

The Locality of Maximal Matching

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standard synchronous message-passing model of distributed computing

• undirected graph G = (V, E), n nodes, maximum degree Δ



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- each round, every node
 - receives messages (sent in previous round)
 - performs some computation
 - sends message to all its neighbors



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- each round, every node
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- unbounded message size



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 round complexity of a problem in the LOCAL

round complexity of a problem in the LOCAL model characterizes its locality every problem is trivially solvable in *O*(diameter) rounds



















Easy centralized problems: greedy solutions.





Matching: set of non-incident edges



Matching: set of non-incident edges

Maximal: no edge can be added



<u>Matching:</u> set of non-incident edges

<u>Maximal:</u> no edge can be added

greedy property!



Centralized (Sequential) Algorithm

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 $\mathbb{E}[$ #removed edges per round $] \geq c|E_i|$



 $\mathbb{E}[\text{#removed edges per round}] \ge c|E_i|$ $O(\log n)$ rounds w.h.p.

Our Result





improving over



improving over

O(log⁴ *n*) Hańćkowiak, Karoński, Panconesi [SODA'98, PODC'99]



improving over

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 $O(\Delta + \log^* n)$ Panconesi, Rizzi [DIST'01]

Overview of Results

Maximal Matching

- Maximal Matching
- Randomized Maximal Matching

Approximate Matching

- $(2 + \varepsilon)$ Approximate Maximum Matching
- $(2 + \epsilon)$ Approximate Maximum Weighted Matching
- $(2 + \varepsilon)$ Approximate Maximum B-Matching
- $(2 + \varepsilon)$ Approximate Maximum Weighted B-Matching
- ε Maximal Matching
- $(2 + \epsilon)$ Approximate Minimum Edge Dominating Set

 $O(\log^2 \Delta \cdot \log n)$ $O(\log^3 \log n + \log \Delta)$

$$O\left(\log^2 \Delta \cdot \log \frac{1}{\epsilon} + \log^* n\right)$$
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 $O(\log^2 \Delta)$ rounds

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I) 4 - Approximate Fractional Matching

 $O(\log \Delta)$ rounds

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 $O(\log \Delta)$ rounds

II) Rounding Fractional Bipartite Matching

 $O(\log^2 \Delta)$ rounds, O(1) loss
$O(\log \Delta)$ rounds







LOCAL Greedy Algorithm
$x_e = 2^{-\lceil \log \Delta \rceil}$ for all $e \in E$
repeat until all edges are blocked
mark half-tight nodes
block its edges
double value of unblocked edges



Fractional Maximum Matching
$$\max \sum_{e \in E} x_e$$

 $value of v$ $s.t.$ $\sum_{e \in E(v)} x_e \leq 1$
 $x_e \in [0,1]$ for all $v \in V$ $x_e \in [0,1]$

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$\max \sum_{e \in E} x_e$ value of v
s.t. $\sum_{e \in E(v)} x_e \le 1$ for all $v \in V$
$x_e \in [0,1]$ for all $e \in E$
v is half-tight if its value is $\geq \frac{1}{2}$
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using Locally Balanced Splitting, inspired by Hańćkowiak, Karoński, Panconesi [SODA'98,PODC'99]



Iterated Factor-2-Rounding using Locally Balanced Splitting

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Locally Balanced Splitting:

2-edge-coloring so that every node roughly balanced



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no constraint violated & no loss in total value (i.e., **perfect rouding**)



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no constraint violated & no loss in total value (i.e., **perfect rouding**) Perfect Splitting not possible in case of...

$O(\log^2 \Delta)$ rounds, O(1) loss



$O(\log^2 \Delta)$ rounds, O(1) loss





$O(\log^2 \Delta)$ rounds, O(1) loss





Sequential Perfect Splitting*

Repeat until all edges colored pick arbitrary cycle alternate

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LOCAL Almost-Perfect Splitting*

Decompose into edge-disjoint cycles In parallel, for all cycles

A) **Short cycles** of length $O(\log \Delta)$



B) Long cycles

chop at length $\Theta(\log \Delta)$

set boundary to 0

alternate **I** in between

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Repeat until all edges colored pick arbitrary cycle alternate

LOCAL Almost-Perfect Splitting*

Decompose into edge-disjoint cycles In parallel, for all cycles

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- B) Long cycles

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* by Hańćkowiak, Karoński, Panconesi [SODA'98,PODC'99] in O(log Δ)
* bipartite and even degree!

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alternate 🔳 🗌

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* by Hańćkowiak, Karoński, Panconesi [SODA'98,PODC'99] in $O(\log \Delta)$

Over all $O(\log \Delta)$ rounding iterations, total loss still constant!

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 $O(\log^2 \Delta)$ rounds



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 $O(\log^2 \Delta)$ rounds











Maximal

 $O(\log^2 \Delta)$ rounds

$$Oig(\log^2 \varDelta \cdot \log n ig)$$

Maximal

$O(\log^2 \Delta)$ rounds



 $O(\log^2 \Delta)$ rounds

Maximal



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 $O(\log^2 \Delta)$ rounds

Maximal



 $O(\log^2 \Delta)$ rounds

Maximal

 $O(\log^2 \Delta \cdot \log n)$



maximum matching size in remainder graph decreases by constant factor

 $O(\log^2 \Delta)$ rounds

Maximal

 $O(\log^2 \Delta \cdot \log n)$



maximum matching size in remainder graph decreases by constant factor

after $O(\log n)$ iterations, maximum matching size is 0, hence graph empty

Lower bound:
$$\Omega\left(\min\left\{\sqrt{\frac{\log n}{\log\log n}}, \frac{\log \Delta}{\log\log \Delta}\right\}\right)$$

Kuhn, Moscibroda, Wattenhofer [PODC'04]

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<u>Open Question:</u> $O(\log \Delta \cdot \log n)$?

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What is Locality of Maximal Matching?



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Thank you!

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