{-1}$. Then, recalling (4.27) and (8.2),

$$\begin{aligned}1 - \overline{\pi}(t_2') &\le p(r_{\max})\Pr\left[\mathrm{Bin}(t_2', p) < r_{\max}\right](1 + o(1)) \\ &= \mathcal{O}\left((t_2'p)^{r_{\max}-1}e^{-t_2'p}\right) = \mathcal{O}\left((np)^{r_{\max}-1}e^{-c_1 np}\right) \\ &= o\left((np)^{-1}\right).\end{aligned}$$

Thus, $\mathbb{E}\left[n - S_n(t_2')\right] = n\left(1 - \overline{\pi}(t_2')\right) = o\left(p^{-1}\right)$, and therefore w.h.p. $n - S_n(t_2') < p^{-1}$, that is, $S_n(t_2') > t_3'$.

Case 5: $t \in \left[n - p^{-1}, n - b^*\right]$. We have, recalling (4.27),

$$\begin{aligned} 1 - \overline{\pi}(t_3') &\leq p(r_{\max})\Pr\left[\mathrm{Bin}(\lfloor t_3' \rfloor, p) < r_{\max}\right](1 + o(1)) \\ &= \mathcal{O}\left((t_3' p)^{r_{\max}-1} e^{-t_3' p}\right) = \mathcal{O}\left((np)^{r_{\max}-1} e^{-np}\right) \\ &= \mathcal{O}\left(\frac{b_c}{n}\right). \end{aligned}$$

Hence, $\mathbb{E}\left[n - S_n(t_3')\right] = n\left(1 - \overline{\pi}(t_3')\right) = \mathcal{O}(b_c) = o(b^*)$, and thus w.h.p. $n - S_n(t_3') < b^*$, that is, $S_n(t_3') > n - b^*$. □

Remark 5.5 *[cf. [30, Remark 8.3]] The proof of this lemma shows that if the bootstrap percolation process has reached an active set of size p^{-1}, then it will w.h.p. grow up to at least $n - b^*$ active vertices in no more than 3 rounds. Hence, the size then is $n - \mathcal{O}_p(b_c)$, see Lemma 2.1.*

It remains to show that the bottleneck can be escaped in the supercritical case, that is, $A(t) > t$ for $t < 3t_c$.

Theorem 5.6 *[cf. [30, Theorem 3.1(ii)]] If $\frac{a}{a_c} \geq 1 + \delta$ for some $\delta > 0$, then $A^* = n - \mathcal{O}_p\left(b_c\right) = n - o_p(n)$.*

Proof For $0 \leq t \leq 3t_c$ we may assume $a \leq 3t_c$, since otherwise $A(t) > t$ trivially. In this case, Lemmas 5.1 and 2.12 (with $x = \frac{t}{t_c}$) yield that w.h.p., uniformly in $t \leq 3t_c$,

$$\begin{aligned} A(t) = a + S(t) &\geq (1 + \delta)\left(1 - r_{\min}^{-1}\right) t_c + \frac{1}{r_{\min}}\left(\frac{t}{t_c}\right)^{r_{\min}} t_c - o(t_c) \\ &\geq \delta\left(1 - r_{\min}^{-1}\right) t_c + t - o(t_c) > t. \end{aligned}$$

This and Lemma 5.4 show that w.h.p. $A(t) > t$ for all $t \leq n - b^*$, and thus $A^* > n - b^*$. Hence, for any choice of $\omega(n)$, w.h.p. $n - A^* < b^* = b_c\omega(n)$, which is by Lemma 2.1 equivalent to $n - A^* = \mathcal{O}_p\left(b_c\right)$. □

We have shown that the process indeed grows up to at least $n - b^*$ active vertices. It remains to differentiate between almost and complete percolation. It is clear that vertices with smaller degree than activation thresholds will never be activated. For that reason, we first give an estimate for the number of these vertices.

Remark 5.7 *[cf. [30, Remark 3.3]] Let*

$$\mathcal{B} := \left|\{v \in V_n \colon deg(v) < r_v\}\right| = \sum_{v \in V_n} [deg(v) < r_v]$$

be the set of vertices v with degrees less than their activation thresholds r_v. Obviously, these vertices only can be active if they are in the initial active set $\mathcal{A}(0)$.

For $a = o(n)$ the probability for such a vertex to be active is thus $\frac{a}{n} \to 0$. Hence, $A^* \leq n - |\mathcal{B}| \left(1 - o_p(1)\right)$. We have, see (4.29) *and* (4.32),

$$\begin{aligned}\mathbb{E}\left[|\mathcal{B}|\right] &= \mathbb{E}\left[\sum_{v \in V} \left[deg(v) < r_v\right]\right] \\ &= \sum_{v \in V_n} Pr\left[deg(v) < r_v\right] = \sum_{v \in V_n} \sum_{r \in \mathcal{R}} p(r) Pr\left[deg(v) < r\right] \\ &= \sum_{v \in V_n} \mathbb{E}\left[Pr\left[deg(v) < r_v\right]\right] = n\mathbb{E}\left[Pr\left[Bin(n-1,p) < r_v\right]\right] \\ &= n\left(1 - \overline{\pi}(n-1)\right) \sim b_c + o\left(b_c + 1\right).\end{aligned}$$

The next theorem states a concrete condition for almost and complete percolation when $a \leq \frac{n}{2}$.

Theorem 5.8 *[cf. [30, Theorem 3.1(iii)]] Under the same assumptions as in Theorem 5.6 and further $a \leq \frac{n}{2}$, we have $A^* = n$ w.h.p. if and only if $b_c \to 0$, that is, if and only if*

$$np - (\log(p(r_{max})n) + (r_{max} - 1)\log\log n) \to \infty.$$

Proof If $b_c \to 0$, we may choose $b^* = 1$. Then w.h.p. $n - A^* < 1$, so $A^* = n$. Conversely, if $b_c \nrightarrow 0$, and thus, cf. Remark 5.7, $\mathbb{E}\left[|\mathcal{B}|\right] \nrightarrow 0$, then the probability that there exists a vertex with degree $< r_{\min}$ is positive, and thus with probability $1 - \frac{a}{n}$ this vertex will never be activated, so $A^* < n$. □

For $a = o(n)$, we provide more detailed results to differentiate between almost and complete percolation.

Theorem 5.9 *[cf. [30, Theorem 3.2]] Suppose that $a = o(n)$ and $A^* = n - o_p(n)$. Then:*

(i) If $np - (\log(p(r_{max})n) + (r_{max} - 1)\log\log n) \to -\infty$, so $b_c \to \infty$ by (4.34)*, then $A^* = n - b_c\left(1 + o_p(1)\right)$.*

(ii) If $np - (\log(p(r_{max})n) + (r_{max} - 1)\log\log n) \to \infty$, so $b_c \to 0$ by (4.34)*, then w.h.p. $A^* = n$.*

(iii) If $np - (\log(p(r_{max})n) + (r_{max} - 1)\log\log n) \to \beta$ for $\beta \in (-\infty, \infty)$, so $b_c \to b > 0$ by (4.34)*, then $n - A^* \xrightarrow{d} Po(b)$. In particular, $Pr\left[A^* = n\right] \to e^{-b} \in (0,1)$.*

Proof Choose $b^* := npb_c \gg b_c$. By (4.31),

$$b^* p = \frac{p(r_{\max})(np)^{r_{\max}+1} e^{-np}}{(r_{\max} - 1)!} \to 0.$$

Hence, $(n-b^*)\,p = np + o(1) \to \infty$, and, recalling (4.28),

$$\begin{aligned} 1-\overline{\pi}(n-b^*) &\sim p(r_{\max})\Pr\left[\mathrm{Bin}\left(\lfloor n-b^*\rfloor, p\right) \le r_{\max}-1\right] \\ &\sim p(r_{\max})\frac{((n-b^*)p)^{r_{\max}-1}}{(r_{\max}-1)!}(1-p)^{n-b^*} \\ &\sim 1-\overline{\pi}(n). \end{aligned}$$

Consequently, see (4.29),

$$\begin{aligned} \mathbb{E}\left[A(n)-A(n-b^*)\right] &= \mathbb{E}\left[S(n)-S(n-b^*)\right] \\ &\le n\left(\overline{\pi}(n)-\overline{\pi}(n-b^*)\right) \\ &= o\left(n\left(1-\overline{\pi}(n)\right)\right) = o(b_c'). \end{aligned} \tag{5.5}$$

By assumption and Lemma 5.4, w.h.p. $T > n-b^*$, and thus $A(n-b^*) \le A(T) \le A(n)$. Hence, (5.5) implies

$$A^* = T = A(T) = A(n) + o_p\left(b_c'\right). \tag{5.6}$$

Further,

$$n - A(n) = n - a - S(n) \in \mathrm{Bin}\left(n-a, 1-\overline{\pi}(n)\right) \tag{5.7}$$

with mean $(n-a)\left(1-\overline{\pi}(n)\right) \sim b_c'$, see (4.29).

If $b_c \to \infty$, then $b_c' \sim b_c$. Thus, (5.7) implies $n - A(n) = b_c + o_p(b_c)$, and (5.6) yields (i). If $b_c \to b < \infty$, then $b_c' = b_c + o(1) \to b$. Thus, (5.6) gives $A^* = A(n) + o_p(1)$ and hence $A^* = A(n)$ w.h.p., since the variables are integer. In this case, (5.7) implies $n - A(n) \xrightarrow{d} \mathrm{Po}(b)$, from which (ii) and (iii) follow. □

Chapter 6

Probability threshold

Whether the process percolates or not depends mainly on two quantities, namely the size of the starting set and the density p of the graph. In the previous chapters we have analyzed the percolation threshold a_c for the size of the starting set a, given a fixed edge probability $p = p(n)$. In this chapter, we provide the probability threshold p_c for percolation, given a fixed size $a = a(n)$ of the starting set $\mathcal{A}(0)$, in the case of an Erdős-Rényi random graph model. The probability threshold can be interpreted in the following way. If $p \leq (1-\delta)p_c$ for some $\delta > 0$, then $A^* \leq 2a = o(n)$ w.h.p., while if $p \geq (1+\delta)p_c$, then $A^* = n - o(n)$ w.h.p.

We only provide some results, as the two percolation thresholds a_c and p_c are basically identical. That means that equation (6.1) is the inverse to equation (4.24) in the sense that the functions $a \mapsto p_c$ and $p \mapsto a_c$ are the inverses of each other. Consequently, the thresholds can be easily transformed into each other.

The two subsequent corollaries are hence easy consequences from Theorems 5.2, 5.6, and 5.8.

The following corollary provides an estimate for the percolation threshold p_c.

Corollary 6.1 *[cf. [30, Theorem 3.4]] Suppose that $a \to \infty$ with $a = o(n)$. Then the percolation threshold for p is*

$$p_c := \left(\frac{(r_{min}-1)^{r_{min}-1}\,(r_{min}-1)!}{p(r_{min})\,r_{min}^{r_{min}-1}} \right)^{\frac{1}{r_{min}}} \left(n a^{r_{min}-1} \right)^{-\frac{1}{r_{min}}}. \tag{6.1}$$

Note that thus $n^{-1} \ll p_c \ll n^{-\frac{1}{r_{\min}}}$.

The next result states the condition on the threshold for complete percolation.

Corollary 6.2 *[cf. [30, Theorem 3.4]]*

Suppose that $a \to \infty$ *and* $p \geq (1+\delta)p_c$ *for some* $\delta > 0$*. Then w.h.p.* $A^* = n$ *if and only if* $p = \frac{\log(p(r_{max})n)+(r_{max}-1)\log\log n+\omega(n)}{n}$ *for some* $\omega(n) \to \infty$*.*

Chapter 7

Number of rounds

In this chapter, we analyze the number of rounds τ the bootstrap percolation process takes until completion in the case of an Erdős-Rényi random graph model, closely following [30, Chapter 10]. Although one could provide such results for every size of the starting set, we focus on the supercritical case only.

In the following, we give upper bounds on the number of rounds needed for each of the three phases, that is, starting, explosion, and end phase. Example 3.11 in [30] demonstrates that each of these can be the time-dominant one.

Combining the main results of the subsequent sections, we can provide, by adding the respective upper bounds for each phase, an asymptotic upper bound on the number of rounds until percolation.

Corollary 7.1 *[cf. [30, Theorem 3.10]] Suppose that $a = o(n)$ and $a \geq (1+\delta)a_c$ for some $\delta > 0$ (so that we have w.h.p. almost percolation according to Theorem 5.6). Then w.h.p.*

$$\tau \sim \frac{1}{\log r_{min}} \left(1 + \log\log(np) - \log_+ \log \frac{a}{a_c}\right) + \frac{\log\left(p(r_{max})n\right)}{np} + \mathcal{O}_p(1).$$

Proof The statement immediately follows from Theorems 7.2, 7.5, and 7.8.□

We define a variable that allows us to express the number of rounds needed for arbitrary subphases of the bootstrap percolation process. For any $m \leq n$ we define

$$\tau(m) := \inf\left\{j \colon T_j \geq m\right\},$$

with $\tau(m) = \infty$ if this set of j is empty, that is, if $m > A^* = T$. Thus, $\tau(m)$ is the number of rounds needed for m steps, or, stated differently, the number of rounds required to achieve at least m active vertices. Obviously, we have $\tau(m) \leq m$, and in fact, we expect $\tau(m)$ to be much smaller than m.

7.1 Starting phase

In the starting phase, the active set grows up to a bottleneck at $t \approx t_c$, where the process dies out in the subcritical case. In the supercritical case, however, there is actually no bottleneck at all. We consider first $\tau(3t_c)$, that is, the number of rounds required to survive the bottleneck.

The following theorem shows that in the supercritical case the bottleneck can be overcome in a constant number of rounds.

Theorem 7.2 *[cf. [30, Proposition 10.1]] Assume $a \geq (1+\delta)a_c$ for some $\delta > 0$. Then w.h.p. $\tau(3t_c) = \mathcal{O}(1)$.*

Proof Lemmas 5.1 and 2.12 imply with $x = \frac{t}{t_c}$ that, uniformly for $0 \leq t \leq 3t_c$,

$$\begin{aligned} A(t) - t = S(t) - t + a &= \left(\frac{x^{r_{\min}}}{r_{\min}} - x\right) t_c + a + o_p(t_c) \\ &\geq -a_c + (1+\delta)a_c + o_p(a_c), \end{aligned}$$

and thus w.h.p.

$$A(t) - t \geq \frac{\delta}{2} a_c \geq \frac{\delta}{4} t_c.$$

Hence, in this range, w.h.p. each generation has size at least $\frac{\delta}{4}t_c$, and the numbers of rounds $\tau(3t_c)$ required to reach $3t_c$ is thus w.h.p. bounded by $\frac{12}{\delta}$. □

7.2 Explosion phase

If the bootstrap percolation process has survived the bottleneck, then the active set size grows, beginning from $3t_c$, doubly exponentially fast up to a point where p^{-1} vertices are active.

In order to show this result, we approximate $A(t)$ by deterministic functions.

Lemma 7.3 *[cf. [30, Lemma 10.2]] Given a function $F\colon [0,\infty) \to [0,\infty)$, we define the iterates $T^F_{j+1} := F(T^F_j)$ with $T^F_0 := 0$. Thus, $T_j = T^A_j$.*

For every j, if $A \leq F$, then $T_j \leq T^F_j$, and if $A \geq F$, then $T_j \geq T^F_j$.

Proof The proof is identical to the one of Lemma 10.2 in [30], which is why we omit it here. □

For any $\delta \in \mathbb{R}$, using (4.23), we define

$$F_\delta(t) := np(r_{\min})\frac{(pt)^{r_{\min}}}{r_{\min}!}(1+\delta) = \left(\frac{t}{t_c}\right)^{r_{\min}-1} \frac{t}{r_{\min}}(1+\delta). \tag{7.1}$$

The following lemma shows that this function is indeed a good estimate for $A(t)$.

Lemma 7.4 *[cf. [30, Lemma 10.6]] For every $\delta > 0$, there are positive constants ε and K such that w.h.p. $F_{-\delta}(t) \le A(t) \le F_\delta(t)$ for all $t \in \left[K(t_c + a), \frac{\varepsilon}{p}\right]$.*

Proof By (4.21) and (7.1), for $Kt_c \le t \le \frac{\varepsilon}{p}$ (with $\varepsilon \le 1$), if n is large enough so $t_c \ge 1$,

$$\overline{\pi}(t) = p(r_{\min})\frac{(pt)^{r_{\min}}}{r_{\min}!}\left(1 + \mathcal{O}\left(\varepsilon + K^{-1}\right)\right) = \frac{1}{n}F_0(t)\left(1 + \mathcal{O}\left(\varepsilon + K^{-1}\right)\right).$$

We may thus choose ε and K such that for all such t and large n

$$F_{-\frac{\delta}{4}}(t) \le n\overline{\pi}(t) \le F_{\frac{\delta}{4}}(t). \tag{7.2}$$

For $t \ge K(t_c + a)$, (7.1) implies

$$F_0(t) = \left(\frac{t}{t_c}\right)^{r_{\min}-1}\frac{t}{r_{\min}} \ge K^{r_{\min}}\frac{a}{r_{\min}},$$

so $a \le \frac{\delta}{4}F_0(t)$ for all $t \in \left[K(t_c + a), \frac{\varepsilon}{n}\right]$, when choosing K large enough. Thus, by (7.2),

$$F_{-\frac{\delta}{4}}(t) - a \le \mathbb{E}\left[S(t)\right] = (n-a)\overline{\pi}(t) \le F_{\frac{\delta}{4}}(t) \le F_{\frac{\delta}{2}}(t) - a.$$

Hence, using Chebyshev's inequality, (4.9), and (7.2),

$$\begin{aligned}\Pr\left[A(t) \notin \left[F_{-\frac{3\delta}{4}(t)}, F_{\frac{3\delta}{4}(t)}\right]\right] &= \Pr\left[S(t) \notin \left[F_{-\frac{3\delta}{4}(t)} - a, F_{\frac{3\delta}{4}(t)} - a\right]\right] \\ &\le \frac{n\overline{\pi}(t)}{\left(\frac{\delta F_0(t)}{4}\right)^2} \le \frac{F_{\frac{\delta}{4}}(t)}{\left(\frac{\delta F_0(t)}{4}\right)^2} = \frac{16\left(1 + \frac{\delta}{4}\right)}{\delta^2 F_0(t)}.\end{aligned} \tag{7.3}$$

Define $t_j := \left(1 + \frac{\delta}{5}\right)^{\frac{j}{r_{\min}}} K(t_c + a)$. Then (7.3) and (7.1) show that, assuming $\delta \le 1$,

$$\begin{aligned}\sum_{j \ge 0:\, t_j \le \frac{\varepsilon}{p}} \Pr\left[A(t_j) \notin \left[F_{-\frac{3\delta}{4}}(t_j), F_{\frac{3\delta}{4}}(t_j)\right]\right] &\le \sum_{j \ge 0}\frac{20}{\delta^2 F_0(t_j)} \\ &= \sum_{j \ge 0}\frac{20}{\delta^2 F_0(t_0)}\left(1 + \frac{\delta}{5}\right)^{-j} \\ &= \frac{100\left(1 + \frac{\delta}{5}\right)}{\delta^3 F_0(t_0)} \to 0,\end{aligned}$$

since, using (7.1) again and (4.25),

$$F_0(t_0) = F_0\left(K(t_c + a)\right) \geq F_0(t_c) = \frac{t_c}{r_{\min}} \to \infty.$$

Consequently, w.h.p. $A(t_j) \in \left[F_{-\frac{3\delta}{4}}(t_j), F_{\frac{3\delta}{4}}(t_j)\right]$ for all $j \geq 0$ with $t_j \leq \frac{\varepsilon}{p}$. However, if $t_j \leq t \leq t_{j+1}$, then $F_0(t_j) \leq F_0(t) \leq F_0(t_{j+1}) = \left(1 + \frac{\delta}{5}\right) F_0(t_j)$, and it follows that, since both $A(t)$ and $F_0(t)$ are monotone, w.h.p.

$$\left(1 + \frac{\delta}{5}\right)^{-1} F_{-\frac{3\delta}{4}}(t) \leq A(t) \leq \left(1 + \frac{\delta}{5}\right) F_{\frac{3\delta}{4}}(t)$$

for all $t \in \left[K(t_c + a), \left(1 + \frac{\delta}{5}\right)^{-\frac{1}{r_{\min}}} \frac{\varepsilon}{p}\right]$, which, provided δ is small and ε is replaced by $\frac{\varepsilon}{2}$, say, yields the result. □

Next, we provide an upper bound on the number of rounds needed for the explosion phase.

Theorem 7.5 *[cf. [30, Proposition 10.7]] Assume $a \geq (1 + \delta)a_c$ for some $\delta > 0$. Then w.h.p.*

$$\tau\left(\frac{1}{p}\right) - \tau(3t_c) = \frac{1}{\log r_{min}} \left(\log\log(np) - \log_+ \log \frac{a}{a_c}\right) + \mathcal{O}(1).$$

Proof Choose a fixed $0 < \delta < 1$, and choose ε and K as in Lemma 7.4. In this proof, we do not have to let $\delta \to 0$, so we can take $\delta = \frac{1}{2}$, say. First, $\tau\left(K(t_c + a)\right) - \tau(3t_c)$, the number of rounds from $3t_c$ to $K(t_c + a)$, is w.h.p. $\mathcal{O}(1)$. Indeed, after $\tau(3t_c)$ rounds we have at least $\max\{3t_c, a\}$ active vertices, and in each of the following rounds until well beyond $K(t_c + a)$ the number is w.h.p. multiplied by at least 1.3, say, by the proof of Lemma 5.1. Similarly, $\tau\left(\frac{1}{p}\right) - \tau\left(\frac{\varepsilon}{p}\right) \leq 1$ w.h.p., arguing as in Case 3 of the proof of Lemma 4.26.

Consequently, it suffices to consider $\tau\left(\frac{\varepsilon}{p}\right) - \tau\left(K(t_c + a)\right)$. We define iterates $T_j^{F_\delta}$ as in Lemma 7.3 by $T_{j+1}^{F_\delta} := F_\delta\left(T_j^{F_\delta}\right)$ for $j \geq 0$, but now starting with $T_0^{F_\delta} := K(t_c + a)$. Further, let

$$N_\delta := \min\left\{j \geq 0 \colon T_j^{F_\delta} \geq \frac{\varepsilon}{p}\right\}.$$

By Lemma 7.4, we may assume that $F_{-\delta}(t) \leq A(t) \leq F_\delta(t)$ for all $t \in \left[K(t_c + a), \frac{\varepsilon}{p}\right]$, and then, by induction as in Lemma 7.3, $T_j^{F_{-\delta}} \leq T_{j+\tau(K(t_c+a))} \leq T_{j+1}^{F_\delta}$ for all $j \geq 0$ with $T_{j-1+\tau(K(t_c+a))} \leq \frac{\varepsilon}{p}$. Consequently, w.h.p.

$$N_{-\delta} \geq \tau\left(\frac{\varepsilon}{p}\right) - \tau\left(K(t_c + a)\right) \geq N_\delta - 1. \tag{7.4}$$

To find N_δ, we rewrite (7.1) as

$$\frac{F_\delta(t)}{c_\delta t_c} = \left(\frac{t}{c_\delta t_c}\right)^{r_{\min}},$$

where $c_\delta := \left(\frac{r_{\min}}{1+\delta}\right)^{\frac{1}{r_{\min}-1}}$. Iterating, we see that for $j \geq 0$

$$\frac{T_j^{F_\delta}}{c_\delta t_c} = \left(\frac{T_0^{F_\delta}}{c_\delta t_c}\right)^{r_{\min}^j} = \left(\frac{K(t_c+a)}{c_\delta t_c}\right)^{r_{\min}^j},$$

and thus

$$\log\left(\frac{T_j^{F_\delta}}{c_\delta t_c}\right) = r_{\min}^j \log\left(\frac{K(t_c+a)}{c_\delta t_c}\right)$$

as well as

$$j \log r_{\min} = \log\log\left(\frac{T_j^{F_\delta}}{c_\delta t_c}\right) - \log\log\left(\frac{K(t_c+a)}{c_\delta t_c}\right).$$

Consequently,

$$N_\delta = \left\lceil \frac{\log\log\left(\frac{\frac{\varepsilon}{p}}{c_\delta t_c}\right) - \log\log\left(\frac{K(t_c+a)}{c_\delta t_c}\right)}{\log r_{\min}} \right\rceil. \tag{7.5}$$

In order to simplify this, note that, using (4.22),

$$\log\left(\frac{\frac{\varepsilon}{p}}{c_\delta t_c}\right) = \log\left(\frac{1}{p t_c}\right) + \mathcal{O}(1) = \frac{1}{r_{\min}-1}\log(np) + \mathcal{O}(1),$$

and thus

$$\log\log\left(\frac{\frac{\varepsilon}{p}}{c_\delta t_c}\right) = \log\log(np) + \mathcal{O}(1). \tag{7.6}$$

Further, the assumption on a implies that $a \geq \frac{a_c}{2} \geq \frac{t_c}{4}$. Hence, $\log\left(K(t_c+a)\right) = \log a + \mathcal{O}(1)$ and thus, since also $\log\left(c_\delta t_c\right) = \log a_c + \mathcal{O}(1)$,

$$\log\left(\frac{K(t_c+a)}{c_\delta t_c}\right) = \log a - \log a_c + \mathcal{O}(1) = \log\frac{a}{a_c} + \mathcal{O}(1). \tag{7.7}$$

We may assume that $K \geq e c_\delta$, so $\log\frac{K(t_c+a)}{c_\delta t_c} \geq 1$, and then (7.7) yields

$$\log\log\frac{K(t_c+a)}{c_\delta t_c} = \log_+\log\frac{a}{a_c} + \mathcal{O}(1). \tag{7.8}$$

Finally, (7.5), (7.6) and (7.8) yield

$$N_\delta \log r_{\min} = \log\log np - \log_+ \log \frac{a}{a_c} + \mathcal{O}(1).$$

Note that the right-hand side depends on δ only in the error term $\mathcal{O}(1)$. Hence, we have the same result for $N_{-\delta}$, and the result follows by (7.4) and the comments at the beginning of the proof. □

7.3 End phase

After the explosion phase, p^{-1} vertices are active. We analyze the number of rounds needed until percolation.

We assume that $b^* \ll \frac{1}{p}$, which happens w.h.p. by Theorem 5.6, as $pb_c \to 0$ by (4.33). Remark 5.5 implies that $\tau(n - b^*) \le \tau\left(\frac{1}{p}\right) + 3$ w.h.p., so it suffices to consider the evolution when less than b^* vertices remain.

Lemma 7.6 *[cf. [30, Lemma 10.8]] For any t and u with $0 \le t \le t + u \le n$ the conditional distribution of $A(t + u) - A(t) = S(t + u) - S(t)$ given $A(t)$ is Bin $(n - A(t), \overline{\pi}(t; u))$, where*

$$\overline{\pi}(t; u) := \frac{\overline{\pi}(t + u) - \overline{\pi}(t)}{1 - \overline{\pi}(t)}.$$

If further $n - b^* \le t \le t + u \le n$, then, uniformly in all such t and u,

$$\overline{\pi}(t; u) = pu\,(1 + o(1))\,. \tag{7.9}$$

Proof Conditioned on $A(t)$, we have that $A(t)$ is a given number, and of the $n - a$ summands in (3.2), $n - A(t)$ are 0. For any of these terms, the probability that it changes from 0 at time t to 1 at time $t + u$ is, by (4.3),

$$\Pr\left[Y_i \le t + u \mid Y_i > t\right] = \frac{\Pr\left[t < Y_i \le t + u\right]}{\Pr\left[Y_i > t\right]} = \frac{\overline{\pi}(t + u) - \overline{\pi}(t)}{1 - \overline{\pi}(t)} = \overline{\pi}(t; u).$$

Hence, the conditional distribution of $S(t + u) - S(t)$ is Bin $(n - A(t), \overline{\pi}(t; u))$.

To see the approximation (7.9), note first that for $n - b^* \le t \le n$, since we assume $pb^* \to 0$, we have $b^* \ll \frac{1}{p} \ll n$ so $t \sim n$. Hence, using again $pb^* \to 0$, (4.30), and (4.39),

$$\begin{aligned}\overline{\pi}(t + 1) - \overline{\pi}(t) = \Pr\left[Y_i = t + 1\right] &\sim p(r_{\max}) \binom{t}{r_{\max} - 1} p^{r_{\max}} (1 - p)^{t + 1 - r_{\max}} \\ &\sim p(r_{\max}) \frac{n^{r_{\max} - 1}}{(r_{\max} - 1)!} p^{r_{\max}} (1 - p)^n = \frac{pb'_c}{n}.\end{aligned}$$

Furthermore, see (4.28) and (4.29), still for $n - b^* \le t \le n$,

$$\begin{aligned} 1 - \overline{\pi}(t) &\sim p(r_{\max})\Pr\left[\mathrm{Bin}(t,p) < r_{\max}\right] \\ &\sim p(r_{\max})\Pr\left[\mathrm{Bin}(t,p) = r_{\max} - 1\right] \\ &\sim p(r_{\max})\frac{n^{r_{\max}-1}}{(r_{\max}-1)!}p^{r_{\max}-1}(1-p)^n = \frac{b'_c}{n}. \end{aligned} \tag{7.10}$$

Consequently, recalling the uniform convergence in (4.39),

$$\overline{\pi}(t+u) - \overline{\pi}(t) = (1+o(1))\,\frac{upb'_c}{n} \tag{7.11}$$

and

$$\overline{\pi}(t;u) = (1+o(1))\,\frac{\frac{upb'_c}{n}}{\frac{b'_c}{n}} = (1+o(1))\,up. \qquad \square$$

The following lemma provides a lower bound on the number of active vertices in the end phase.

Lemma 7.7 *[cf. [30, Lemma 10.9]] Suppose that $a = o(n)$. If $b_c \to \infty$ and $n - b^* \le t \le n$, then $A(t) = n - b_c\left(1 + o_p(1)\right)$. In particular, $n - A(t) < 2b_c$ w.h.p.*

Proof We have, using (4.8) and (7.10), since $b_c \to \infty$ implies $b'_c \sim b_c$,

$$\mathbb{E}\left[n - A(t)\right] = n - a - \mathbb{E}\left[S(t)\right] = (n-a)\left(1 - \overline{\pi}(t)\right) \sim (n-a)\frac{b'_c}{n} \sim b_c,$$

and similarly, using (4.9),

$$\mathrm{Var}\left[n - A(t)\right] = \mathrm{Var}\left[S(t)\right] \le (n-a)\left(1 - \overline{\pi}(t)\right) \sim b_c.$$

Thus, using Chebyshev's inequality, since $b_c \to \infty$,

$$n - A(t) = (1+o(1))\,b_c + \mathcal{O}\left(b_c^{\frac{1}{2}}\right) = \left(1 + o_p(1)\right)b_c. \qquad \square$$

Finally, the next theorem supplies an upper bound on the number of rounds needed to activate the last few vertices in the end phase.

Theorem 7.8 *[cf. [30, Proposition 10.10]] Suppose that $a = o(n)$. When $A^* \ge 3t_c$,*

$$\tau - \tau\left(\frac{1}{p}\right) = (1+o(1))\,\frac{\log\left(p(r_{max})n\right)}{np} + \mathcal{O}_p(1).$$

In particular, if further $p \ge c\frac{\log(p(r_{max})n)}{n}$ for some $n \ge 0$, then $\tau - \tau\left(\frac{1}{p}\right) = \mathcal{O}_p(1)$. Furthermore, when $A^* = n$, w.h.p. $\tau - \tau\left(\frac{1}{p}\right) \le 3$.

Proof By Remark 5.5, after $\tau\left(\frac{1}{p}\right)+3$ rounds, the active size is $T_{\tau\left(\frac{1}{p}\right)+3} \geq n-b^*$ w.h.p.

If $b_c \to 0$, we can choose $b^* = \frac{1}{2}$, so w.h.p. $T_{\tau\left(\frac{1}{p}\right)+3} = n$ and $\tau \leq \tau\left(\frac{1}{p}\right)+3$.

More generally, if $b_c = \mathcal{O}(1)$, we have, by (7.11),

$$\begin{aligned}\mathbb{E}\left[S(n)-S(n-b^*)\right] &\leq n\left(\overline{\pi}(n)-\overline{\pi}(n-b^*)\right) \sim nb^*\frac{pb_c'}{n}\\ &= pb^*b_c = \mathcal{O}(pb^*) = o(1).\end{aligned}$$

Hence, w.h.p. $S(n) = S(n-b^*)$, which means that no further activations occur after $n-b^*$. Consequently, in this case too, w.h.p. $\tau = \tau(n-b^*) \leq \tau\left(\frac{1}{p}\right)+3$. In particular, this proves that $\tau \leq \tau\left(\frac{1}{p}\right)+3$ w.h.p. when $A^* = n$, since w.h.p. $A^* < n$ if $b_c \to \infty$, by Theorem 5.9.

Further, when $b_c = \mathcal{O}(1)$, (4.34) implies that $np \geq \log\left(p(r_{\max})n\right)$ for large n, so $\frac{\log(p(r_{\max})n)}{np} \leq 1$, and the result follows in this case.

Now assume that $b_c \to \infty$. For convenience, we modify the counting of rounds and start at $t = n-b^*$, regarding the active but unused vertices at $n-b^*$ as *generation* 0. We may assume that b^* is an integer. We recursively define

$$\begin{aligned}T_0' &:= n-b^*,\\ T_{j+1}' &:= A(T_j') \text{ for } j \geq 0,\\ \Delta_j &:= T_{j+1}' - T_j' = A(T_j') - T_j',\end{aligned}$$

and

$$\tau' := \max\left\{j \geq 0 \colon \Delta_j > 0\right\}.$$

Since w.h.p. $T_{\tau\left(\frac{1}{p}\right)-1} \leq \max\left\{\frac{1}{p}, a\right\} < n-b^* \leq T_{\tau\left(\frac{1}{p}\right)+3}$, it follows by induction that $T_{\tau\left(\frac{1}{p}\right)-1+j} \leq T_j' \leq T_{\tau\left(\frac{1}{p}\right)+3+j}$ for $j \geq 0$, and thus w.h.p.

$$\tau' + \tau\left(\frac{1}{p}\right) - 1 \leq \tau \leq \tau' + \tau\left(\frac{1}{p}\right) + 3. \tag{7.12}$$

Consequently, it suffices to estimate τ'.

By Lemma 7.6, conditioned on $A\left(T_j'\right)$, for large n

$$\mathbb{E}\left[\Delta_{j+1} \;\middle|\; A\left(T_j'\right)\right] = \left(n-A(T_j')\right)\overline{\pi}\left(T_j';\Delta_j\right) \leq \left(n-A(T_0')\right)2p\Delta_j,$$

and thus, by induction, since $\Delta_0 \leq n-T_0' = b^*$,

$$\mathbb{E}\left[\Delta_j \mid A\left(T_0'\right)\right] \leq \left(2\left(n-A(T_0')\right)p\right)^j\Delta_0 \leq \left(2\left(n-A(T_0')\right)p\right)^j b^*. \tag{7.13}$$

Further, Lemma 7.7 yields $n - A(T_0') = n - A(n - b^*) < 2b_c$ w.h.p. Consequently, (7.13) implies, w.h.p. for all $j \geq 0$ simultaneously

$$\mathbb{E}\left[\Delta_j \mid A\left(T_0'\right)\right] \leq (4pb_c)^j b^*. \tag{7.14}$$

Recall that $pb_c \to 0$ by (4.33), so we may assume $4pb_c < 1$. If j is chosen such that $(4pb_c)^j b^* \to 0$, then (7.14) implies that w.h.p. $\Delta_j = 0$ and thus $\tau' < j$. Hence, for any $\omega' = \omega'(n) \to \infty$, w.h.p.

$$\tau' \leq \frac{\log b^*}{|\log(pb_c) + \log 4|} + \omega'(n),$$

which is, see Lemma 2.1, equivalent to

$$\tau' \leq \frac{\log b^*}{|\log(pb_c) + \log 4|} + \mathcal{O}_p(1) = \frac{\log b^*}{|\log(pb_c)|}\left(1 + o(1)\right) + \mathcal{O}_p(1). \tag{7.15}$$

For a lower bound, fix ε with $0 < \varepsilon < 1$, and define the deterministic numbers Δ_j^- by

$$\Delta_j^- := (1-\varepsilon)^{j+1}(pb_c)^j b^*. \tag{7.16}$$

Let $\omega'' := \frac{1}{pb_c} \to \infty$. We claim that w.h.p.

$$\Delta_j \geq \Delta_j^- \text{ for all } j \geq 0 \text{ such that } \Delta_j^- \geq \omega''. \tag{7.17}$$

By our assumption $4pb_c < 1$, we have $\frac{\Delta_{j+1}^-}{\Delta_j^-} < \frac{1}{4}$, so $\Delta_j^- \to 0$ geometrically fast.

By Lemma 7.7 and $\frac{b_c}{b^*} \to 0$, w.h.p.

$$\Delta_0 = A(T_0') - (n - b^*) = A(T_0') - n + b^* \geq b^* - 2b_c \geq (1-\varepsilon)b^* = \Delta_0^-,$$

so (7.17) holds w.h.p. for $j = 0$.

Say that $j \geq 0$ is *good* if $\Delta_j \geq \Delta_j^-$ and *fat* if $A(T_j') > n - \left(1 - \frac{\varepsilon}{4}\right) b_c$. Let $j \geq 0$. At time T_j' we have $A(T_j') - T_j' = \Delta_j$ active but unused vertices. Further, by Lemma 7.6, we have, conditioned on $A\left(T_j\right)$ (which specifies both T_j' and Δ_j),

$$\Delta_{j+1} = T_{j+2}' - T_{j+1}' = A(T_j' + \Delta_j) - A(T_j') \in \mathrm{Bin}\left(n - A(T_j'), \overline{\pi}(T_j'; \Delta_j)\right).$$

By Lemma 7.6, $\overline{\pi}(T_j'; \Delta_j) = p\Delta_j\left(1 + o(1)\right) \geq p\Delta_j\left(1 - \frac{\varepsilon}{4}\right)$ for large n, so if j is good but not fat,

$$\begin{aligned}\mathbb{E}\left[\Delta_{j+1} \;\middle|\; A\left(T_j'\right)\right] &= \left(n - A(T_j')\right)\overline{\pi}(T_j'; \Delta_j) \geq \left(1 - \frac{\varepsilon}{4}\right)^2 b_c p \Delta_j \\ &\geq \left(1 - \frac{\varepsilon}{2}\right) b_c p \Delta_j^- \geq \left(1 + \frac{\varepsilon}{2}\right)\Delta_{j+1}^-,\end{aligned}$$

and Chebyshev's inequality yields, since $x \mapsto \frac{x}{(x-a)^2}$ is decreasing for $x > a$,

$$\begin{aligned}\Pr\left[\Delta_{j+1} < \Delta_{j+1}^- \;\middle|\; A\left(T_j'\right)\right] &\le \frac{\operatorname{Var}\left[\Delta_{j+1} \;\middle|\; A\left(T_j'\right)\right]}{\left(\mathbb{E}\left[\Delta_{j+1} \;\middle|\; A\left(T_j'\right)\right] - \Delta_{j+1}^-\right)^2} \\ &\le \frac{\mathbb{E}\left[\Delta_{j+1} \;\middle|\; A\left(T_j'\right)\right]}{\left(\mathbb{E}\left[\Delta_{j+1} \;\middle|\; A\left(T_j'\right)\right] - \Delta_{j+1}^-\right)^2} \\ &\le \frac{\left(1+\frac{\varepsilon}{2}\right)\Delta_{j+1}^-}{\left(\frac{\varepsilon\Delta_{j+1}^-}{2}\right)^2} = \mathcal{O}\left(\frac{1}{\Delta_{j+1}^-}\right).\end{aligned}$$

Say that j is *bad* if j is not good and that j *fails* if j is fat or bad. Then, by stopping at the first j that fails, we see that

$$\begin{aligned}\Pr\left[\text{some } j \le \omega'' \text{ fails}\right] &\le \Pr\left[\text{some } j \le \omega'' \text{ is fat}\right] + \Pr\left[0 \text{ is bad}\right] \\ &\quad + \sum_{j>0:\, \Delta_j^- \ge \omega''} \Pr\left[j \text{ is bad} \mid j-1 \text{ is good and not fat}\right] \\ &\le \Pr\left[A(n) > n - \left(1 - \frac{\varepsilon}{4} b_c\right)\right] + o(1) \\ &\quad + \sum_{j:\, \Delta_j^- \ge \omega''} \mathcal{O}\left(\frac{1}{\Delta_j^-}\right) \\ &= o(1),\end{aligned}$$

since $A(n) < n - \left(1 - \frac{\varepsilon}{4}\right) b_c$ w.h.p. by Lemma 7.7. The final sum is $\mathcal{O}\left(\frac{1}{\omega''}\right) = o(1)$ because the terms $\frac{1}{\Delta_j^-}$ increase geometrically, so the sum is dominated by its largest (and last) term.

We have shown that w.h.p., if $\Delta_j^- \ge \omega$, then $\Delta_j \ge \Delta_j^- > 0$ and thus $\tau' \ge j$. Hence, by (7.17) and (7.16), w.h.p.

$$\tau' \ge \left\lfloor \frac{\log\left((1-\varepsilon)\frac{b^*}{\omega''}\right)}{\left|\log\left((1-\varepsilon)pb_c\right)\right|} \right\rfloor = \frac{\log b^*}{\left|\log(pb_c)\right|}\left(1 + o(1)\right) + \mathcal{O}(1). \tag{7.18}$$

Combining the upper bound (7.15) and the lower bound (7.18), we find

$$\tau' = \frac{\log b^*}{\left|\log(pb_c)\right|}\left(1 + o(1)\right) + \mathcal{O}(1). \tag{7.19}$$

By (4.31), $\log(pb_c) = -\left(np - r_{\max}\log(np) + \mathcal{O}(1)\right)$ and

$$\begin{aligned}\log\left(p(r_{\max})n\right) &\ge \log b^* + \log p(r_{\max}) \ge \log b_c + \log p(r_{\max}) \\ &\ge \log n - np + \log p(r_{\max}) - \mathcal{O}(1),\end{aligned}$$

and thus $\log b^* = \log\left(p(r_{\max})n\right) + \mathcal{O}(np)$. Hence, finally (7.19) yields

$$\begin{aligned}\tau' &= \frac{\log\left(p(r_{\max})n\right) + \mathcal{O}(np)}{np - r_{\max}\log(np) + \mathcal{O}(1)}\left(1 + o(1)\right) + \mathcal{O}_p(1) \\ &= \frac{\log\left(p(r_{\max})n\right)}{np}\left(1 + o(1)\right) + \mathcal{O}_p(1).\end{aligned}$$

The result follows from (7.12). □

Chapter 8

Boundary cases

In this chapter, we treat the cases when the assumption (4.1) does not hold for an Erdős-Rényi random graph $G_{n,p}$. That is, we analyze the bootstrap percolation process either when $p = \mathcal{O}\left(n^{-1}\right)$, thus for a sparse graph, or when $p = \Omega\left(n^{-\frac{1}{r_{\min}}}\right)$, hence for a dense graph. The main observation is that, at least in the non-trivial boundary cases, a kind of threshold phenomenon occurs. These results are based on [30, Chapters 5 and 11].

Note that a different strategy has to be applied, since the approximation provided in Chapter 4 in general do not work anymore.

8.1 Sparse case

The following theorem shows, that in the extremely sparse case, when $p \ll n^{-1}$, the bootstrap percolation process dies out fast.

Theorem 8.1 *Suppose that $p \ll n^{-1}$ and $a = o(n)$. Then w.h.p. $A^* = o(n)$.*

Proof The expected degree of a vertex i is

$$\mathbb{E}\left[\deg(i)\right] = (n-1)p = o(1),$$

and thus, using Markov's inequality,

$$\Pr\left[\deg(i) \geq r_{\min}\right] \leq \frac{(n-1)p}{r_{\min}} = o(1).$$

Hence, $\mathbb{E}\left[A^*\right] \leq a + (n-a)\Pr\left[\deg(i) \geq r_{\min}\right] = o(n)$. Consequently, for every $\delta(n) > \varepsilon n$ with $\varepsilon > 0$ we have, using Markov's inequality,

$$\Pr\left[A^* \geq \delta(n)\right] \leq \frac{\mathbb{E}\left[A^*\right]}{\delta(n)} = o(1),$$

and the claim follows. □

In the moderately sparse case, when $p \sim \frac{c}{n}$ for some $c > 0$, then t_c and a_c are of order n. That is why processes with starting sets significantly smaller than n will die out for sure. This suggests that the interesting case is when a positive fraction of all vertices are initially active. Scalia-Tomba [36] showed for constant activation threshold r and large enough c, thus dense enough graphs, that there is also a sort of sharp threshold phenomenon. Although at the end always a linear fraction of vertices will be active, there can be observed a sudden jump from a small to a large fraction of n as final active set size.

In the following, we provide an alternative approximation of $\overline{\pi}(t)$ by a Poisson distribution, which is simpler to analyze. These results are based on [30, Section 3.2]. We define

$$\widetilde{\pi}^r(t) := \Pr\left[\mathrm{Po}(tp) \geq r\right] = \psi(tp, r) := \sum_{j=r}^{\infty} \frac{(pt)^j}{j!} e^{-pt} \tag{8.1}$$

for $r \geq 2$ and

$$\overline{\widetilde{\pi}}(t) := \sum_{r \in \mathcal{R}} p(r) \widetilde{\pi}^r(t) = \sum_{r \in \mathcal{R}} p(r) \Pr\left[\mathrm{Po}(tp) \geq r\right] =: \overline{\psi}(tp). \tag{8.2}$$

To warrant readability, we introduce the notation

$$\widetilde{\pi}(t) := \widetilde{\pi}^{r_{\min}}(t). \tag{8.3}$$

Note that $\psi(\cdot, r)$ is a differentiable, increasing function on $(0, \infty)$ and that, see [30, (3.14)],

$$\frac{d}{dt} \widetilde{\pi}^r(t) = p\psi'(tp, r) = p \frac{(pt)^{r-1}}{(r-1)!} e^{-pt} = \frac{p^r t^{r-1}}{(r-1)!} e^{-pt} \tag{8.4}$$

for $r \geq 2$.

By a standard estimate for the Poisson approximation of a binomial distribution, see (2.2) and [30, (3.15)], we get

$$\left|\pi^r(t) - \widetilde{\pi}^r(t)\right| \leq \mathrm{d_{TV}}\left(\mathrm{Bin}(t, p), \mathrm{Po}(tp)\right) < p \tag{8.5}$$

for all $r \geq 2$.

This result can easily be extended to an upper bound on the statistical distance between $\overline{\pi}(t)$ and $\overline{\widetilde{\pi}}(t)$.

Lemma 8.2 *[cf. [30, (3.15)]] We have* $\left|\overline{\pi}(t) - \overline{\widetilde{\pi}}(t)\right| < p$.

Proof Recalling (4.3) and (8.2), it follows from the triangle inequality and (8.5) that

$$\begin{aligned}\left|\overline{\pi}(t)-\overline{\widetilde{\pi}}(t)\right| &= \left|\sum_{r\in\mathcal{R}} p(r)\left(\pi^r(t)-\widetilde{\pi}^r(t)\right)\right| \leq \sum_{r\in\mathcal{R}} p(r)|\pi^r(t)-\widetilde{\pi}^r(t)| \\ &< \sum_{r\in\mathcal{R}} p(r)p = p.\end{aligned}$$

□

We define

$$\begin{aligned}f(x,c,\theta) &:= (1-\theta)\sum_{r\in\mathcal{R}} p(r)\mathrm{Pr}\left[\mathrm{Po}(cx)\geq r\right]+\theta-x \\ &= (1-\theta)\sum_{r\in\mathcal{R}} p(r)\sum_{j=r}^{\infty}\frac{(cx)^j}{j!}e^{-cx}-x+\theta \\ &= 1-x-(1-\theta)\sum_{r\in\mathcal{R}} p(r)\mathrm{Pr}\left[\mathrm{Po}(cx)\leq r-1\right] \\ &= 1-x-(1-\theta)\sum_{r\in\mathcal{R}} p(r)\sum_{j=0}^{r-1}\frac{(cx)^j}{j!}e^{-cx}\end{aligned} \tag{8.6}$$

for $x, c \geq 0$ and $\theta \in [0,1]$. We let $x_0(\theta)$ be the smallest root $x \geq 0$ and $x_1(\theta)$ be the largest root in $[0,1]$ of

$$f(x,c,\theta) = 0. \tag{8.7}$$

Since $f(0,c,\theta) = \theta \geq 0$ and $f(1,c,\theta) = -(1-\theta)\sum_{r\in\mathcal{R}} p(r)\mathrm{Pr}\left[\mathrm{Po}(c)\leq r-1\right] \leq 0$, there is always at least one root in $[0,1]$, and $0 \leq x_0(\theta) \leq x_1(\theta) \leq 1$. When $0 < \theta < 1$, then $0 < x_0(\theta) \leq x_1(\theta) < 1$, whereas we have $x_0(0) = 0$ and $x_0(1) = x_1(1) = 1$. With

$$y_c := \frac{\sum_{r\in\mathcal{R}} p(r)\frac{1}{(r-2)!}}{\sum_{r\in\mathcal{R}} p(r)\frac{1}{(r-1)!}} \tag{8.8}$$

we also define

$$c_c := y_c + \frac{1-\overline{\psi}(y_c)}{\sum_{r\in\mathcal{R}} p(r)\frac{y_c^{r-1}}{(r-1)!}e^{-y_c}}. \tag{8.9}$$

In the following, we analyze the roots of $f(\cdot,c,\theta)$.

Lemma 8.3 *[cf. [30, Lemma 5.1]]*

(i) If $0 \leq c \leq c_c$, then (8.7) has a unique root $x = x_0(\theta) \in [0,1]$ for every $\theta \in [0,1]$, and $x_0(\theta)$ is a continuous strictly increasing function of θ.

(ii) If $c > c_c$, then there exists $\theta_c^- = \theta_c^-(c)$ and $\theta_c = \theta_c(c)$ with $0 \leq \theta_c^- < \theta_c < 1$ such that (8.7) has three roots in $[0,1]$ when $\theta \in (\theta_c^-, \theta_c)$ and a unique root

when $\theta \in [0, \theta_c^-)$ or $\theta \in (\theta_c, 1]$. If $\theta = \theta_c^- > 0$ or $\theta = \theta_c$, there are two roots, one of them double. The smallest root $x_0(\theta)$ is strictly increasing and continuous on $[0,1]$ except at θ_c where it has a jump from $x_0(\theta_c)$ to $x_1(\theta_c) > x_0(\theta_c)$, where $x_1(\theta_c) = x_0(\theta_c^+) := \lim_{\theta \searrow \theta_c} x_0(\theta)$ is the other root for $\theta = \theta_c$. Furthermore, if $\theta = \theta_c$, then $f(x, c, \theta) \geq 0$ for $x \in [0, x_1(\theta)]$, and $x_0(\theta)$ is a double root.

Proof By the implicit function theorem, at least locally, the root $x_0(\theta)$ is smooth except at points where

$$f(x, c, \theta) = \frac{\delta}{\delta x} f(x, c, \theta) = 0. \tag{8.10}$$

We begin by studying such critical points. Let

$$g(y) := \sum_{r \in \mathcal{R}} p(r) \Pr\left[\mathrm{Po}(y) \leq r - 1\right] = 1 - \overline{\psi}(y),$$

cf. (8.3) and (8.2). Differentiations yield, see (8.4),

$$g'(y) = -\sum_{r \in \mathcal{R}} p(r) \Pr\left[\mathrm{Po}(y) = r - 1\right] = -\sum_{r \in \mathcal{R}} p(r) \frac{y^{r-1}}{(r-1)!} e^{-y}$$

and, recalling (8.8),

$$g''(y) = -g'(y) - \sum_{r \in \mathcal{R}} p(r) \frac{y^{r-2}}{(r-2)!} e^{-y} = \left(\frac{y_c}{y} - 1\right) g'(y). \tag{8.11}$$

We have, see (8.6),

$$f(x, c, \theta) = 1 - x - (1 - \theta) g(cx),$$

and thus

$$\frac{\delta}{\delta x} f(x, c, \theta) = -1 - c(1 - \theta) g'(cx).$$

Hence, (8.10) holds if and only if

$$\begin{cases} (1 - \theta) g(cx) = 1 - x \\ c(1 - \theta) g'(cx) = -1, \end{cases}$$

which imply $g(cx) = -c(1 - x) g'(cx)$ and thus

$$c = cx - \frac{g(cx)}{g'(cx)}. \tag{8.12}$$

Let

$$h(y) := y - \frac{g(y)}{g'(y)} \text{ for } y > 0,$$

so (8.12) says $c = h(cx)$. Then by (8.11),

$$h'(y) = 1 - \frac{g'(y)}{g'(y)} + \frac{g(y)g''(y)}{g'(y)^2} = \frac{y_c - y}{y}\frac{g(y)}{g'(y)}.$$

Since $g(y) > 0$ and $g'(y) < 0$ for $y > 0$, the function h has a global minimum at $y = y_c$, and the minimum value is, recalling 8.9,

$$\min_{y>0} h(y) = h(y_c) = c_c.$$

Furthermore, $h(y) > y \to \infty$ as $y \to \infty$, and $h(y) \to \infty$ as $y \to 0$, as then $g(y) \to 1$ and $g'(y) \to 0$.

Consequently, if $0 \le c \le c_c$, then (8.12) has no solution $x > 0$, and thus there is no critical point. If $c = c_c$, there is exactly one $x > 0$ satisfying (8.12) (namely $x = \frac{y_c}{c_c}$), and if $c > c_c$, there are two. Since (8.12) implies $c > cx$, these roots are in $(0, 1)$.

To complete the proof, it is perhaps simplest to rewrite (8.7) as $\theta = \vartheta(x)$, with

$$\vartheta(x) := 1 - \frac{1-x}{g(cx)}.$$

Since $g(y) > 0$ for $y \ge 0$, ϑ is a smooth function on $[0, 1]$, with $\vartheta(0) = 0$ and $\vartheta(1) = 1$. Moreover, $f(x, c, \theta) = g(cx)(\theta - \vartheta(x))$, which implies that $f(x, c, \theta) = \frac{\delta}{\delta x} f(x, c, \theta) = 0$ if and only if $\theta = \vartheta(x)$ and $\vartheta'(x) = 0$. Consequently, by the results above, if $c < c_c$, then $\vartheta' \neq 0$, so $\vartheta'(x) > 0$ for $x \ge 0$. In this case, ϑ is strictly increasing and thus a bijection $[0, 1] \to [0, 1]$, and x_0 is its inverse.

If $c = c_c$, then $\vartheta' = 0$ only at a single point, and it follows again that ϑ is a strictly increasing function and x_0 is its inverse.

If $c > c_c$, then $\vartheta'(x) = 0$ at two values x_1 and x_2 with $0 < x_1 < x_2 < 1$ and $cx_1 < c_c < cx_2$. It can be seen, for example, using (8.11), that $\vartheta''(x_1) < 0 < \vartheta''(x_2)$, and thus ϑ is decreasing on the interval $[x_1, x_2]$. The result follows with $\theta_c = \vartheta(x_1)$, $\theta_c^- = \max\{\vartheta(x_2), 0\}$ and $x_0(\theta_c) = x_2$. Note that $\vartheta(x_2) = \min_{x\in[0,1]} \vartheta(x) < 0$ if c is large enough. □

The next lemma provides a probabilistic upper bound on the number of activated vertices at the beginning of the process in the moderately sparse case.

Lemma 8.4 *[cf. [30, Lemma 11.2]] Suppose that $p = \mathcal{O}(\frac{1}{n})$ and $pt = o(1)$. Then $S(t) = o_p(t)$.*

Proof Note that $S(t) = 0$ for $t < r_{\min}$. We may assume $1 \le t \le \frac{1}{p}$. Then $\overline{\pi}(t) = \mathcal{O}((pt)^{r_{\min}}) = o(pt)$ by (4.21), and thus the number of activated vertices is $\mathbb{E}[S(t)] = (n-a)\overline{\pi}(t) = o(npt) = o(t)$. □

The next theorem shows the modified sharp threshold result. That is, if the starting set is a positive fraction of n, then the final active set grows to a bigger linear factor but does not percolate, not even almost. If the starting set is negligibly small compared to n, then there will never be a linear factor activated. The jump in the final size, and thus the threshold, is at $\theta_c n$.

Theorem 8.5 *[cf. [30, Theorem 5.2]] Suppose that $p \sim \frac{c}{n}$ and $a \sim \theta n$ for some constants $c \geq 0$ and $\theta \geq 0$.*

(i) If $\theta = 0$, and thus $a = o(n)$, then $\frac{A^*}{a} \xrightarrow{p} 1$.

(ii) If $c = 0$, and thus $p = o(\frac{1}{n})$, then $\frac{A^*}{a} \xrightarrow{p} 1$.

(iii) If $0 \leq c \leq c_c$, then $\frac{A^*}{n} \xrightarrow{p} x_0(\theta)$ (which is the unique non-negative root of $f(\cdot, c, \theta)$).

(iv) If $c > c_c$ and $\theta \neq \theta_c(c)$, then $\frac{A^*}{n} \xrightarrow{p} x_0(\theta)$ (which is the smallest non-negative root of $f(\cdot, c, \theta)$).

Proof First, in (i) and (ii), $ap \to \theta c = 0$. Let $\varepsilon > 0$. Taking $t = (1+\varepsilon)a$ in Lemma 8.4, we find w.h.p. $S((1+\varepsilon)a) < \varepsilon a$ and thus

$$A((1+\varepsilon)a) = a + S((1+\varepsilon)a) < (1+\varepsilon)a,$$

whence $A^* = T < (1+\varepsilon)a$. Consequently, $1 \leq \frac{A^*}{a} < 1+\varepsilon$ w.h.p., proving (i) and (ii).

Next, by (3.3), Lemma 4.1 and (4.8), uniformly for all $t \geq 0$,

$$A(t) = a + S(t) = a + \mathbb{E}[S(t)] + o_p(n) = (n-a)\overline{\pi}(t) + a + o_p(n),$$

and thus, using also (8.2),

$$n^{-1}A(t) = (1-\theta)\overline{\pi}(t) + \theta + o_p(1) = (1-\theta)\overline{\overline{\pi}}(t) + \theta + o_p(1).$$

Substituting $t = xn$, we find by (8.2) and (8.1), since $pt = xc + o(x)$, uniformly in all $x \geq 0$,

$$\begin{aligned} n^{-1}A(xn) &= (1-\theta)\sum_{r\in\mathcal{R}} p(r)\Pr[\mathrm{Po}(tp) \geq r] + \theta + o_p(1) \\ &= (1-\theta)\sum_{r\in\mathcal{R}} p(r)\Pr[\mathrm{Po}(cx) \geq r] + \theta + o_p(1). \end{aligned}$$

Further, recalling (8.6), still uniformly in $x \geq 0$,

$$n^{-1}(A(xn) - xn) = f(x, c, \theta) + o_p(1). \tag{8.13}$$

Let $\varepsilon > 0$. Since $f(x, c, \theta) > 0$ for $x \in [0, x_0(\theta))$, and thus by compactness $f(\cdot, c, \theta)$ is bounded from below on $[0, x_0(\theta) - \varepsilon]$, (8.13) implies that w.h.p. $A(xn) - xn > 0$ on $[0, x_0(\theta) - \varepsilon]$, and thus $T > (x_0(\theta) - \varepsilon)n$. Furthermore, both in (iii) and (iv) with $\theta \neq \theta_c$, we have $\frac{\delta}{\delta x} f(x_0(\theta), c, \theta) \neq 0$ and thus, if $\varepsilon > 0$ is small enough, $f(x_0(\theta) + \varepsilon, c, \theta) < 0$, so (8.13) implies that w.h.p. $A((x_0(\theta) + \varepsilon)n) < (x_0(\theta) + \varepsilon)n$, and thus $T < (x_0(\theta) + \varepsilon)n$. □

8.2 Dense case

In this section, we analyze the bootstrap percolation process in the case of a dense graph, that is, first when $p = \Theta\left(n^{-\frac{1}{r_{\min}}}\right)$, and then when $p \gg n^{-\frac{1}{r_{\min}}}$.

The former assumption implies that t_c and a_c are constant, what suggests that the first few steps determine whether the process will die out immediately or grow very quickly. It happens with constant probability bounded away from 0 and 1 that a starting set of constant size leads to complete percolation.

The next lemma shows that if $\omega(n)$ vertices are active, then the activation spreads w.h.p. to all vertices. To prove this, we will resort to a graph $G_{n,\widehat{p}}$ where $\widehat{p}$ satisfies (4.1), and thus for which Lemma 5.4 applies.

Lemma 8.6 *[cf. [30, Lemma 11.4]] Suppose that $p \geq cn^{-\frac{1}{r_{min}}}$ for some $c > 0$. If $\omega(n) \to \infty$, then w.h.p. $A(t) > t$ for all t with $\omega(n) \leq t \leq n-1$.*

Proof Let $\widehat{p} := \omega(n)^{-\frac{1}{2r_{\min}}} n^{-\frac{1}{r_{\min}}}$. We may assume $\omega(n) \leq n$ and then $n^{-1} \ll \widehat{p} \ll n^{-\frac{1}{r_{\min}}}$, so $\widehat{p} < p$, at least for large n, and we may further assume that $G_{n,\widehat{p}} \subseteq G_{n,p}$. We consider bootstrap percolation on $G_{n,\widehat{p}}$ and $G_{n,p}$ simultaneously, with the same initial set $\mathcal{A}(0)$ of size a. We use $\widehat{\ }$ to denote variables for $G_{n,\widehat{p}}$ and start with families of i.i.d. random indicator variables $\widehat{I}_i(s) \in \mathrm{Be}(\widehat{p})$ and $I_i(s) \in \mathrm{Be}(p)$, where we may assume $\widehat{I}_i(s) \leq I_i(s)$. Then $\widehat{S}(t) \leq S(t)$ and $\widehat{A}(t) \leq A(t)$. The critical time is, recalling (4.22),

$$\widehat{t_c} = \mathcal{O}\left((n\widehat{p}^{r_{\min}})^{-\frac{1}{r_{\min}-1}}\right) = \omega(n)^{\frac{1}{2(r_{\min}-1)}} = o\left(\omega(n)\right).$$

Further, $\widehat{p} \geq n^{-\frac{3}{2r_{\min}}} \geq n^{-\frac{3}{4}}$, so by (4.31) $\widehat{b_c} \to 0$ and we may choose $\widehat{b^*} \to 0$. Hence, Lemma 5.4, applied to $G_{n,\widehat{p}}$, shows that w.h.p. $A(t) \geq \widehat{A}(t) > t$ for $t \in \left[3\widehat{t_c}, n - \widehat{b^*}\right]$, and the result follows because $3\widehat{t_c} \leq \omega(n)$ for large n and $n - \widehat{b^*} > n-1$. □

The next theorem provides limits for the probability of complete percolation. Because Lemma 8.6 implies that once this threshold $\omega(n)$ is reached, the process w.h.p. percolates completely, it only remains to show that the process activates $\omega(n)$ vertices with a probability in $(0,1)$ for that purpose.

Theorem 8.7 *[cf. [30, Theorem 5.6]] Suppose that $p \sim cn^{-\frac{1}{r_{min}}}$ for a constant $c > 0$.*

(i) If $a \geq r_{min}$ is fixed, then

$$Pr\left[A^* = n\right] \to \zeta(a,c)$$

for some $\zeta(a,c) \in (0,1)$. Furthermore, there exist numbers $\zeta(a,c,k) > 0$ for $k \geq a$ such that $Pr[A^* = k] \to \zeta(a,c,k)$ for each fixed $k \geq a$, and $\sum_{k=a}^{\infty} \zeta(a,c,k) + \zeta(a,c) = 1$.

(ii) If $a \to \infty$, then $Pr[A^* = n] \to 1$, that is, $A^* = n$ w.h.p.

Proof For (ii), we apply Lemma 8.6 (if necessary with a smaller c). Taking $\omega(n) = a$, we see that w.h.p. $A(t) > t$ for all $t \in [a, n-1]$. Since also $A(t) \geq a$, we have $A(t) > t$ for all $t \leq n-1$, and thus $A^* = T = n$.

For (i), suppose $p \sim cn^{-\frac{1}{r_{\min}}}$ and let $a \geq r_{\min}$ be some constant. We consider the probability that a vertex is activated at a given time k. Note that the estimate 4.4 still applies. Hence, by (4.38),

$$\begin{aligned} \Pr[Y_i = k] &\sim p(r_{\min}) \binom{k-1}{r_{\min}-1} p^{r_{\min}} (1-p)^{k-r_{\min}} \\ &\sim p(r_{\min}) \binom{k-1}{r_{\min}-1} \frac{c^{r_{\min}}}{n}. \end{aligned} \tag{8.14}$$

For any fixed K the random variables

$$X_k := A(k) - A(k-1) = S(k) - S(k-1) = \sum_{i=1}^{n-a} [Y_i = k]$$

for $k \in [K]$ form together with

$$X_{K+1} := n - a - A(K) = \sum_{i=1}^{n-a} [Y_i > K]$$

a random vector with the multinomial distribution $\mathrm{Mul}\left(n-a, (p_k)_{k=1}^{K+1}\right)$ with $p_k := \Pr[Y_i = k]$ for $k \leq K$ and $p_{K+1} := \Pr[Y_i \geq K+1]$. By (8.14), $(n-a)p_k \to p(r_{\min})\binom{k-1}{r_{\min}-1} c^{r_{\min}}$ for $k \leq K$, and it follows, see Section 2.3.3, that X_k for $k \leq K$ have a joint Poisson limit

$$(X_k)_{k=1}^K \xrightarrow{d} (\xi_k)_{k=1}^K \text{ with } \xi_k \in \mathrm{Po}\left(p(r_{\min})\binom{k-1}{r_{\min}-1} c^{r_{\min}}\right) \text{ independent.} \tag{8.15}$$

The limiting probabilities can be expressed as hitting probabilities of an inhomogeneous random walk, see Section 2.4.3 and [30, Remark 5.7]. For independent $\xi_k \in \mathrm{Po}\left(p(r_{\min})\binom{k-1}{r_{\min}-1} c^{r_{\min}}\right)$ for $k \in \mathbb{N}$ and with the notation

$$\widetilde{S}_k := \sum_{j=1}^{k} (\xi_j - 1)$$

and

$$\widetilde{T} := \min\left\{k \geq 1 \colon a + \widetilde{S}_k = 0\right\} \in \mathbb{N} \cup \{\infty\},$$

we have

$$\zeta(a,c) = \Pr\left[\widetilde{T} = \infty\right] = \Pr\left[a + \widetilde{S}_k \geq 1 \text{ for all } k \geq 1\right]$$

as well as $\zeta(a,c,k) = \Pr\left[\widetilde{T} = k\right]$. Consequently, Theorem 8.7(i) is equivalent to $\mathrm{d_{TV}}\left(A^*, \min\left(\widetilde{T}, n\right)\right) \to 0$.

We obtain

$$A(k) \xrightarrow{d} a + \sum_{j=1}^{k} \xi_j = a + k + \widetilde{S}_k \text{ for } k \in [t] \text{ jointly,}$$

and thus $\Pr[T = k] \to \Pr\left[\widetilde{T} = k\right]$ for $k \leq K$, and $\Pr[T > K] \to \Pr\left[\widetilde{T} > K\right]$.

Since K is arbitrary, we have shown $\Pr[A^* = k] = \Pr[T = k] \to \Pr\left[\widetilde{T} = k\right] = \zeta(a,c,k)$ for every finite $k \geq 1$. Furthermore, $\Pr[T > K] - \Pr\left[\widetilde{T} > K\right] \to 0$ for any fixed K, and a standard argument shows that there exists a sequence $K_n \to \infty$ such that $\Pr[T > K_n] - \Pr\left[\widetilde{T} > K_n\right] \to 0$, and thus $\Pr[T > K_n] \to \Pr\left[\widetilde{T} = \infty\right]$. On the other hand, Lemma 8.6 with $\omega(n) = K_n$ shows that $\Pr[K_n \leq T < n] \to 0$. Consequently, $\Pr[T = n] = \Pr[T > K_n] + o(1) \to \Pr\left[\widetilde{T} = \infty\right] = \zeta(a,c)$.

It is clear that $\zeta(a,c,k) = \Pr\left[\widetilde{T} = k\right] > 0$ for every $k \geq a$. To see that also $\zeta(a,c) = \Pr\left[\widetilde{T} = \infty\right] > 0$, note that, see (8.15), $\mathbb{E}[\xi_k] = p(r_{\min})\binom{k-1}{r_{\min}-1}c^{r_{\min}} \to \infty$ as $k \to \infty$. Hence, there is some K_0 such that $\mathbb{E}[\xi_{K_0}] > 1$. Since ξ_k stochastically dominates ξ_{K_0} for $k \geq K_0$, it follows that if the process reaches K_0 without stopping, the continuation dominates (up to a change of time) a Galton-Watson branching process with offspring distribution ξ_{K_0}, which is supercritical and thus has a positive probability of living forever, see Section 2.4.2. Hence, $\Pr\left[\widetilde{T} = \infty\right] > 0$. □

Under the assumption that $p \gg n^{-\frac{1}{r_{\min}}}$, we show that the initial set percolates as long as $a \geq r_{\min}$.

Theorem 8.8 *[cf. [30, Theorem 5.8]] Suppose that $p \gg cn^{-\frac{1}{r_{min}}}$ and $a \geq r_{min}$. Then $A^* = n$ w.h.p.*

Proof It suffices to consider $a = r_{\min}$. Thus assume $a = r_{\min}$ and consider the vertices activated in the first round, that is, at time $t = r_{\min}$. There are $S(r_{\min}) \in \mathrm{Bin}\,(n - r_{\min}, p^r_{\min})$ such vertices. Note that, see (4.3), $\overline{\pi}(r_{\min}) = \sum_{r\in\mathcal{R}} p(r)\Pr[\mathrm{Bin}\,(r_{\min}, p) = r_{\min}] = p^{r_{\min}}$. Consequently, $\mathbb{E}[S(r_{\min})] = (n -$

$r_{\min})p^{r_{\min}} \to \infty$. Let $\omega(n) = \frac{\mathbb{E}[S(r_{\min})]}{2}$, so $\omega(n) \to \infty$. It follows from the Chernoff bounds that

$$\Pr\left[S(r_{\min}) \leq \omega(n)\right] \leq e^{-\frac{(n-r_{\min})p^{r_{\min}}}{6}} \to 0,$$

that is, w.h.p. $S(r_{\min}) > \omega(n)$. Hence, w.h.p. for all $t \in [r_{\min}, \omega(n)]$, we have $A(t) \geq A(r_{\min}) > S(r_{\min}) > \omega(n) \geq t$. Together with the trivial $A(t) \geq a = r_{\min} > t$ for $t < r_{\min}$ and Lemma 8.6, this shows that w.h.p. $A(t) > t$ for all $t \leq n-1$, and thus $A^* = T = n$. □

Chapter 9

Other random graph models

In this chapter, we analyze the bootstrap percolation process for a different underlying random graph model, namely a directed random graph with an arbitrary distribution for the out-degrees. First, we refine the step model introduced in Section 3.1.1 to adapt to this specific graph model and then, under the assumptions listed in Section 9.2, provide approximations for the relevant quantities to model the bootstrap percolation process, similar to the ones in Chapter 4.3, in Section 9.3. Using these estimates, we prove a slightly modified sharp threshold result in Section 9.4.

9.1 Model

We introduce the formal model for the bootstrap percolation process with an underlying directed random graph $G_n = (V_n, E)$, where the direction of an edge indicates the course of spread of activation.

Each vertex $i \in V_n$ independently draws its expected out-degree $d(i)$ from a certain distribution. Then it selects each vertex independently with probability $p_{d(i)} := \frac{d(i)}{n-1}$ as its neighbor, also see [34]. Let $\mu := \mathbb{E}[d(i)]$ be the mean and $\mathcal{D} \subseteq \{0, 1, \ldots, n-1\}$ be the support of the degree distribution, and use $p(d)$ to denote the corresponding probability density function. We partition the set of active, used vertices $\mathcal{Z}(t)$ into sets of vertices with equal degrees, as illustrated in Figure 9.1. Let

$$\mathcal{Z}_d(t) := \{v \in \mathcal{Z}(t) \colon \deg(v) = d\}$$

denote the set of all active vertices with degree d and $Z_d(t) := |\mathcal{Z}_d(t)|$ its size. Note that $(Z_d(t))_{d\in\mathcal{D}}$, recalling $Z(t) = t$, are multinomial-distributed with parameters t and $(p(d))_{d\in\mathcal{D}}$. Consequently,

$$\mathbb{E}[Z_d(t)] = tp(d) \tag{9.1}$$

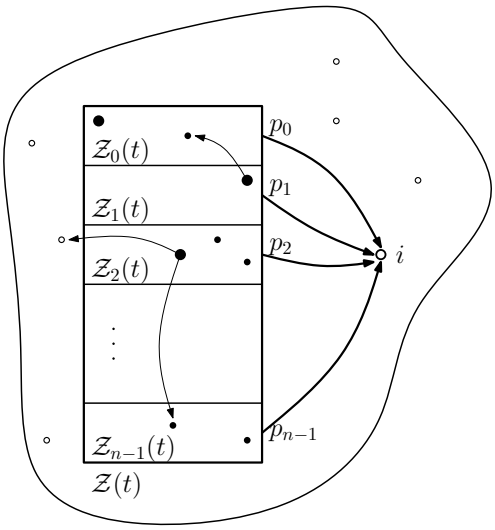

Figure 9.1: The unused vertices are drawn as circles, and the used ones are depicted as disks, summarized in box $\mathcal{Z}(t)$. We subdivide this box into smaller boxes $\mathcal{Z}_d(t)$ of vertices with the same degree d. Exponents of these groups are drawn as larger disks. From the view of vertex i, each vertex in the subbox $\mathcal{Z}_d(t)$ is connected to it with the same probability $p_d := \frac{d}{n-1}$.

and $\sum_{d\in\mathcal{D}} Z_d(t) = t$. Let $X_d(t) \in \text{Bin}\left(Z_d(t), \frac{d}{n-1}\right)$ be the number of edges from $\mathcal{Z}_d(t)$ to an inactive vertex. Then the total number of edges to a vertex i originating from $\mathcal{Z}(t)$ can be expressed as

$$M_i(t) = \sum_{d\in\mathcal{D}} X_d(t).$$

The probability that vertex i is active in step t is thus given by

$$\widehat{\pi}_i(t) := \Pr\left[M_i(t) \geq r_i\right].$$

To simplify notation, we define

$$\widehat{\pi}^r(t) := \Pr\left[M_i(t) \geq r\right].$$

Analogously to the calculation in (4.6) we can get rid of the dependence on i, thus write

$$\widehat{\pi}_i(t) = \sum_{r\in\mathcal{R}} p(r)\widehat{\pi}^r(t) = \widehat{\pi}(t),$$

where $\widehat{\pi}(t) := \mathbb{E}\left[\widehat{\pi}^{r_i}(t)\right]$. Hence,

$$S(t) \in \text{Bin}\left(n-a, \widehat{\pi}(t)\right). \tag{9.2}$$

9.2 Assumptions

In this section, we state the assumptions we make about the probability distribution for the out-degrees of the vertices. To simplify notation, we

first introduce some definitions and then list the conditions the probability distribution has to satisfy.

For a $d_0 \in D$, define $\mathcal{D}_< := \{d \in \mathcal{D} \colon d \leq d_0\}$ and $\mathcal{D}_> := \mathcal{D} \setminus \mathcal{D}_<$. We introduce the set

$$\mathcal{D}_+ := \left\{ d \in \mathcal{D}_< \colon p(d)d \geq \frac{\varepsilon\mu}{d_0} \right\} \tag{9.3}$$

for some $\varepsilon > 0$ and let $\mathcal{D}_- := \mathcal{D} \setminus \mathcal{D}_+$. Further, we define $\mathcal{S}_< := \sum_{d\in\mathcal{D}_<} p(d)d$ and analogously the sets $\mathcal{S}_>$, $\mathcal{S}_+$, and $\mathcal{S}_-$ based on $\mathcal{D}_>$, $\mathcal{D}_+$, and $\mathcal{D}_-$, respectively.

We assume that

$$1 \ll \mu \ll n^{\frac{r_{\min}-1}{r_{\min}}} \tag{9.4}$$

and that there exists a distinguished degree $d_0 \in \mathcal{D}$ such that $1 \ll d_0 \ll n$,

$$\mathcal{S}_< \gg \mathcal{S}_>, \tag{9.5}$$

$$\sum_{d\in\mathcal{D}_+} Z_d(t) = t - o(t), \tag{9.6}$$

and

$$(\log d_0)^2 \, d_0^2 \mu = \mathcal{O}(n). \tag{9.7}$$

Note that (9.6) implies that we can assume w.h.p. $\sum_{d\in\mathcal{D}_+} Z_d(t) = t_0$ for $t_0 \in [t - o(t), t]$, which, put simply, means that almost all used vertices have degrees from $\mathcal{D}_+$. We denote this event that $\sum_{d\in\mathcal{D}_+} Z_d(t) = t_0$ for some $t_0 = t\,(1 + o(1))$ by ξ_{t_0}.

9.3 Approximations

In the following, we first derive an estimate for $M_i(t)$ under the assumptions stated in Section 9.2, and then approximate $\widehat{\pi}(t)$ by a Poisson distribution. Because of the similarity of (4.7) and (9.2), the approximations in Section 4.2 work analogously, replacing $\overline{\pi}(t)$ by $\widehat{\pi}(t)$, wherefore we omit the proofs and just state that Lemmas 4.1, 4.2, and 4.3 also hold in the case of a graph with arbitrary degree distribution.

To simplify calculations, we introduce some notation that will be used throughout the subsequent sections. We define

$$p := \frac{1}{n-1} \sum_{d\in\mathcal{D}_+} p(d)d, \tag{9.8}$$

and $\lambda(t) := pt$. Further, we let

$$t_c(r) := \left(\frac{(r-1)!}{np^r} \right)^{\frac{1}{r-1}}. \tag{9.9}$$

9.3.1 Restriction to a subset of all possible degrees

In this section, we prove some properties of the set $\mathcal{D}_+$, defined by (9.3), which allow us to only consider degrees in $\mathcal{D}_+$ in order to analyze the asymptotic behavior of the bootstrap percolation process.

The following lemma shows that the mean of the degree distribution is dominated by the degrees in $\mathcal{D}_+$.

Lemma 9.1 *We have* $\mathcal{S}_+ = \Theta(\mu)$.

Proof Assumption (9.5) implies that $\mathcal{S}_< = \mu - \mathcal{S}_> = \mu - o(\mu)$. Hence, $\mathcal{S}_+ + \mathcal{S}_- = \mathcal{S}_< = \mu(1 - o(1))$. Recalling the definition of $\mathcal{D}_+$ in (9.3), we get that $\mathcal{S}_- < \sum_{d \in \mathcal{D}_-} \frac{\varepsilon\mu}{d_0} = \frac{\varepsilon\mu|\mathcal{D}_-|}{d_0} < \varepsilon\mu$ and thus $\mathcal{S}_+ > \mu(1 - \varepsilon - o(1))$. Therefore $\mathcal{S}_+ = \Theta(\mu)$. □

The preceding lemma and (9.8) yield

$$p = \Theta\left(\frac{\mu}{n}\right). \tag{9.10}$$

Next, we show that for all contributing degrees in $\mathcal{D}_+$ the number of active, used vertices with this degree typically is highly concentrated around the expected number of such vertices.

Lemma 9.2 *We have w.h.p. for all* $d \in \mathcal{D}_+$ *that* $Z_d(t) = (1 + o(1))\, tp(d)$.

Proof We first show that $\mathbb{E}[Z_d(t)] = \omega(1)$ for all $d \in \mathcal{D}_+$ and $t = \Theta(t_c(r))$. Recalling (9.9) and using (9.10), we have

$$t_c(r) = \Theta\left(\left(\frac{1}{p^r n}\right)^{\frac{1}{r-1}}\right) = \Theta\left(\left(\frac{n^{r-1}}{\mu^r}\right)^{\frac{1}{r-1}}\right) = \Theta\left(\frac{n}{\mu^{\frac{r}{r-1}}}\right),$$

and thus, with (9.3), (9.7), and $\max \mathcal{D}_+ \le d_0$,

$$tp(d) = \Theta\left(\frac{n}{\mu^{\frac{r}{r-1}}} p(d)\right) = \Omega\left(\frac{n}{d_0^2 \mu^{\frac{1}{r-1}}}\right) = \omega\left(\frac{n}{d_0^2 \mu}\right) = \omega((\log d_0)^2). \tag{9.11}$$

Recalling (9.1), this yields $\mathbb{E}[Z_d(t)] = \omega\left((\log d_0)^2\right)$. Conditioned on the event ξ_{t_0}, we have

$$\Pr[Z_d(t) = z_d(t) \mid \xi_{t_0}] = \Pr[\mathrm{Bin}(t_0, p(d)) = z_d(t)]$$

for all $d \in \mathcal{D}_+$. Using the Chernoff bounds, we get for all $\delta_d(n) = o(1)$ with $\delta_d(n) \gg \frac{1}{\sqrt{\log(d_0)}}$ a lower bound

$$\begin{aligned}
&1 - \Pr\left[Z_d(t) \leq (1-\delta_d(n))\, t_0 p(d) \mid \xi_{t_0}\right] - \Pr\left[Z_d(t) \geq (1+\delta_d(n))\, t_0 p(d) \mid \xi_{t_0}\right] \\
&= 1 - \Pr\left[\mathrm{Bin}\left(t_0, p(d)\right) \leq (1-\delta_d(n))\, t_0 p(d)\right] \\
&\quad - \Pr\left[\mathrm{Bin}\left(t_0, p(d)\right) \geq (1+\delta_d(n))\, t_0 p(d)\right] \\
&\geq 1 - e^{-\frac{t_0 p(d) \delta_d(n)^2}{2}} - e^{-\frac{t_0 p(d) \delta_d(n)^2}{3}} = 1 - o\left(\frac{1}{d_0}\right)
\end{aligned}$$

for $\Pr\left[Z_d(t) = (1+\delta_d(n))\, t_0 p(d) \mid \xi_{t_0}\right]$, since $t_0 p(d) \delta_d(n)^2 = \omega\left(\log d_0\right)$, by (9.11) and the choice of $\delta_d(n)$. Applying the union bound yields a lower bound

$$1 - \sum_{d=0}^{d_0} \Pr\left[Z_d(t) \neq (1+\delta_d(n))\, t_0 p(d) \mid \xi_{t_0}\right] = 1 - d_0 o\left(\frac{1}{d_0}\right) = 1 - o(1)$$

for the probability

$$\Pr\left[\forall d \in \mathcal{D}_+ \colon Z_d(t) = (1+\delta_d(n))\, t_0 p(d) \mid \xi_{t_0}\right].$$

As the event ξ_{t_0} occurs with probability $1 - o(1)$, the claim follows. □

The preceding lemma implies that we can assume w.h.p.

$$Z_d(t) = tp(d)\,(1 + o(1))$$

for all $d \in \mathcal{D}_+$. Conditioned on values

$$z_d(t) \in \left[tp(d)\,(1-\delta_d(n)), tp(d)\,(1+\delta_d(n))\right]$$

for all $d \in \mathcal{D}_+$, the values X_d are independent. We denote the event that values from these intervals are taken by ξ. Note that this happens with probability $1 - o(1)$.

9.3.2 Approximation of the mean

In this section, we first present a simplified expression for the number of edges leading from active, used vertices to an inactive vertex and then provide an approximation for $\widehat{\pi}$, and hence for $\mathbb{E}\left[S(t)\right]$. By (9.10) and our assumption (9.4), p, defined by (9.8), satisfies (4.1), which suggests that the previous proof structures can be adopted.

Recall that $M_i(t)$ is a sum of dependent random variables with binomial distributions $\mathrm{Bin}\left(Z_d(t), \frac{d}{n-1}\right)$. However, once the random variables $Z_d(t)$

take a fixed value $z_d(t)$, the summands become independent, and thus Le Cam's theorem applies. For some $r \geq r_{\min}$ and $t = \Theta(t_c(r))$, we have w.h.p.

$$\begin{aligned}
\widehat{\pi}^r(t) &= \sum_{j=r}^{t} \Pr[M_i(t) = j] \\
&< \left(\sum_{j=r}^{t} \frac{\lambda(t)^j e^{-\lambda(t)}}{j!} + 2 \sum_{d \in \mathcal{D}_+} tp(d)\,(1+o(1)) \left(\frac{d}{n-1}\right)^2 \right) (1+o(1)) \\
&\leq \sum_{j=r}^{t} \frac{\lambda(t)^j e^{-\lambda(t)}}{j!} \left(1 + \frac{2\sum_{d \in \mathcal{D}_+} (1+o(1)) \left(\frac{d}{n-1}\right)^2}{\frac{\lambda(t)^r e^{-\lambda(t)}}{r!}} \right) \\
&\leq \sum_{j=r}^{t} \frac{\lambda(t)^j e^{-\lambda(t)}}{j!} \left(1 + \mathcal{O}\left(\frac{\frac{d_0}{n} \sum_{d=0}^{d_0} tp(d) \frac{d}{n}}{\left(\sum_{d=0}^{d_0} tp(d) \frac{d}{n}\right)^r} \right) \right) \\
&= \sum_{j=r}^{t} \frac{\lambda(t)^j e^{-\lambda(t)}}{j!} \left(1 + \mathcal{O}\left(\frac{d_0}{n\,(pt)^{r-1}} \right) \right) \\
&= \sum_{j=r}^{t} \frac{\lambda(t)^j e^{-\lambda(t)}}{j!} \left(1 + \mathcal{O}(d_0 p)\right) \\
&= \sum_{j=r}^{t} \frac{\lambda(t)^j e^{-\lambda(t)}}{j!} (1+o(1)) = \sum_{j=r}^{t} \frac{\lambda(t)^j}{j!} (1+o(1)),
\end{aligned}$$

using Theorem 2.5, $\lambda(t) = o(1)$, $(pt)^{r-1} = \Theta\left(\frac{1}{np}\right)$, $t \gg 1$, and $d_0 p = \Theta\left(\frac{d_0 \mu}{n}\right) = o(1)$, by (9.10) and (9.7).

Analogously, we get for $t = \Theta(t_c)$ w.h.p.

$$\widehat{\pi}^r(t) \geq \sum_{j=r}^{t} \frac{\lambda(t)^j}{j!} (1+o(1)).$$

Combining these two bounds yields w.h.p.

$$\widehat{\pi}^r(t) = \sum_{j=r}^{t} \frac{\lambda(t)^j}{j!} (1+o(1)).$$

As in (4.19), since $t \gg 1$ and $\lambda(t) = pt = o(1)$, we can approximate $\widehat{\pi}(t)$ by

$$\widehat{\pi}(t) = \sum_{r \in \mathcal{R}} p(r) \widehat{\pi}^r(t) = p(r_{\min}) \sum_{j=r_{\min}}^{t} \frac{\lambda(t)^j}{j!} (1+o(1)),$$

and thus, analogously to (4.21), we have w.h.p.

$$\widehat{\pi}(t) = p(r_{\min}) \frac{(pt)^{r_{\min}}}{r_{\min}!} (1+o(1)) \tag{9.12}$$

for $tp \ll 1$. Hence, w.h.p. the bootstrap percolation process on the directed random graph can be approximated, at least around the threshold, by the same deterministic function as the one on the Erdős-Rényi graph. This leads w.h.p. to the same estimates a_c and t_c for the threshold. Moreover, Lemma 5.1 can be proved analogously.

9.4 Sharp threshold result

In this section, we prove a modified sharp threshold result. That is, we show that a subcritical starting set leads to no percolation w.h.p., whereas a linear fraction of the vertices are active at the end in the supercritical case.

As mentioned above, the findings from Section 4.2, especially Lemma 4.3, and also Lemma 5.1 can be adopted. This implies that the proof of the sharp threshold result is very similar to the one of Theorems 5.2 and 5.6. Note, however, that we only provide approximations for $t \approx t_c$ and not for $t \approx n$, wherefore we get a weaker result.

9.4.1 Subcritical case

We show that a starting set below the threshold, that is, $a \sim \alpha a_c$ for some $\alpha < 1$, implies that w.h.p. the process does not percolate, as with the Erdős-Rényi random graph model.

Theorem 9.3 *[cf. Theorem 5.2] If* $\frac{a}{a_c} \to \alpha < 1$, *then* $A^* = \big(\varphi(\alpha) + o_p(1)\big)\, t_c$, *where* $\varphi(\alpha)$ *is the unique root in* $[0,1]$ *of* (5.1). *Further,* $\frac{A^*}{a} \xrightarrow{p} \varphi_1(\alpha)$, *where* $\varphi_1(\alpha)$ *is given by* (5.2).

Proof The proof is very similar to the one of Theorem 5.2. □

9.4.2 Supercritical case

In the supercritical case, when $a > (1+\delta)a_c$, we have w.h.p. at least a linear fraction of active vertices at the end of the bootstrap percolation process.

Theorem 9.4 *[cf. Lemma 5.1 and Theorem 5.6] If* $\frac{a}{a_c} \geq 1+\delta$, *for some* $\delta > 0$, *then* $A^* = \Omega(n)$.

Proof An argument similar to the one in the proof of Lemma 5.4, but omitting Case 4 and 5, yields that for any a w.h.p. $A(t) > t$ for all $t \in [3t_c, cn]$, with $c > 0$ a constant. Then, as at the beginning of the proof of Theorem 5.6, we show that $A(t) > t$, uniformly in $t \leq 3t_c$. Combining these results, we have $A(t) > t$ for all $t \leq cn$, and thus $A^* > cn$. □

Chapter 10

Conclusion

In this thesis, we prove that there is no fundamental difference between fixed and individual activation thresholds concerning the sharp threshold phenomenon on an Erdős-Rényi random graph model. The variability of the thresholds leads to the qualitatively same results, only minor quantitative alterations have to be made.

The additional introduction of a directed random graph model, which enables us to model an asymmetric bootstrap percolation process, yields similar findings. We present a sharp threshold result between no percolation with almost no additionally activated vertices and quasi percolation, where at least a linear fraction of all vertices is active at the end.

The straight-forward adaptability to individual activation thresholds is not that surprising, since we only consider a constant range of possible values. However, if we would allow the activation threshold to depend on the number of vertices, presumably a different behavior would arise. In particular, the existence of large degrees would inevitably prevent complete percolation and surely decelerate the percolation process. Likewise worthwhile to analyze is the influence of time-varying thresholds.

A further extension, mainly motivated by neuroscience, considers not only activating (excitatory), but also inhibitory vertices, which leads to a weakening of the spread of information. This idea is currently examined by Einarsson et al. on different graphs with fixed activation thresholds [23]. They succeeded in proposing a model that leads to a behavior completely different from a sharp threshold phenomenon. It would be interesting to enhance this model to permit individual activation thresholds.

Appendix A

Declaration of originality

ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Eigenständigkeitserklärung

Die unterzeichnete Eigenständigkeitserklärung ist Bestandteil jeder während des Studiums verfassten Semester-, Bachelor- und Master-Arbeit oder anderen Abschlussarbeit (auch der jeweils elektronischen Version).

Die Dozentinnen und Dozenten können auch für andere bei ihnen verfasste schriftliche Arbeiten eine Eigenständigkeitserklärung verlangen.

Ich bestätige, die vorliegende Arbeit selbständig und in eigenen Worten verfasst zu haben. Davon ausgenommen sind sprachliche und inhaltliche Korrekturvorschläge durch die Betreuer und Betreuerinnen der Arbeit.

Titel der Arbeit (in Druckschrift):

Bootstrap percolation with individual activation thresholds

Verfasst von (in Druckschrift):
Bei Gruppenarbeiten sind die Namen aller Verfasserinnen und Verfasser erforderlich.

Name(n):	**Vorname(n):**
Fischer	Manuela

Ich bestätige mit meiner Unterschrift:
- Ich habe keine im Merkblatt „Zitier-Knigge" beschriebene Form des Plagiats begangen.
- Ich habe alle Methoden, Daten und Arbeitsabläufe wahrheitsgetreu dokumentiert.
- Ich habe keine Daten manipuliert.
- Ich habe alle Personen erwähnt, welche die Arbeit wesentlich unterstützt haben.

Ich nehme zur Kenntnis, dass die Arbeit mit elektronischen Hilfsmitteln auf Plagiate überprüft werden kann.

Ort, Datum	**Unterschrift(en)**
Zürich, 16.09.2014	M. Fischer

Bei Gruppenarbeiten sind die Namen aller Verfasserinnen und Verfasser erforderlich. Durch die Unterschriften bürgen sie gemeinsam für den gesamten Inhalt dieser schriftlichen Arbeit.

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