A Simple Parallel/Distributed Sampling Technique: Local Glauber Dynamics

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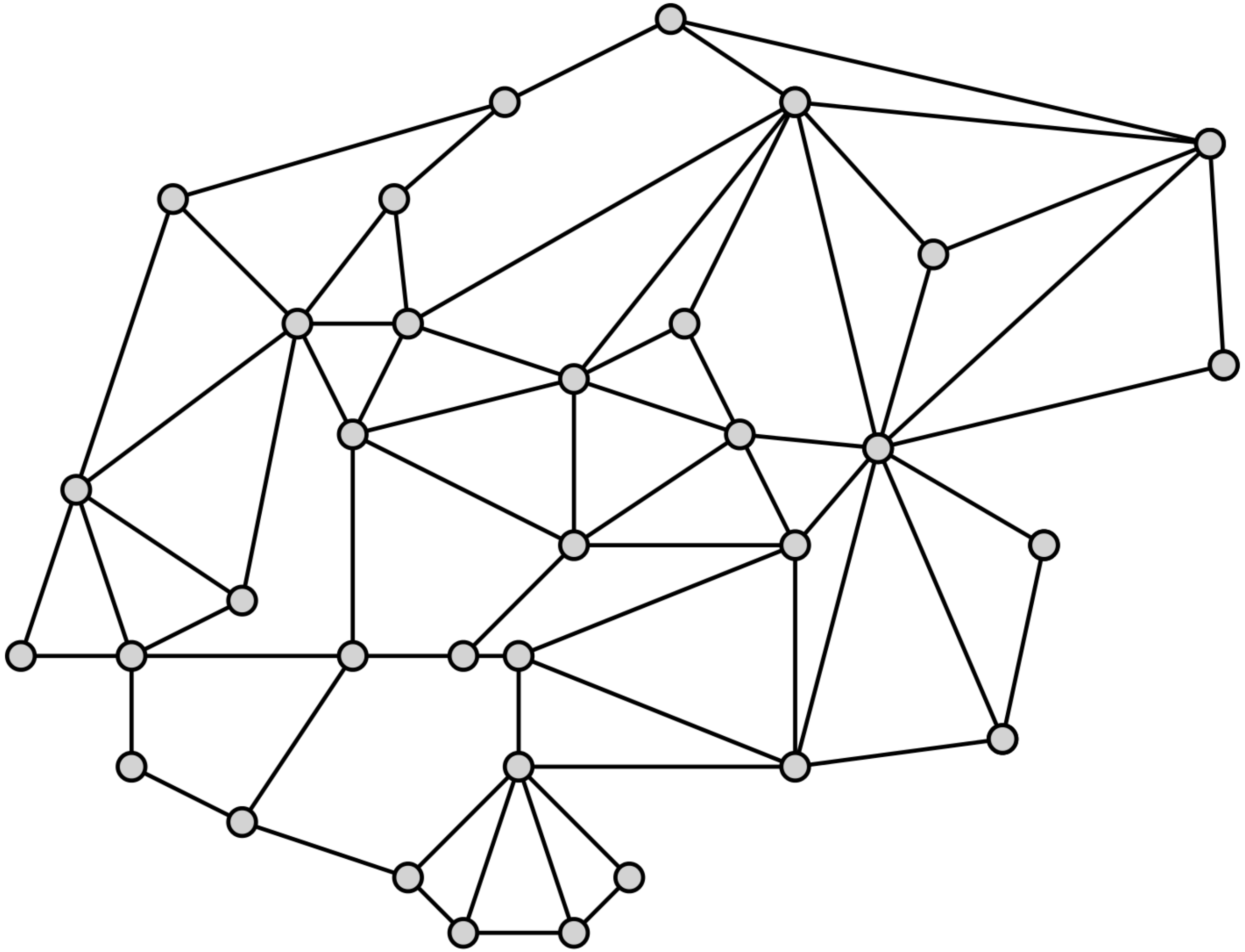
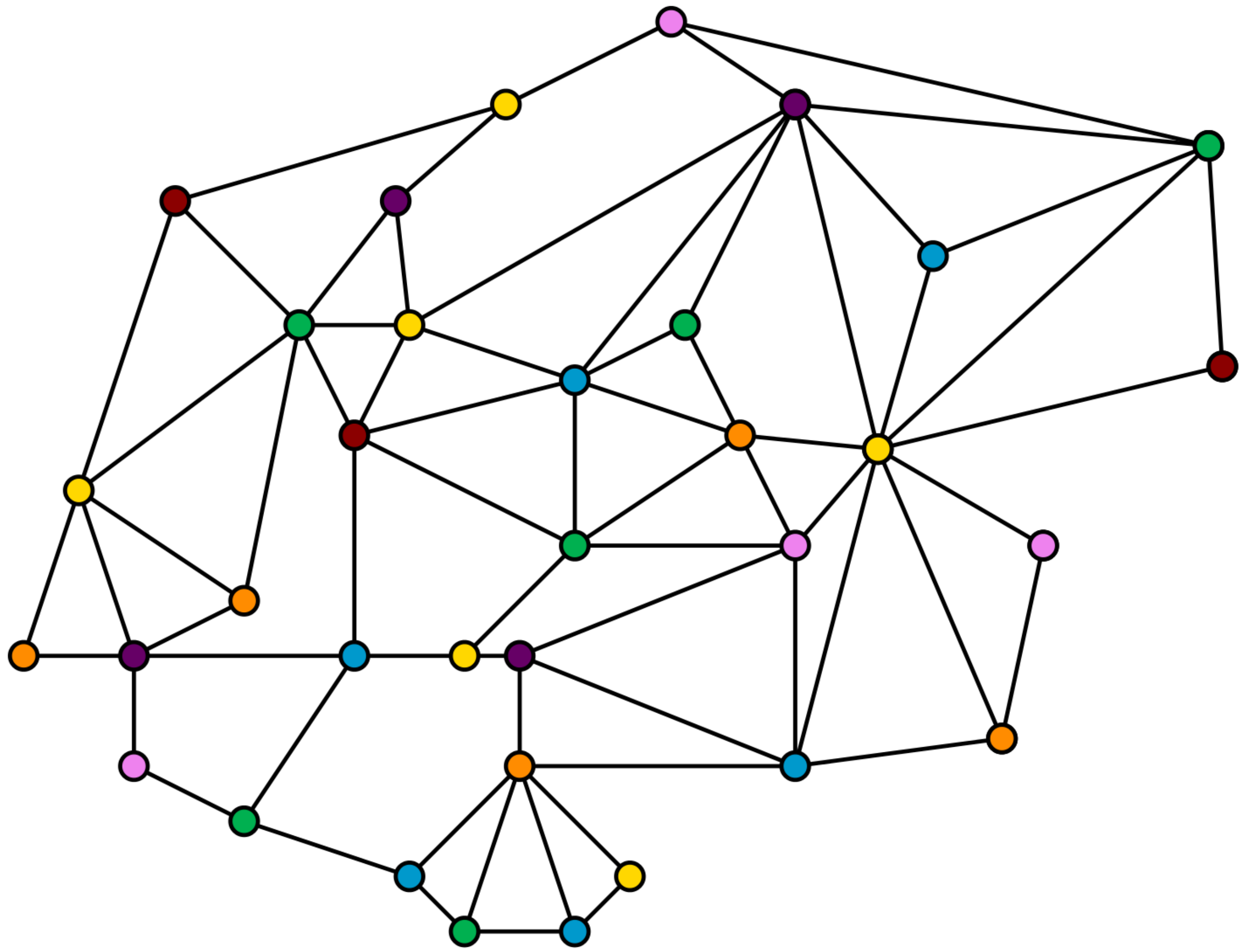
joint work with
Mohsen Ghaffari

Sampling
Proper Colorings

Sampling a Proper q-Coloring



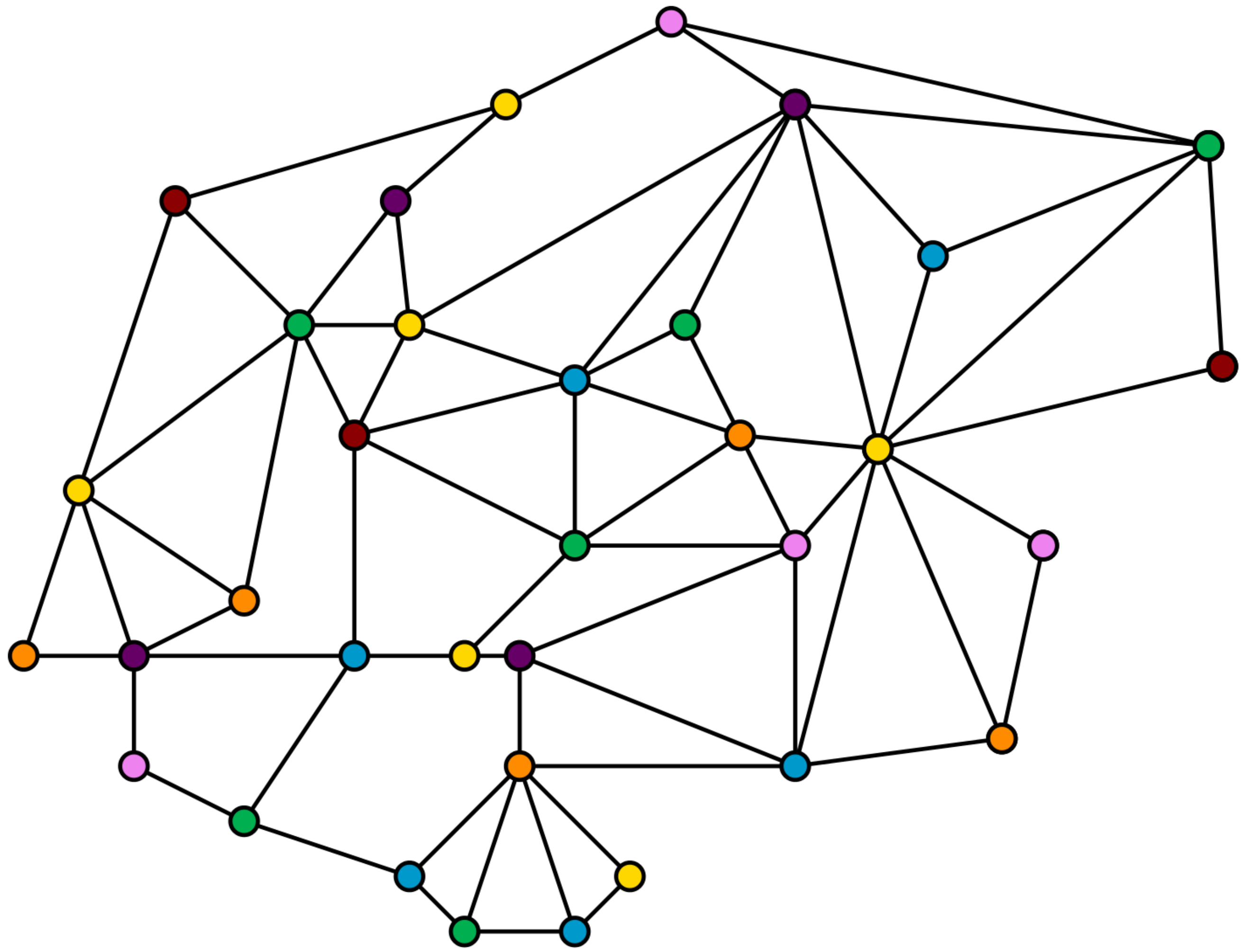
Sampling a Proper q-Coloring



Sampling a Proper q-Coloring

Markov chain

- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing



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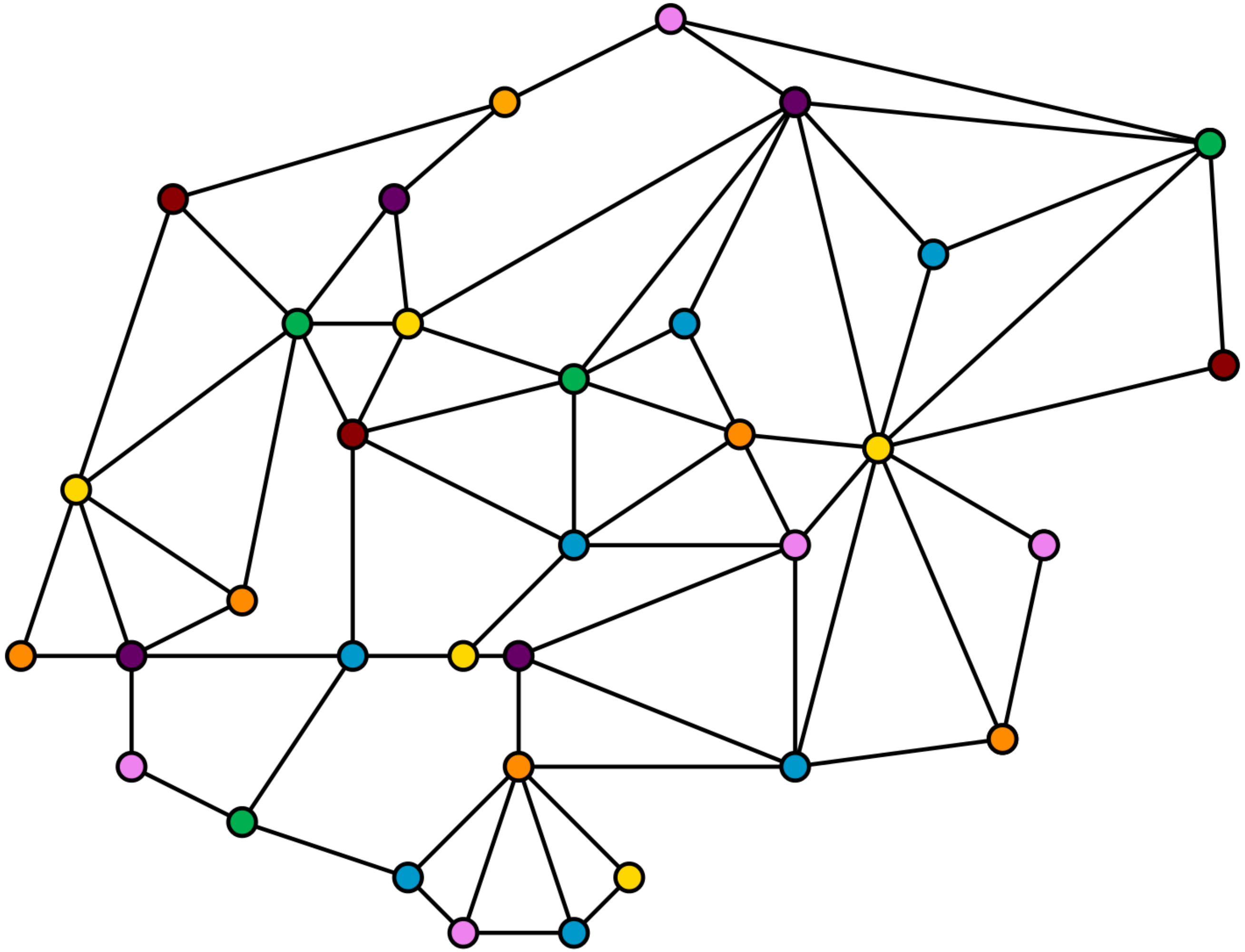

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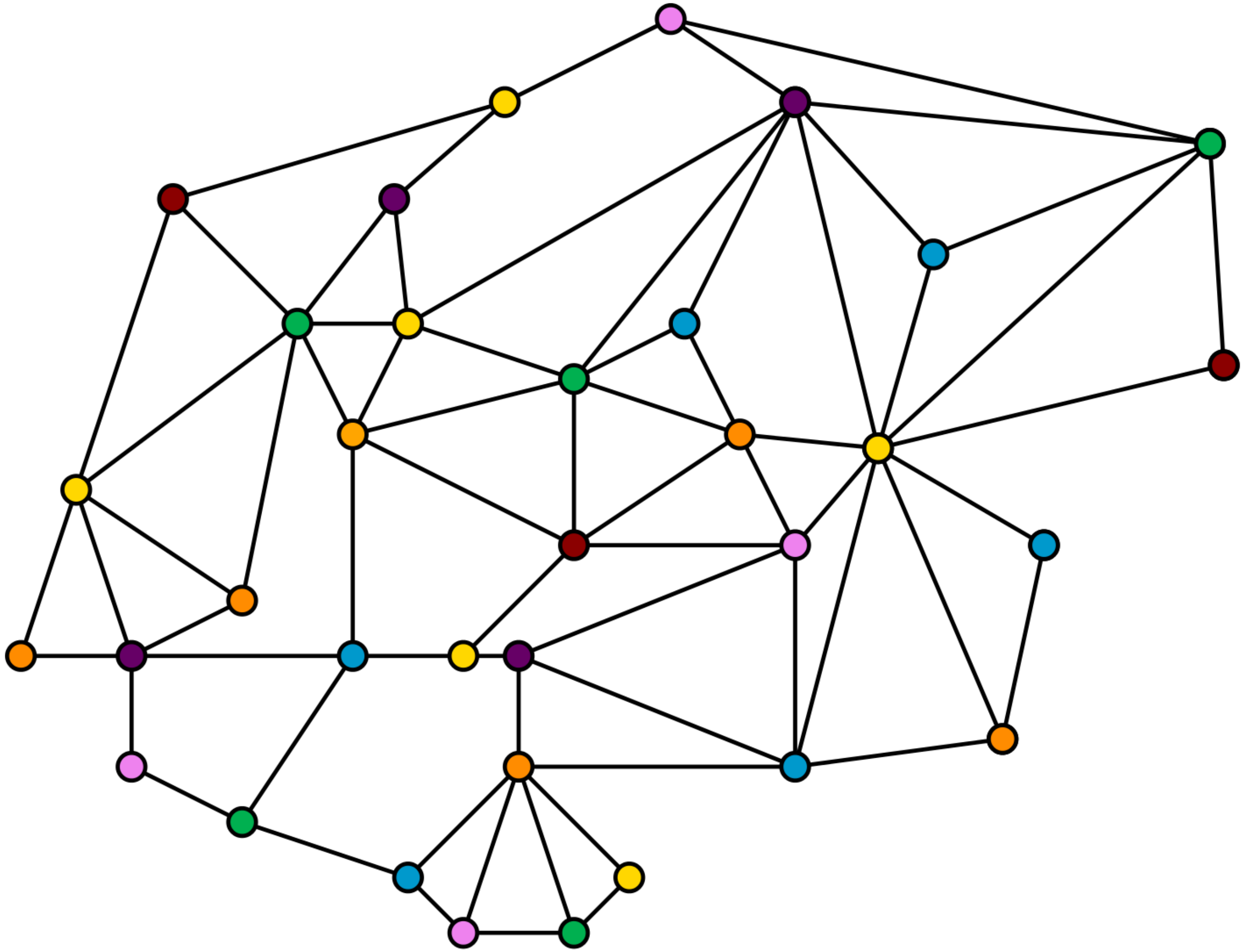

$t = 1$

Sampling a Proper q-Coloring

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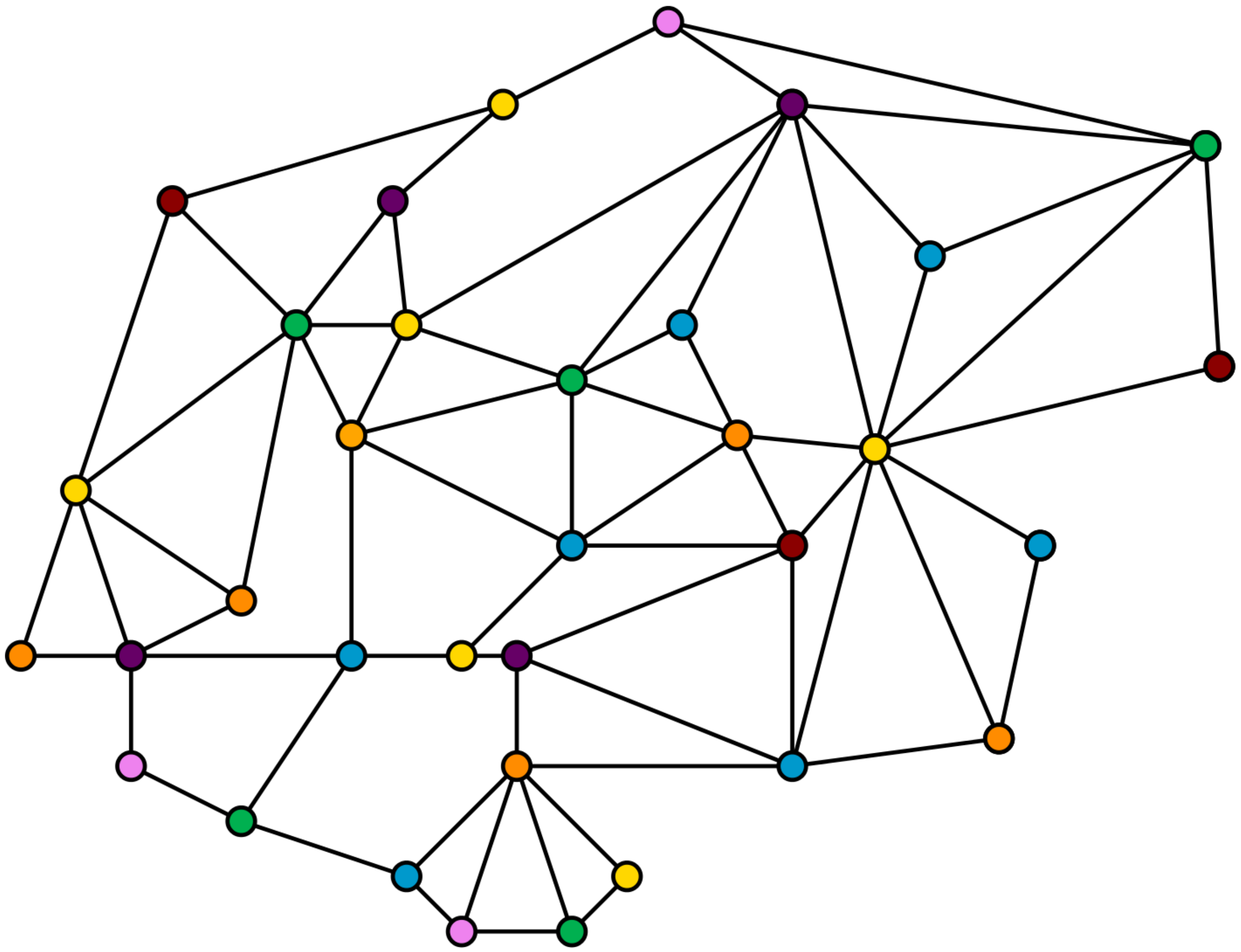

$t = 2$

Sampling a Proper q-Coloring

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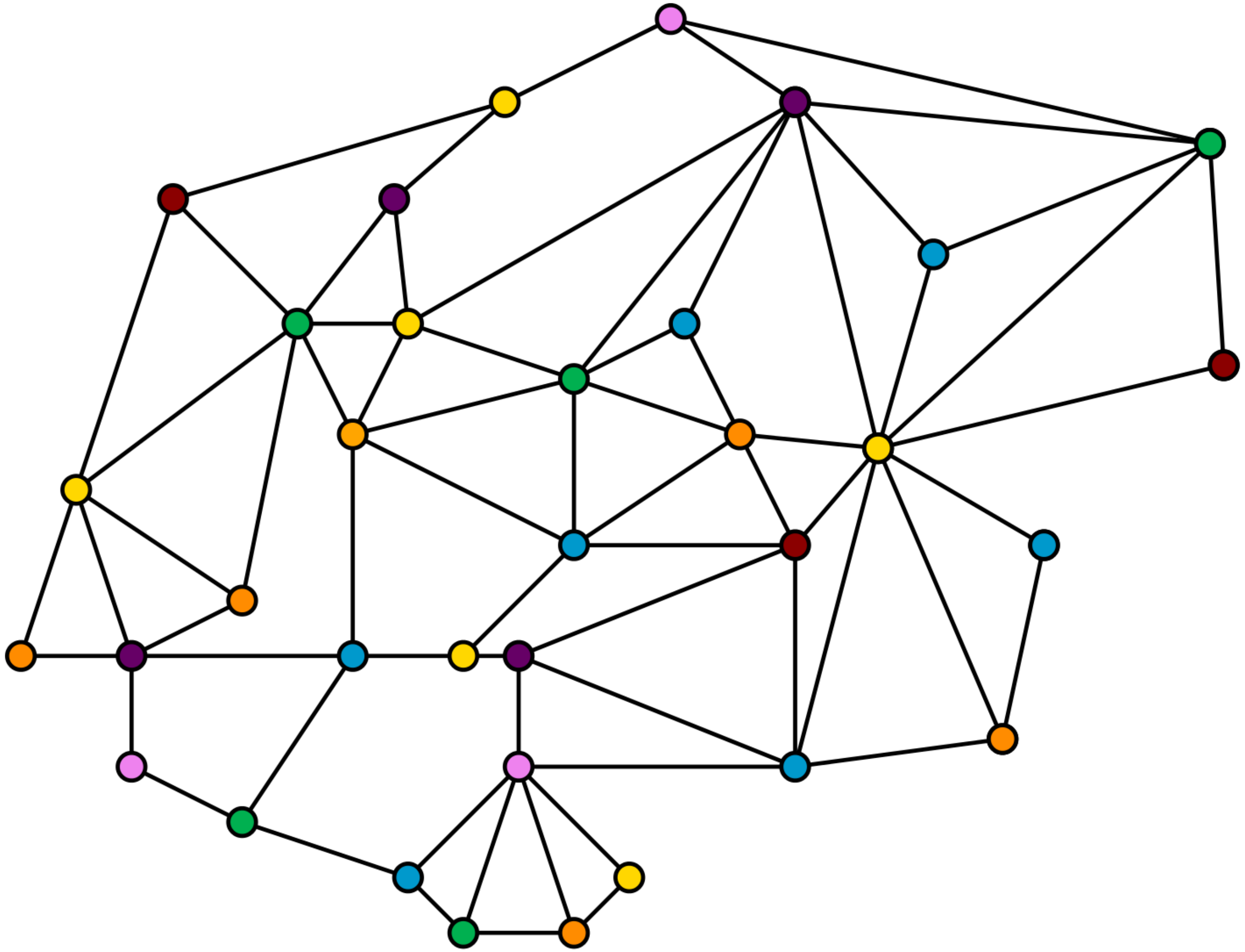
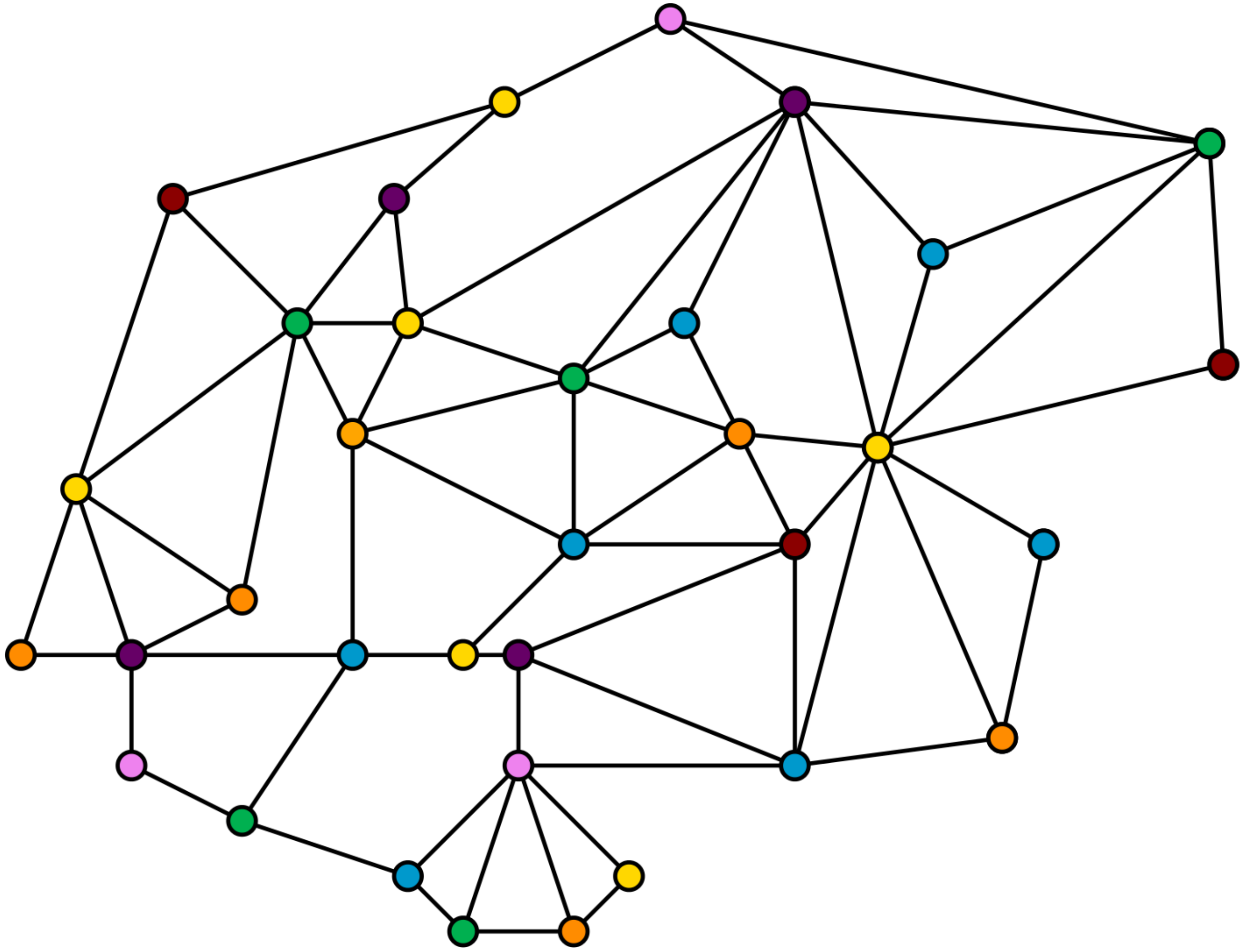


$t = 3$

Sampling a Proper q-Coloring

Markov chain

- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing



$t = 4$

Sampling a Proper q-Coloring

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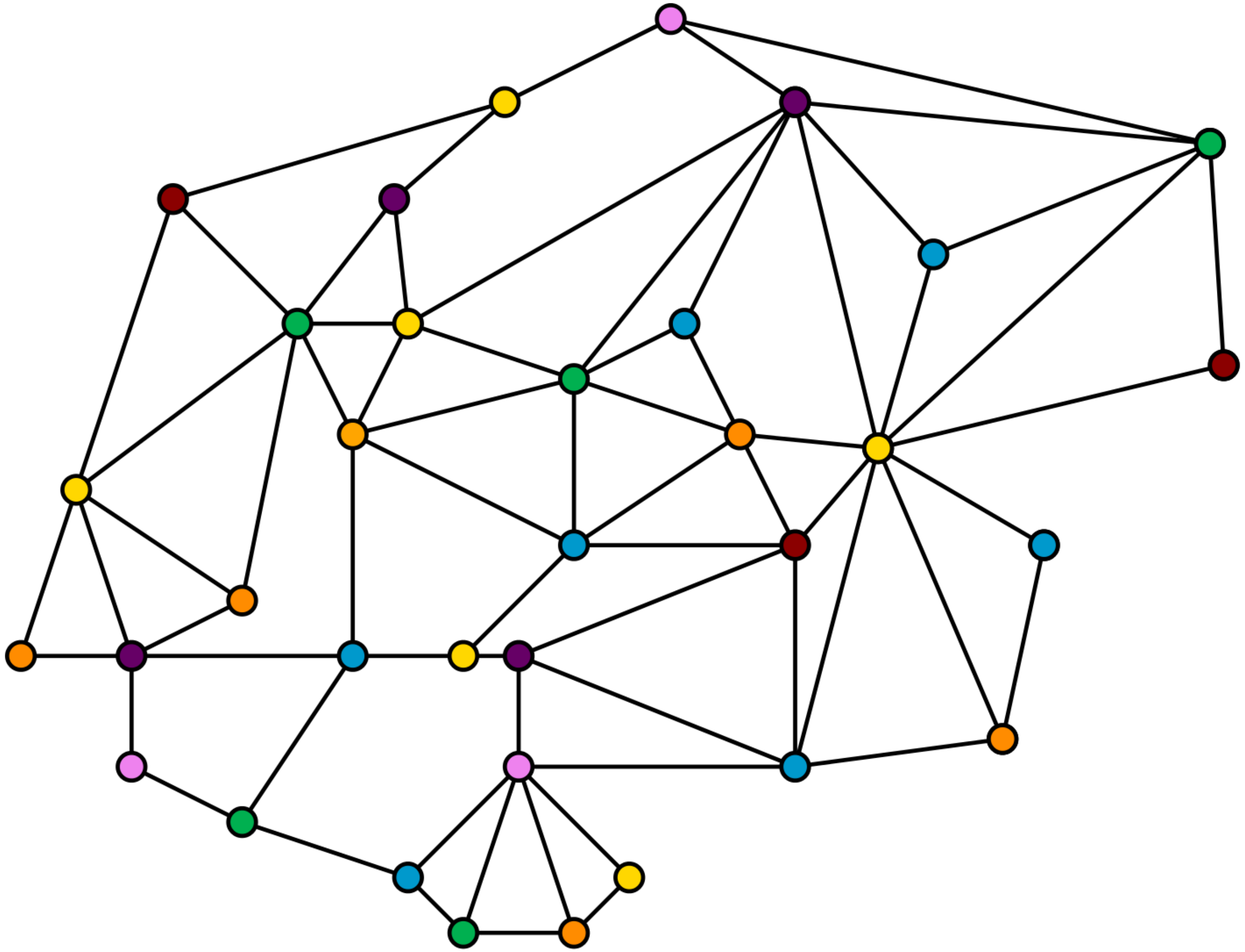

$t \geq t_{mix}$

Single-Site Glauber Dynamics

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update color of **a random node** to a **random color**, if proper

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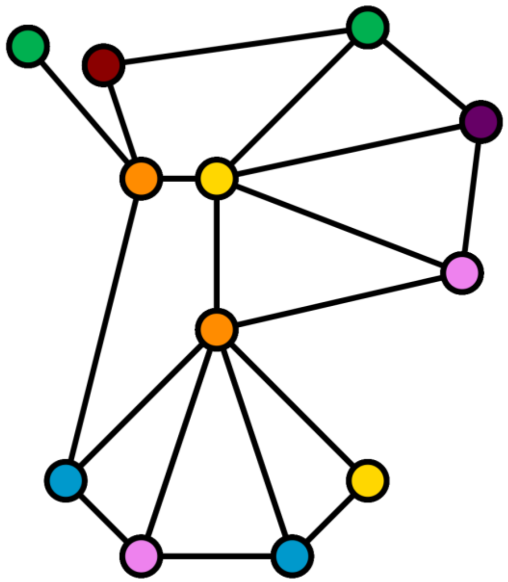

t

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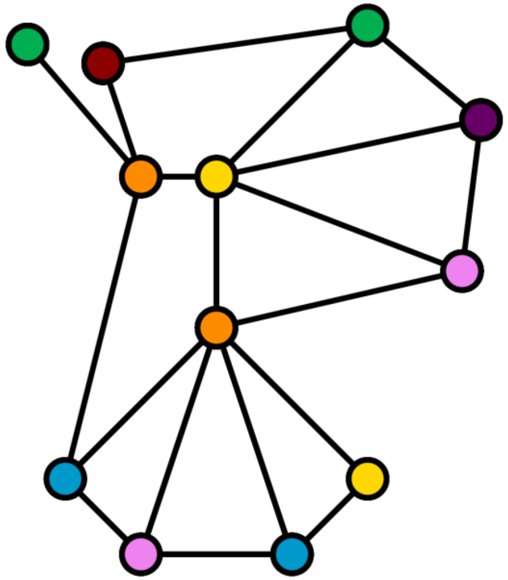
pick node $\mathbf{v}$ uniformly at random



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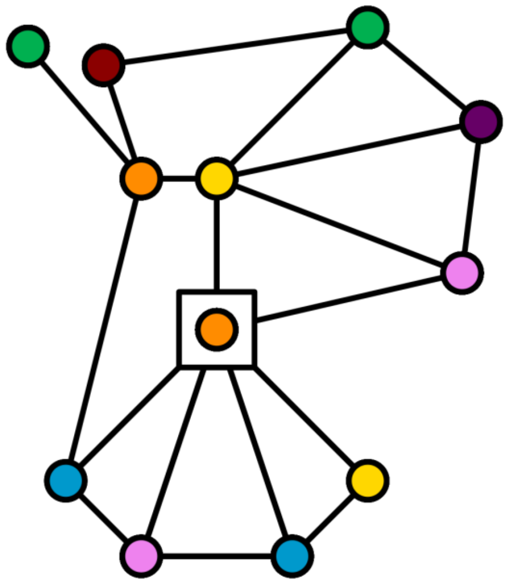

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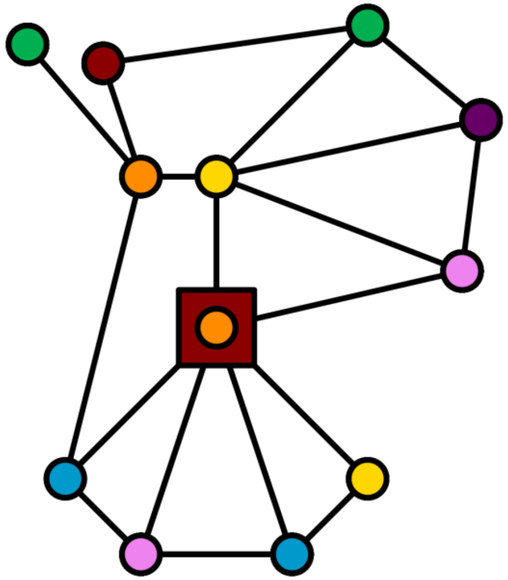

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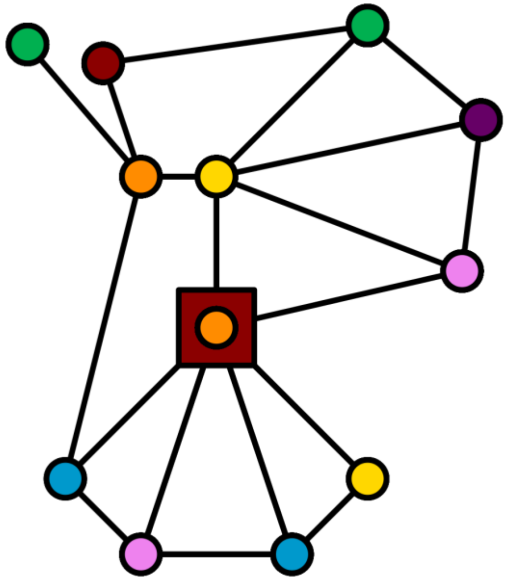

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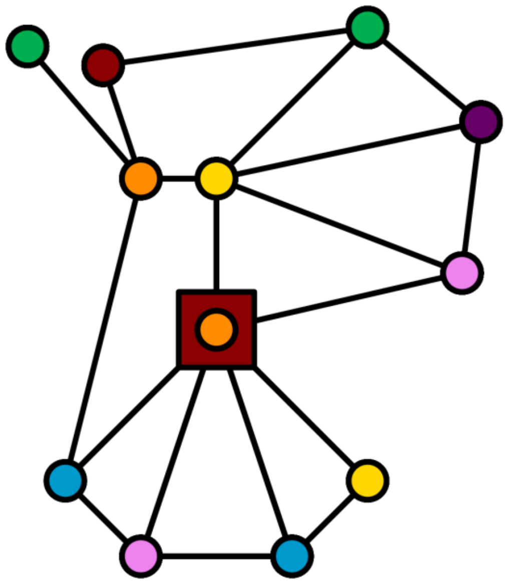

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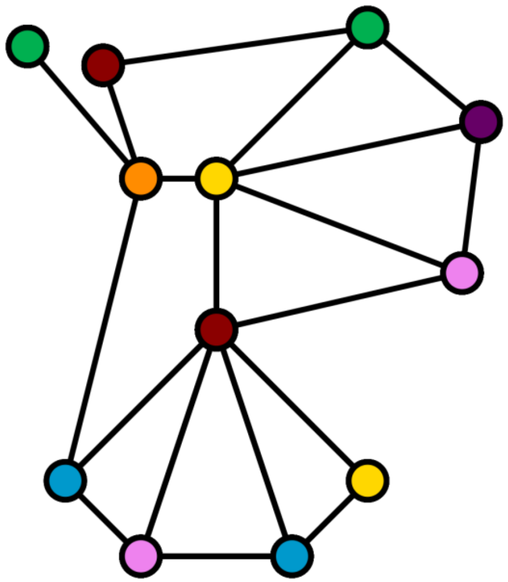

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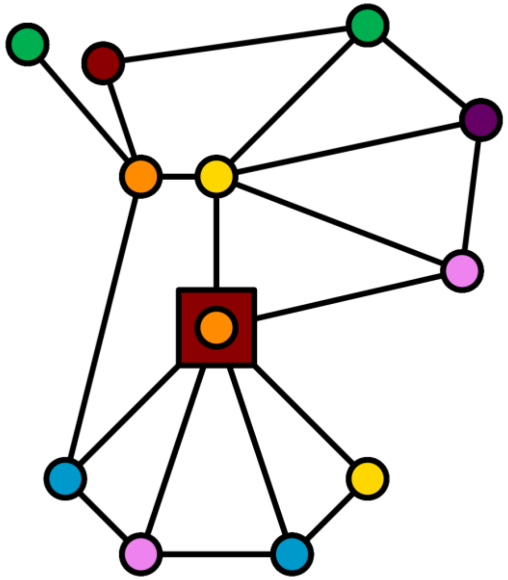

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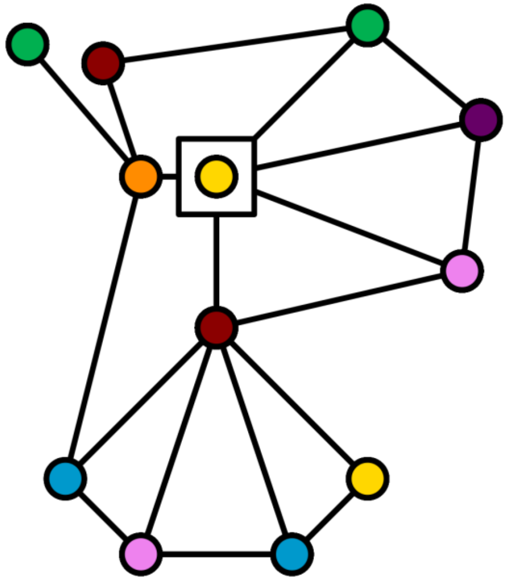

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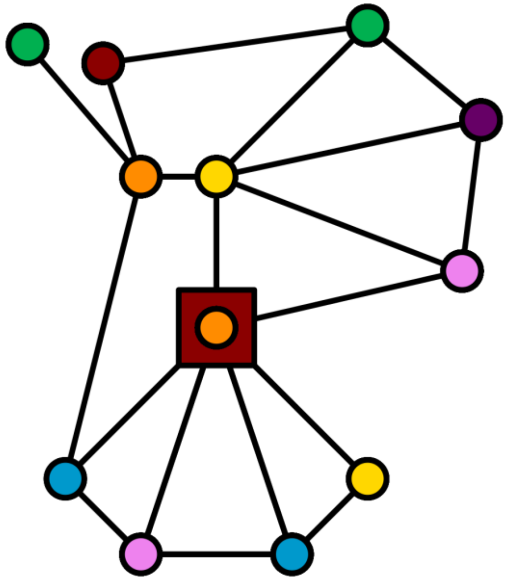

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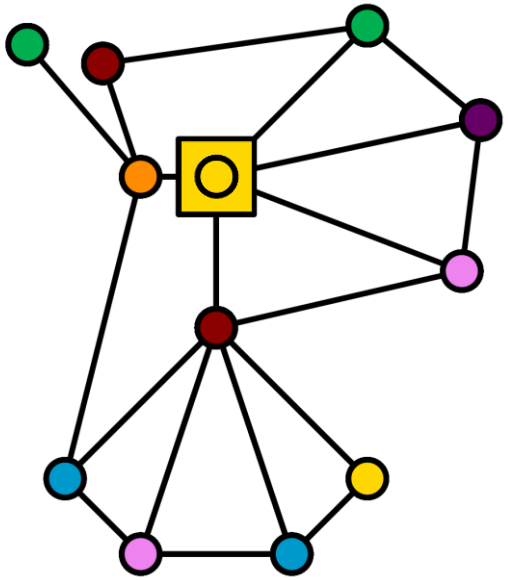

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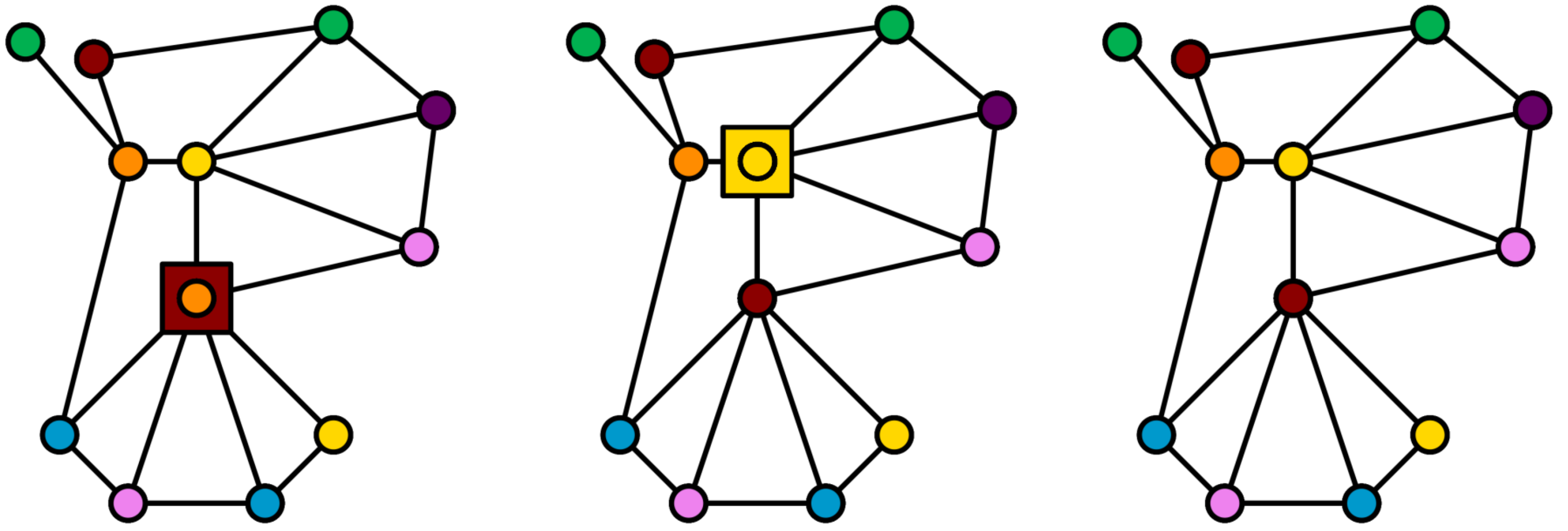

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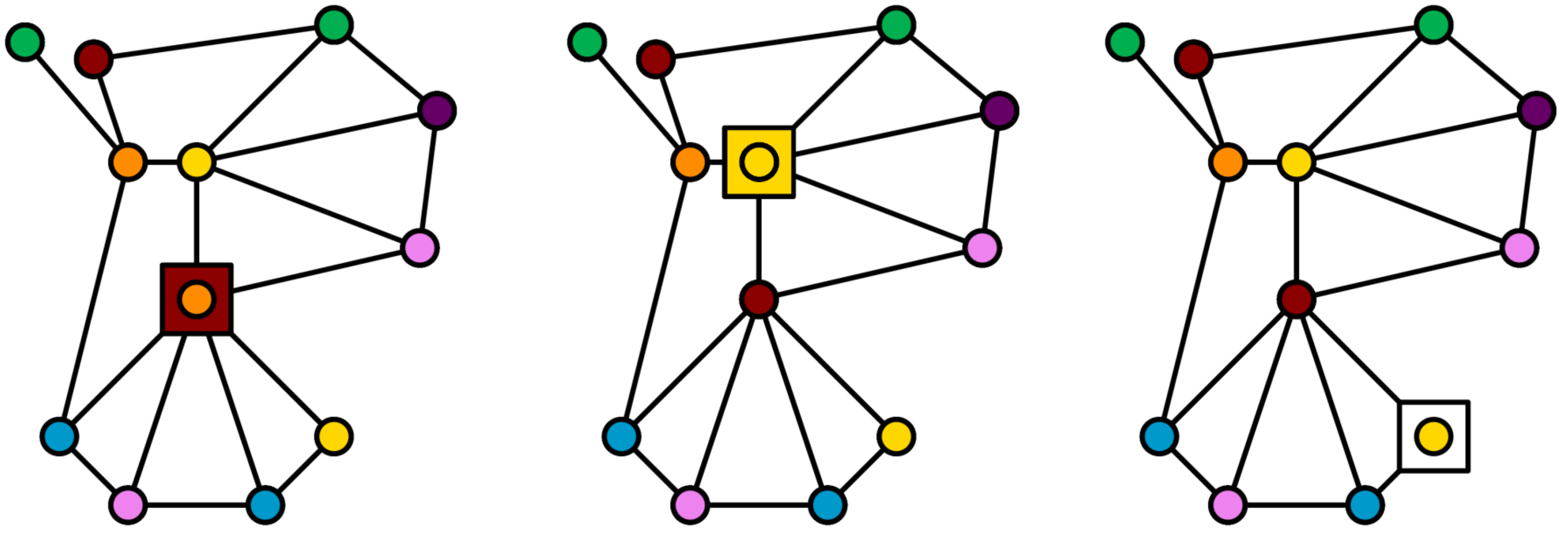

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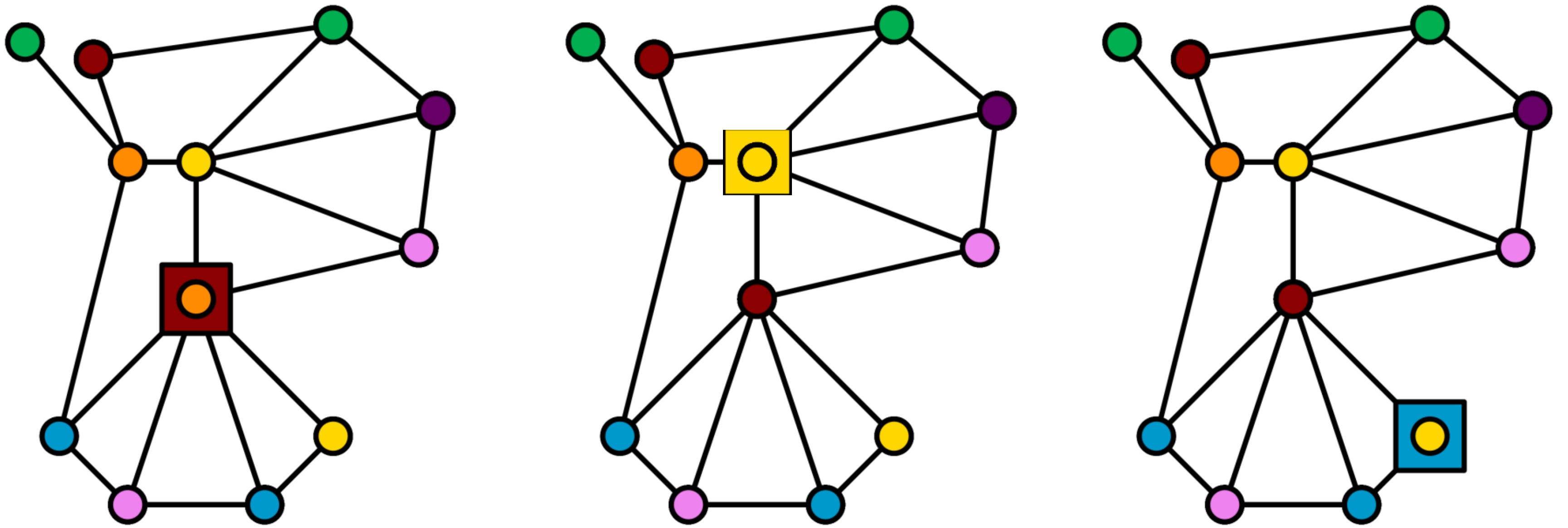

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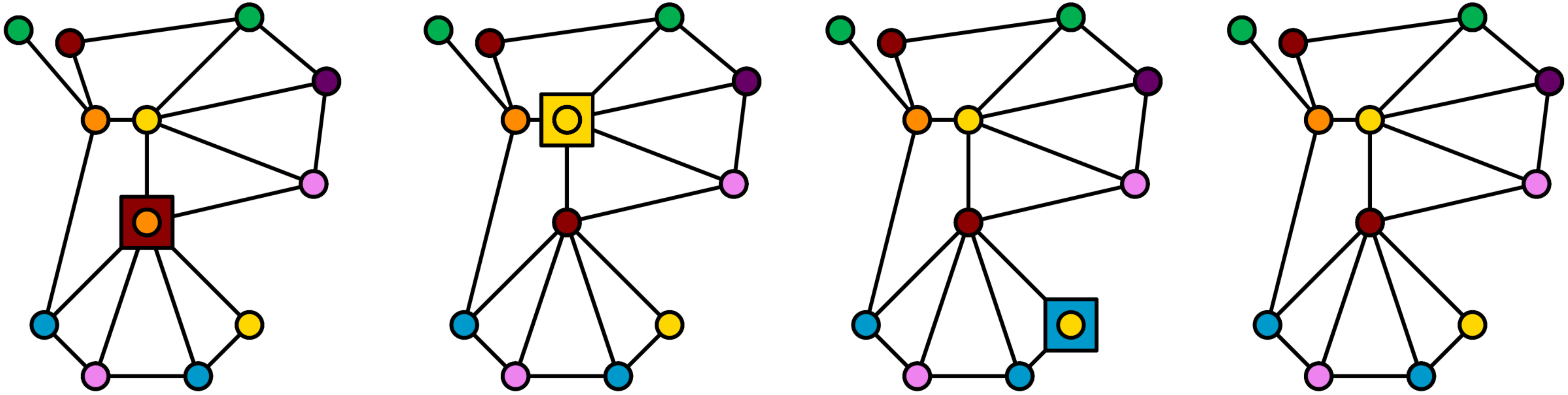

t $\quad$ $t+1$ $\quad$ $t+2$ $\quad$ $t+3$

Centralized

[*Jerrum 1995*]

Single-Site Glauber $\boldsymbol{O(n \log n)}$ steps $\boldsymbol{q \geq 2\Delta + 1}$

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Decentralized

[*Feng, Sun, Yi 2017*]

What can be sampled locally?

local/decentralized sampling techniques?

local/decentralized transition rules for Markov chain?

Centralized

[*Jerrum 1995*]

Single-Site Glauber $\boldsymbol{O(n \log n)}$ steps $\boldsymbol{q \geq 2\Delta + 1}$

Decentralized

[*Feng, Sun, Yi 2017*]

LubyGlauber $\boldsymbol{O(\Delta \log n)}$ steps $\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2}$

Centralized

[*Jerrum 1995*]

update a single node

$\boldsymbol{O(n \log n)}$ steps

$\boldsymbol{q \geq 2\Delta + 1}$

Decentralized

[*Feng, Sun, Yi 2017*]

LubyGlauber

$\boldsymbol{O(\Delta \log n)}$ steps

$\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2}$

Centralized

[*Jerrum 1995*]

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Decentralized

[*Feng, Sun, Yi 2017*]

update an independent set $\boldsymbol{O(\Delta \log n)}$ steps $\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2}$

Centralized

[Jerrum 1995]

update a single node — $O(n \log n)$ steps — $q \geq 2\Delta + 1$

Decentralized

[Feng, Sun, Yi 2017]

update an independent set — $O(\Delta \log n)$ steps — $q \geq \alpha\Delta$ for $\alpha > 2$

[Feng, Sun, Yi 2017]

LocalMetropolis — $O(\log n)$ steps — $q \geq \alpha\Delta$ for $\alpha > 2 + \sqrt{2}$

Centralized

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update a single node — $O(n \log n)$ steps — $q \geq 2\Delta + 1$

Decentralized

[Feng, Sun, Yi 2017]

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update all nodes — $O(\log n)$ steps — $q \geq \alpha\Delta$ for $\alpha > 2 + \sqrt{2}$

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[Feng, Sun, Yi 2017]

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update all nodes $\quad O(\log n)$ steps $\quad q \geq \alpha\Delta$ for $\alpha > 2 + \sqrt{2}$

[F., Ghaffari 2018]

Local Glauber $\quad O(\log n)$ steps $\quad q \geq \alpha\Delta$ for $\alpha > 2$

Centralized

[Jerrum 1995]

update a single node — $O(n \log n)$ steps — $q \geq 2\Delta + 1$

Decentralized

[Feng, Sun, Yi 2017]

update an independent set — $O(\Delta \log n)$ steps — $q \geq \alpha\Delta$ for $\alpha > 2$

[Feng, Sun, Yi 2017]

update all nodes — $O(\log n)$ steps — $q \geq \alpha\Delta$ for $\alpha > 2 + \sqrt{2}$

[F., Ghaffari 2018]

update an almost independent set — $O(\log n)$ steps — $q \geq \alpha\Delta$ for $\alpha > 2$

Local Glauber Dynamics

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update colors of a **small-degree node set** to **random colors**, if still proper

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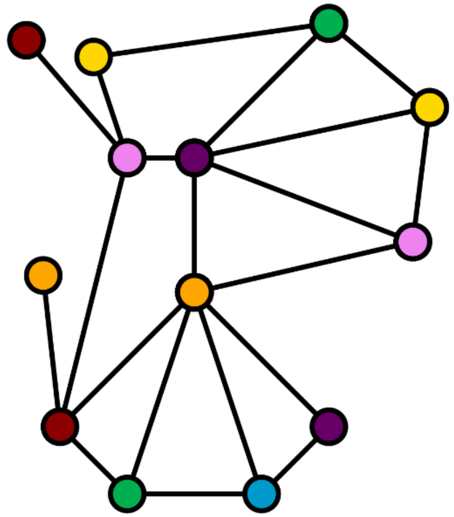

t

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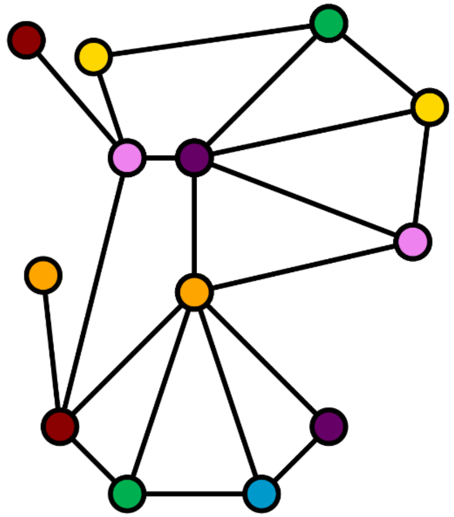
each node v

independently marks itself with probability $\gamma \in (0,1)$



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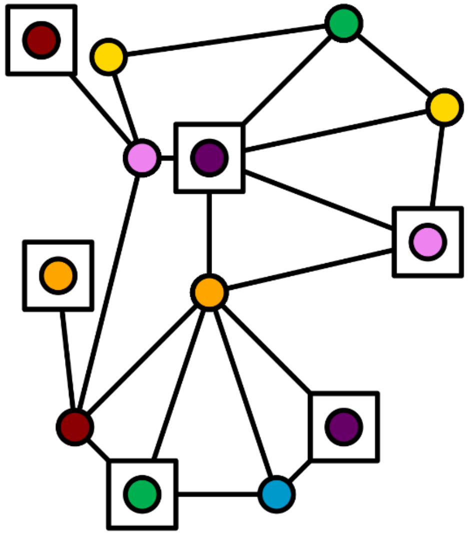

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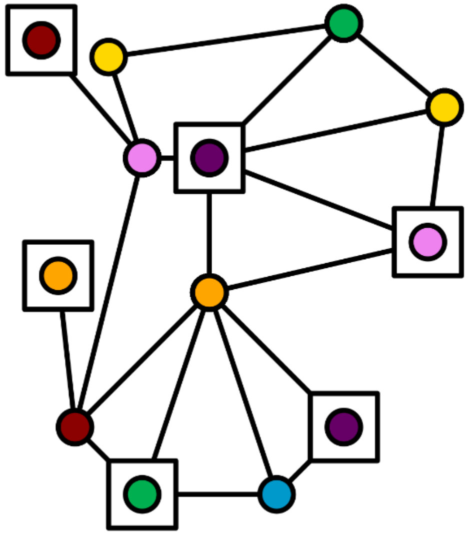

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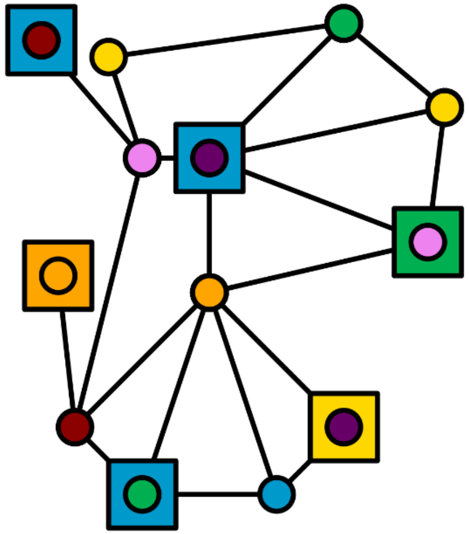

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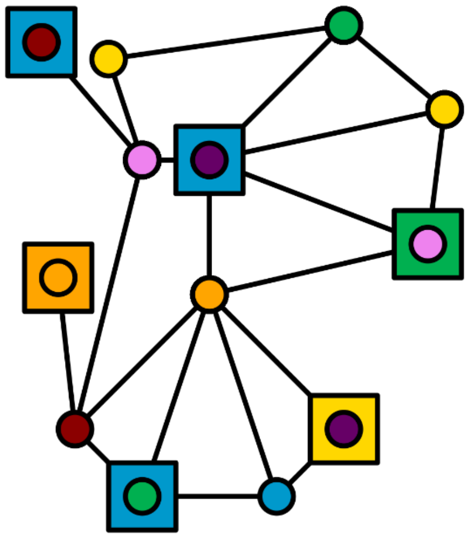

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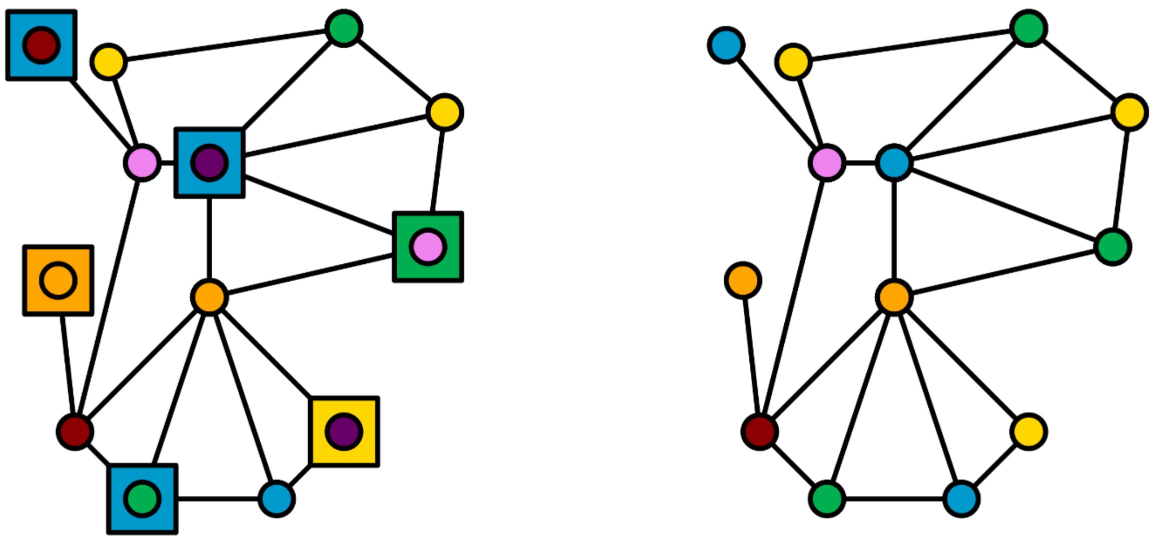

t $\qquad\qquad$ $t+1$

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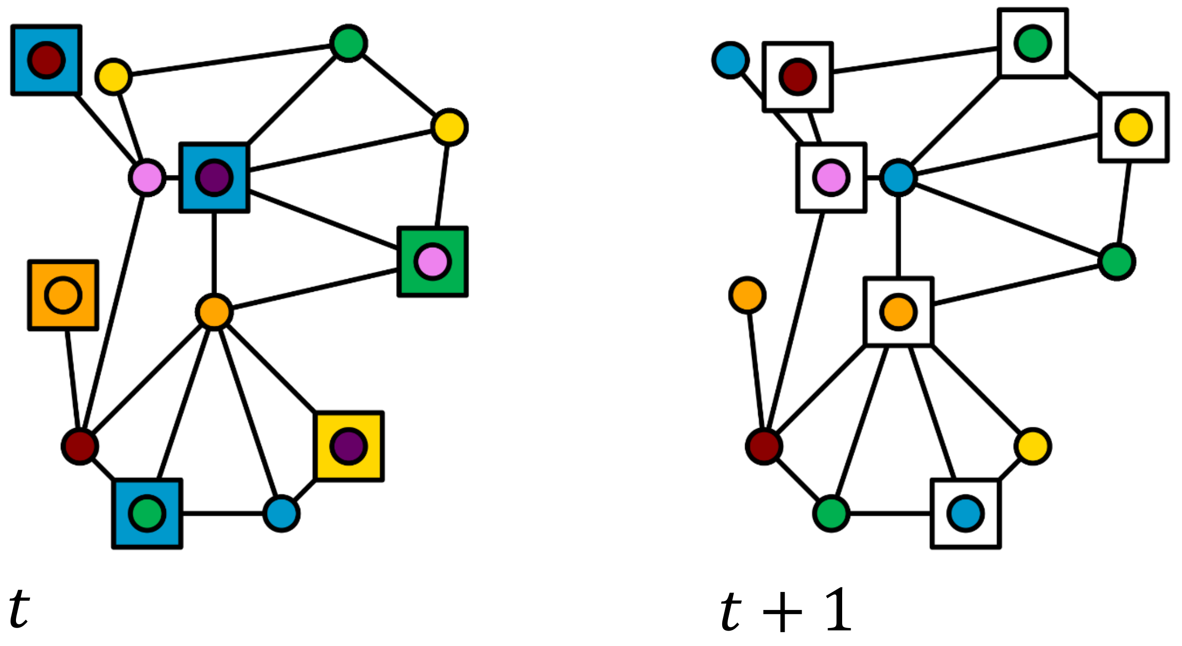

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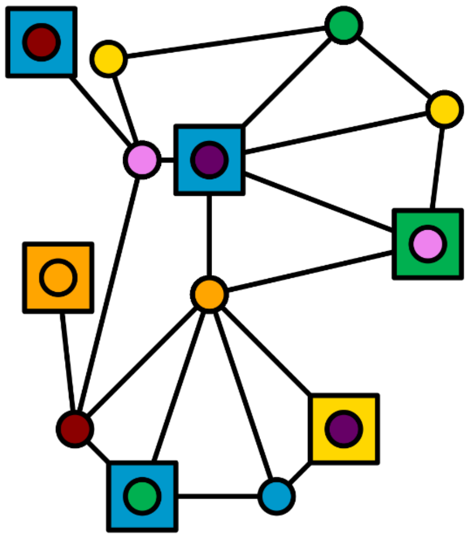

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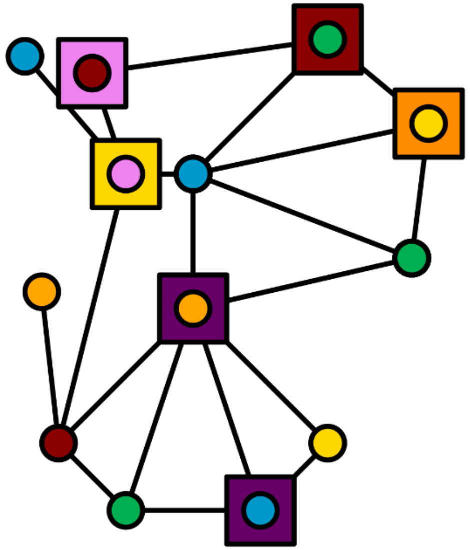

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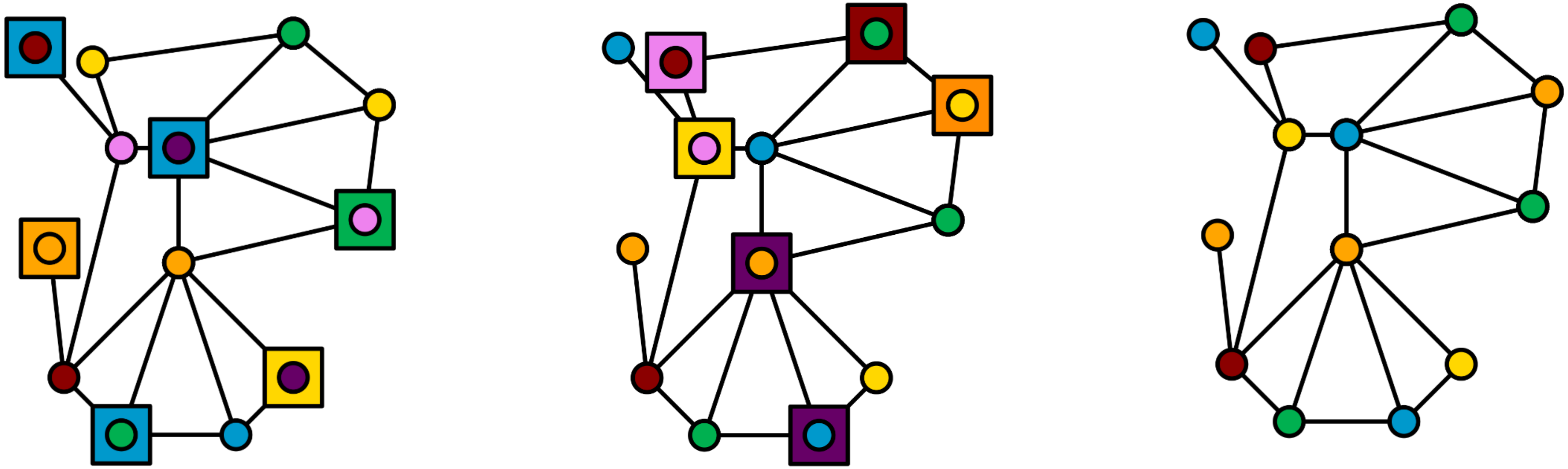

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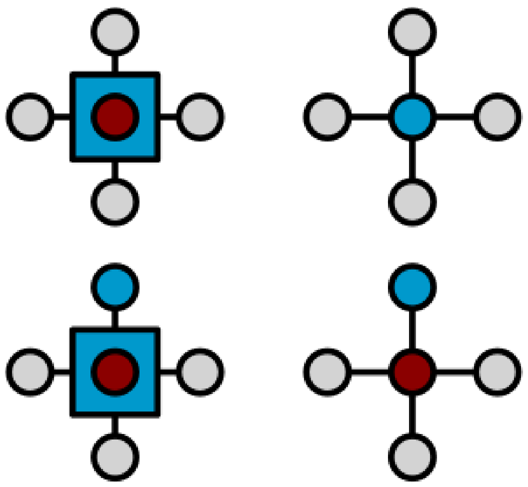

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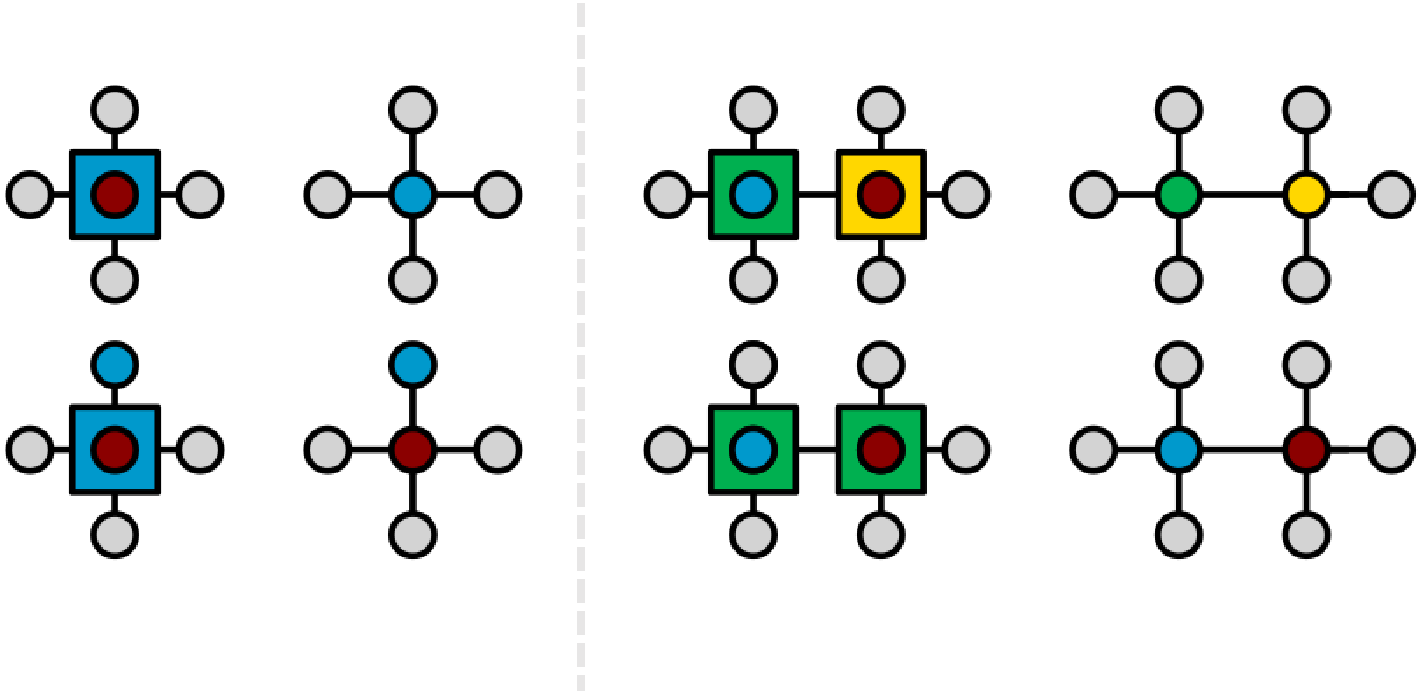

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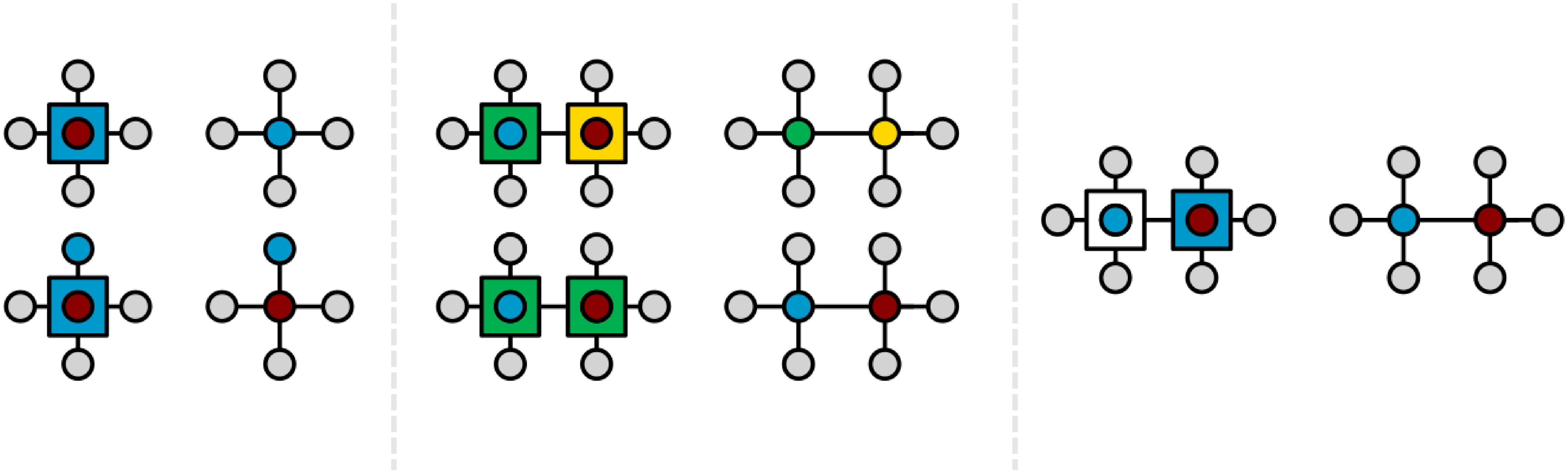

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[*F., Ghaffari 2018*]

Theorem

Local Glauber converges to uniform distribution over proper q-colorings in $O(\log n)$ steps if $q \geq \alpha\Delta$ for $\alpha > 2$.

Proof Sketch

[*Bubley, Dyer 1997*]

Path Coupling Lemma (simplified & informal)

If there is a coupling so that for all pairs of adjacent states
the two Markov chains started in those two states come closer together in expectation,
then the Markov chain is rapidly mixing.

[*Bubley, Dyer 1997*]

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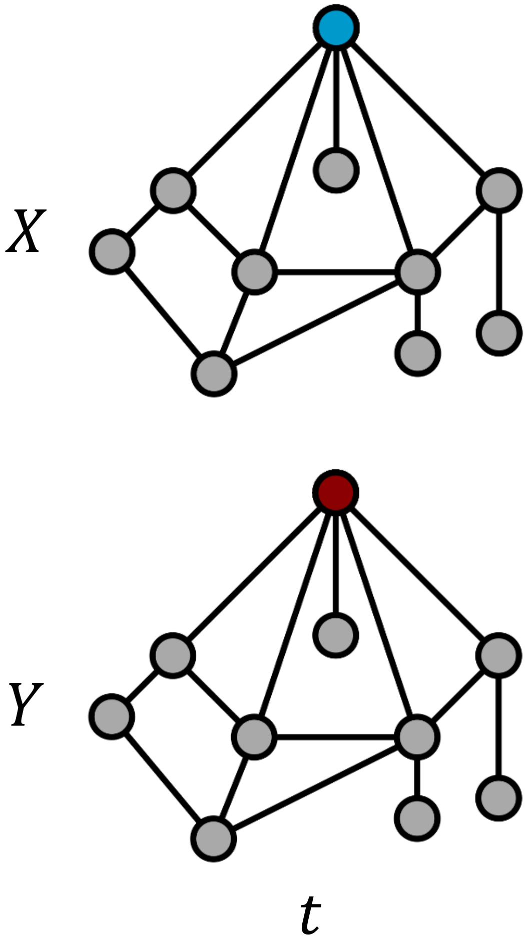

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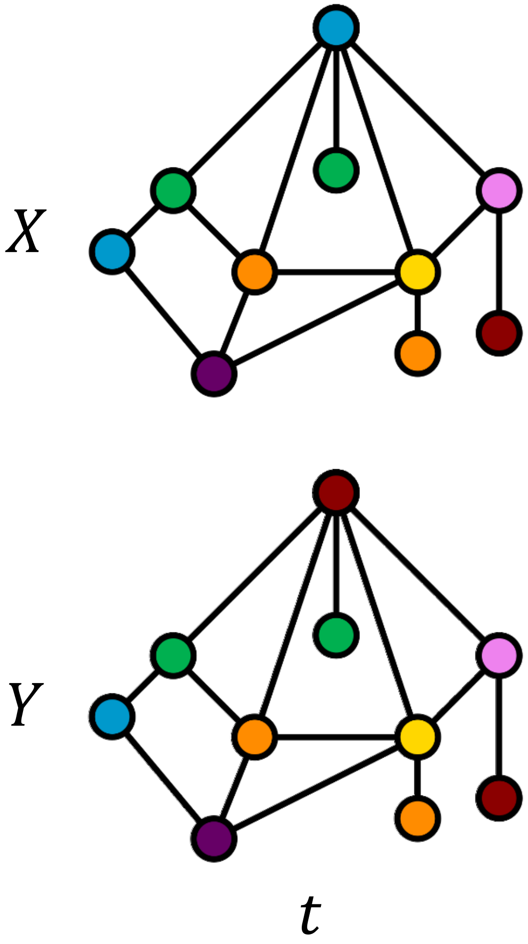

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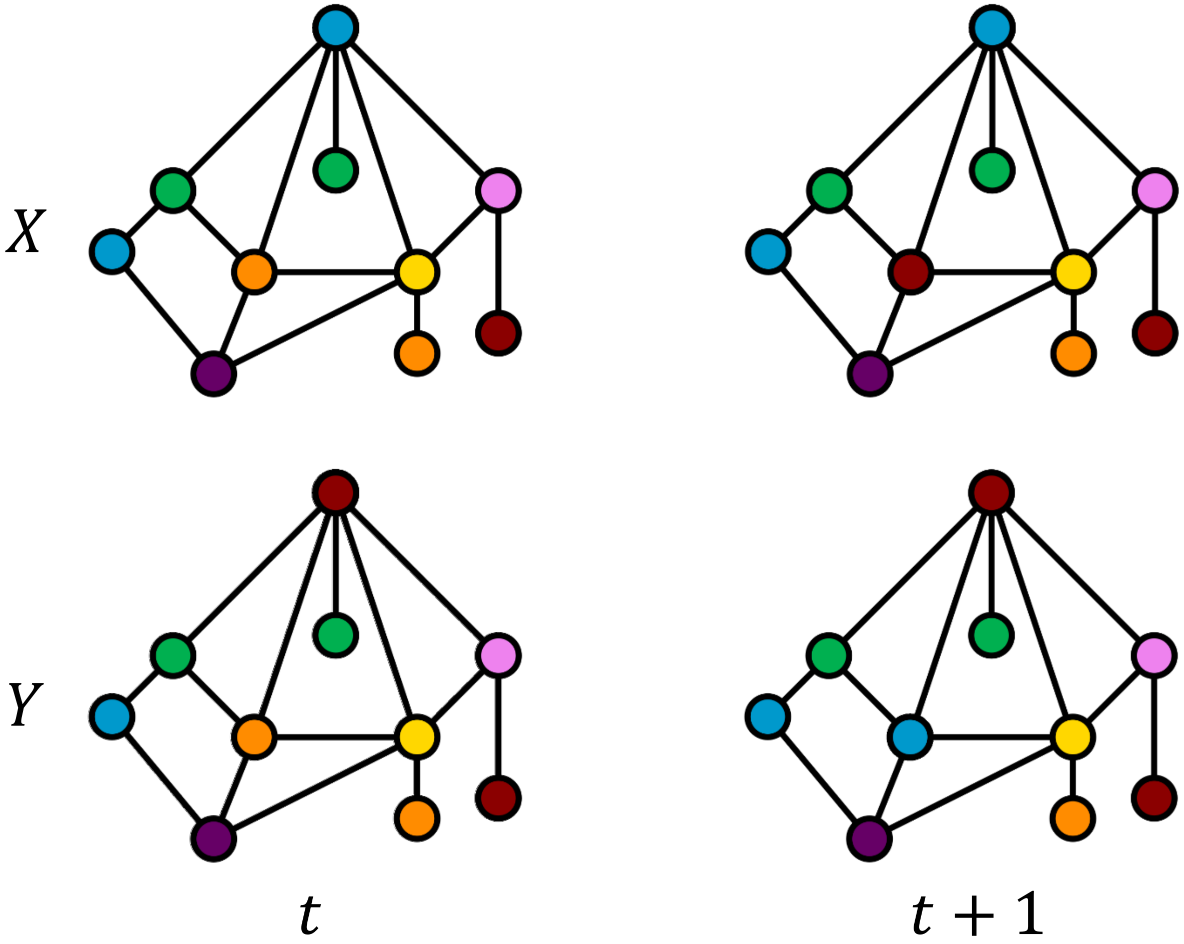

Reviewing
Path Coupling for
Single-Site Glauber Dynamics

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update color of **a random node** to a **random color**, if proper

Reviewing Path Coupling for Single-Site Glauber Dynamics

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pick node v uniformly at random
pick color c uniformly at random
if none of v's neighbors has color c, update v's color to c

Naïve Path Coupling for Single-Site Glauber Dynamics

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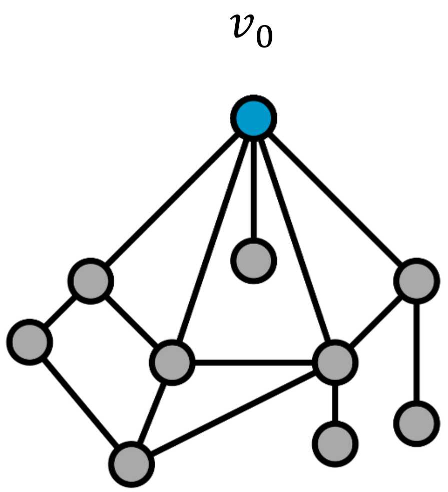

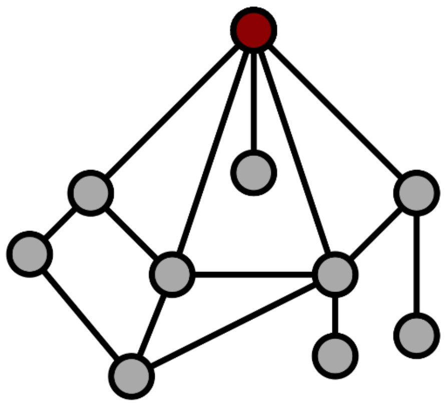

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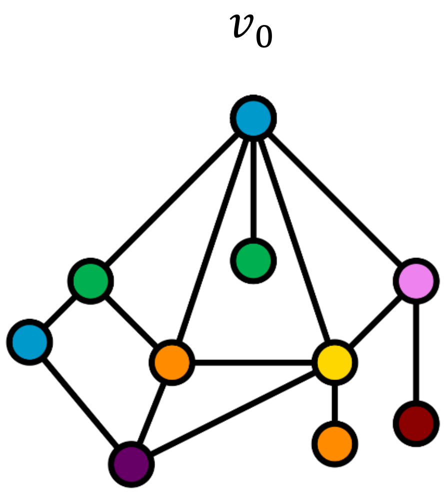

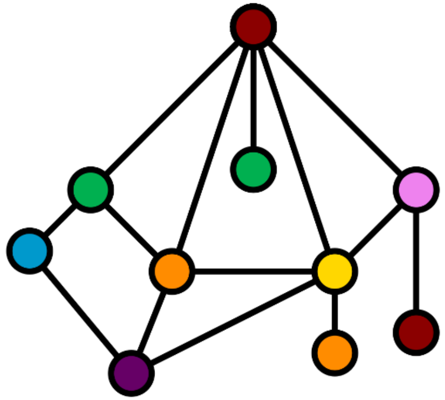

Naïve Path Coupling for Single-Site Glauber Dynamics

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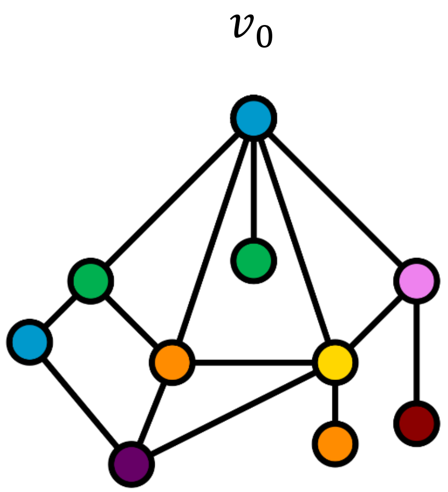

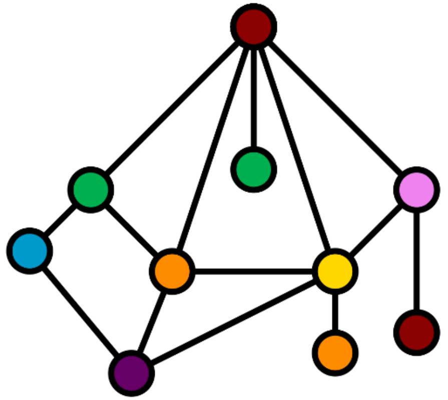

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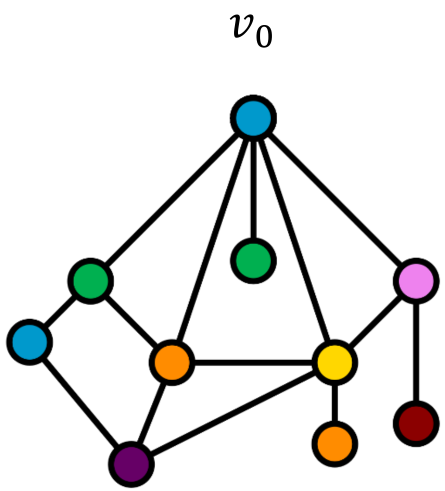

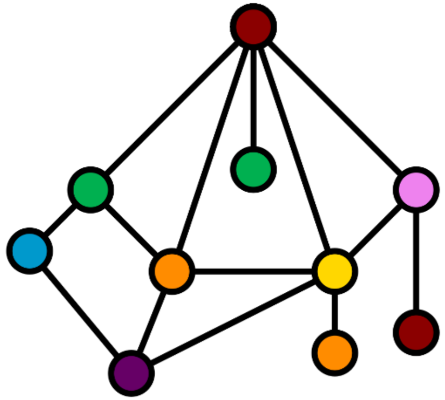

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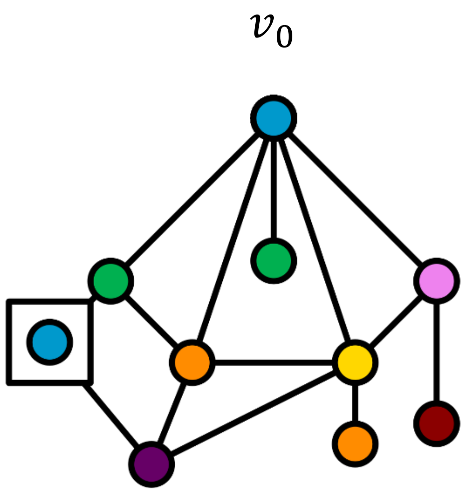

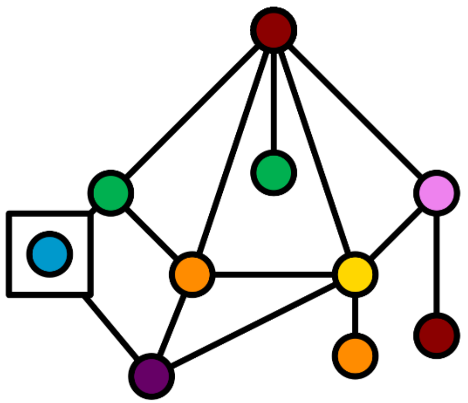

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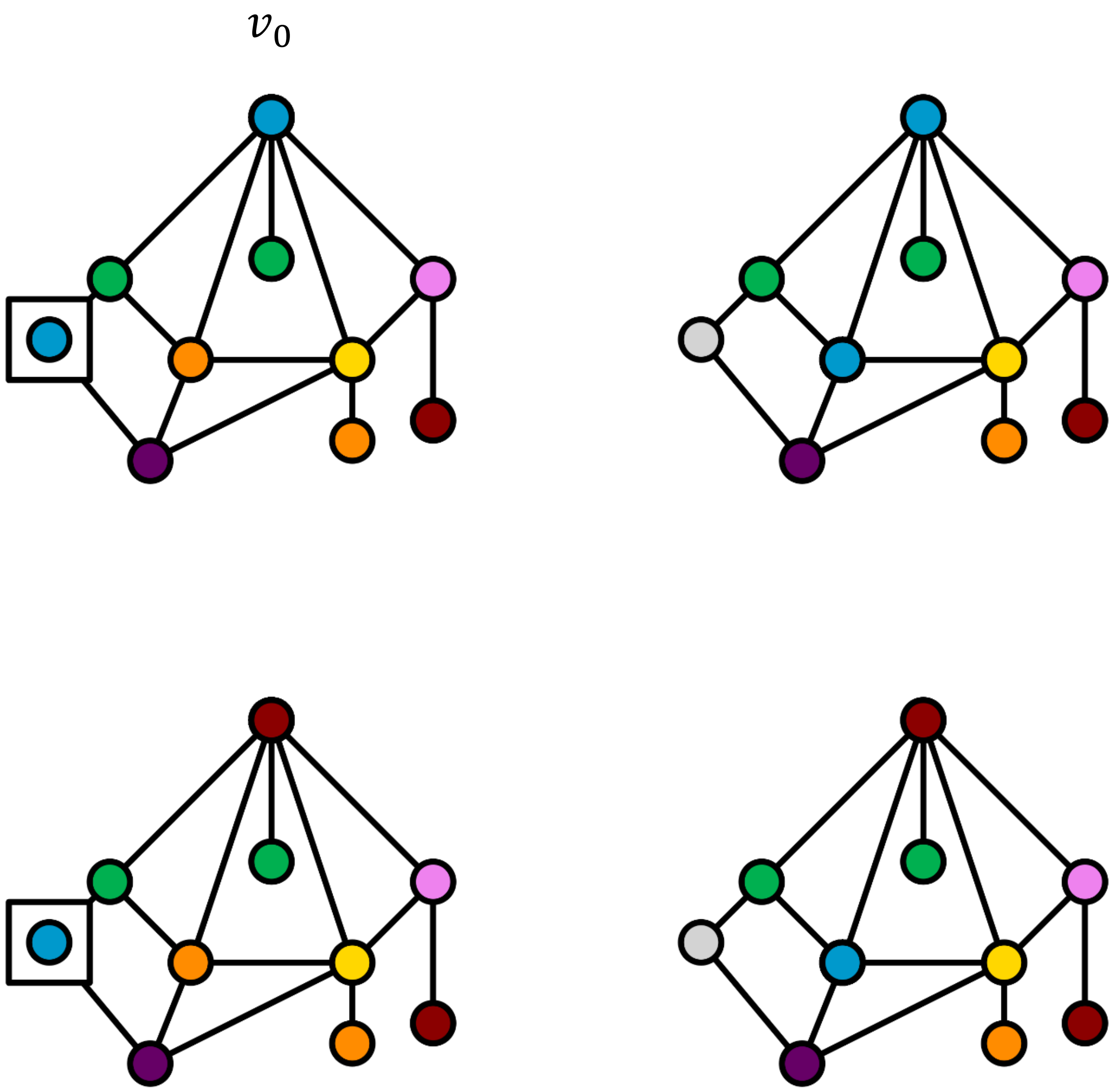

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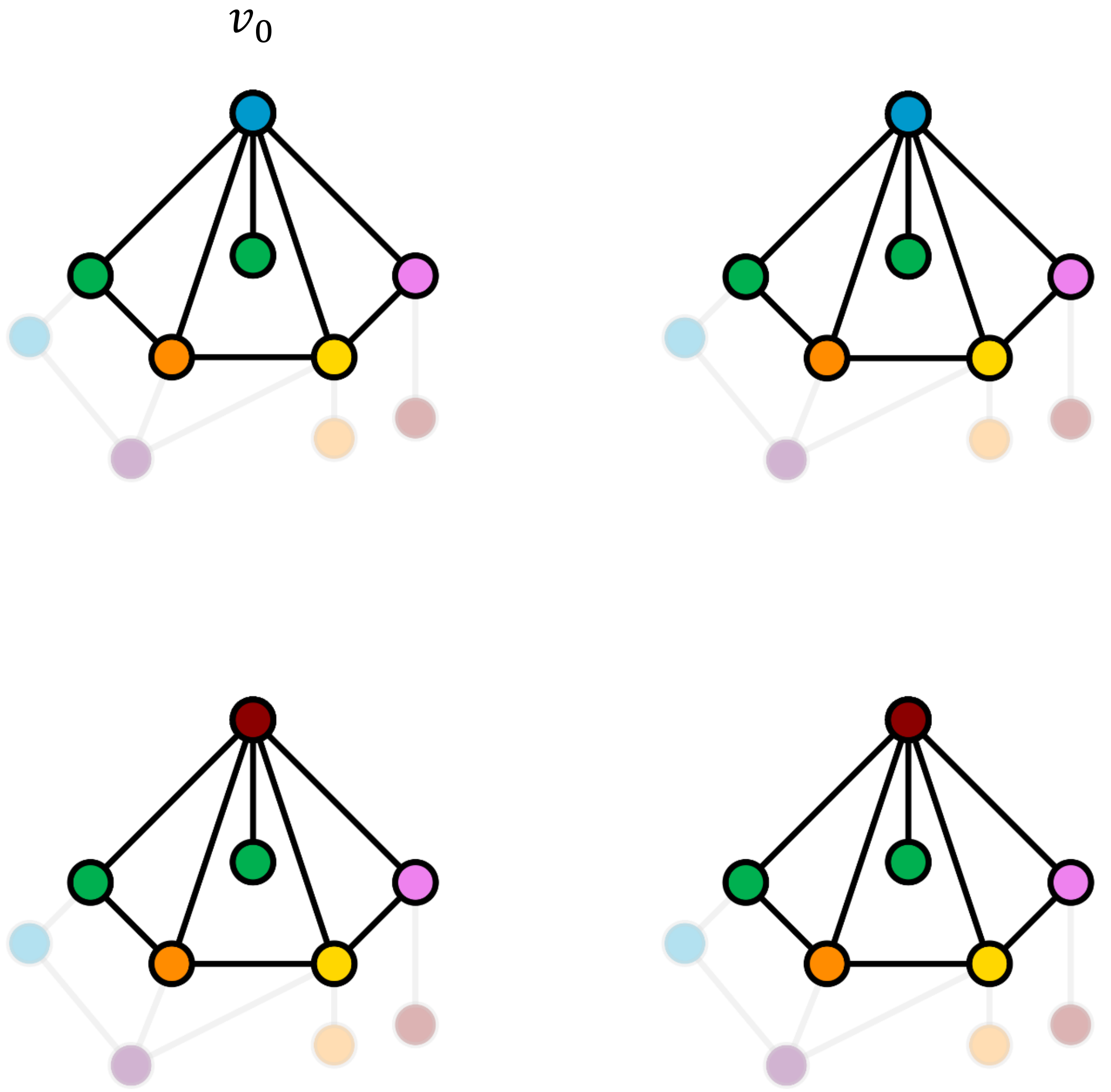

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Naïve Path Coupling for Single-Site Glauber Dynamics

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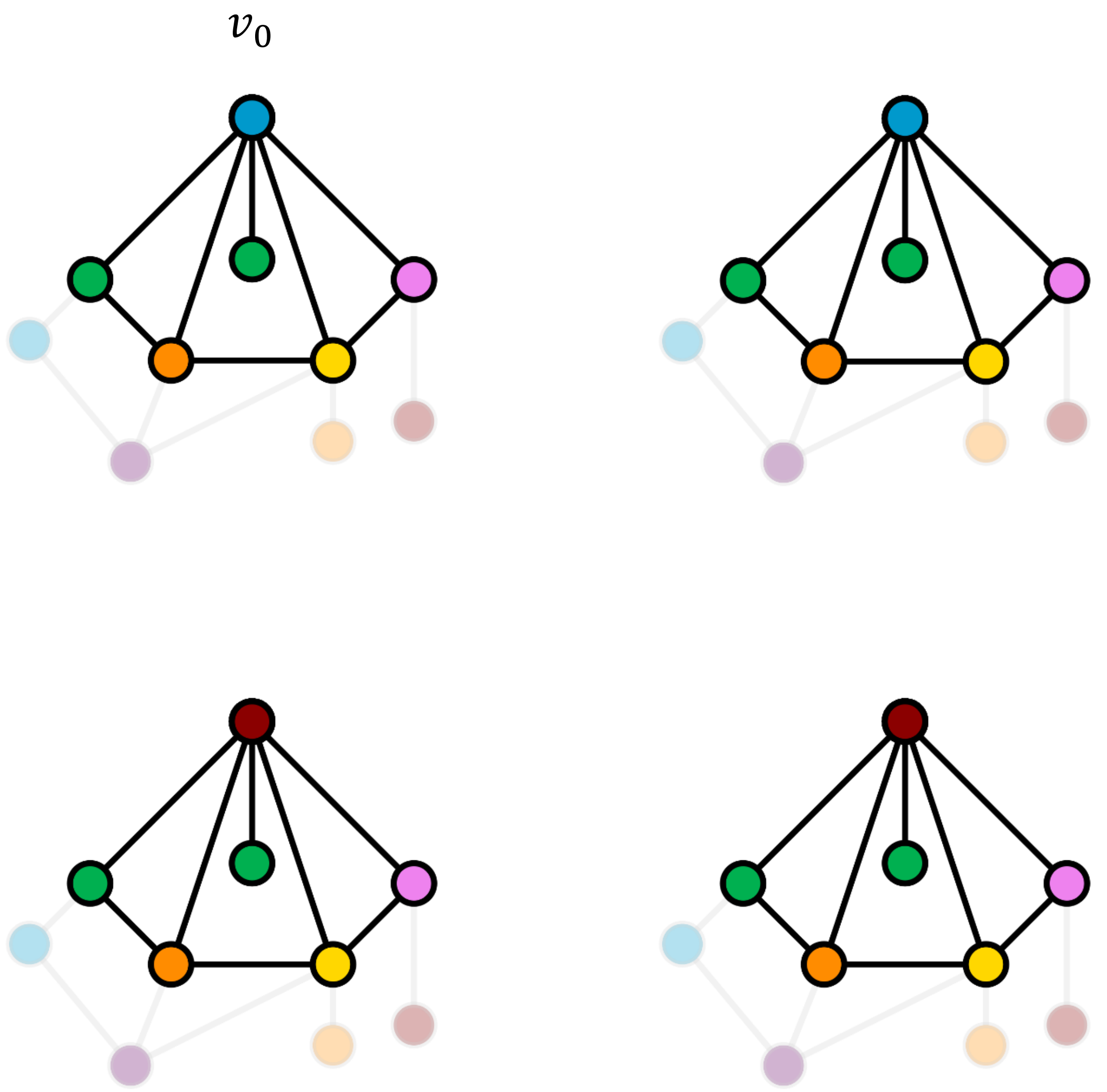
[node v_0]



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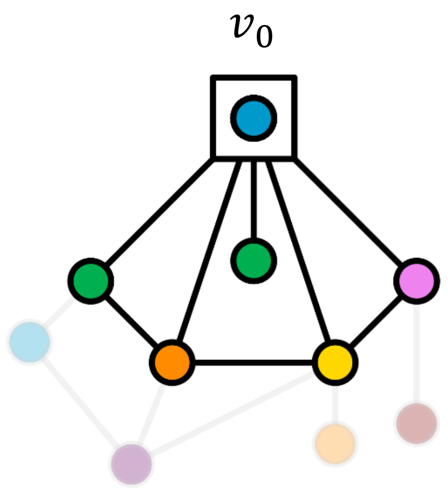

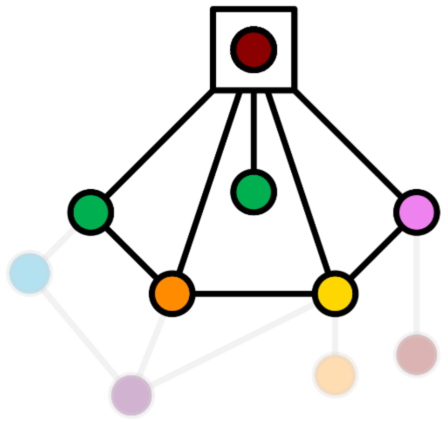

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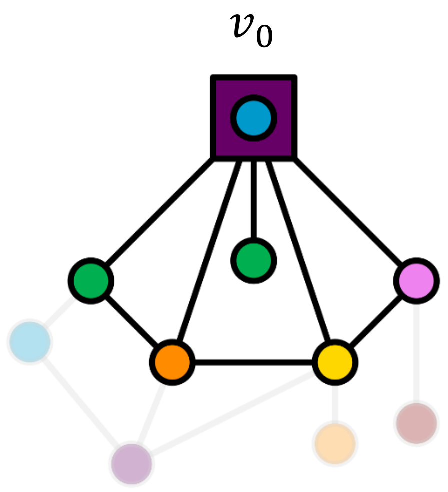

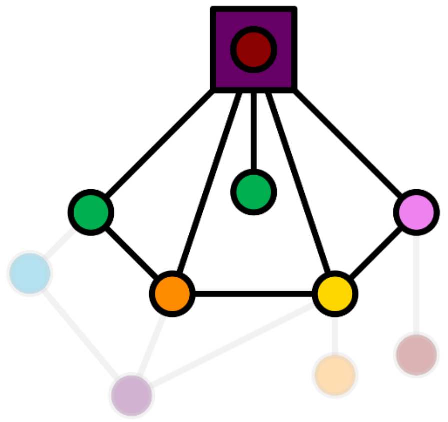

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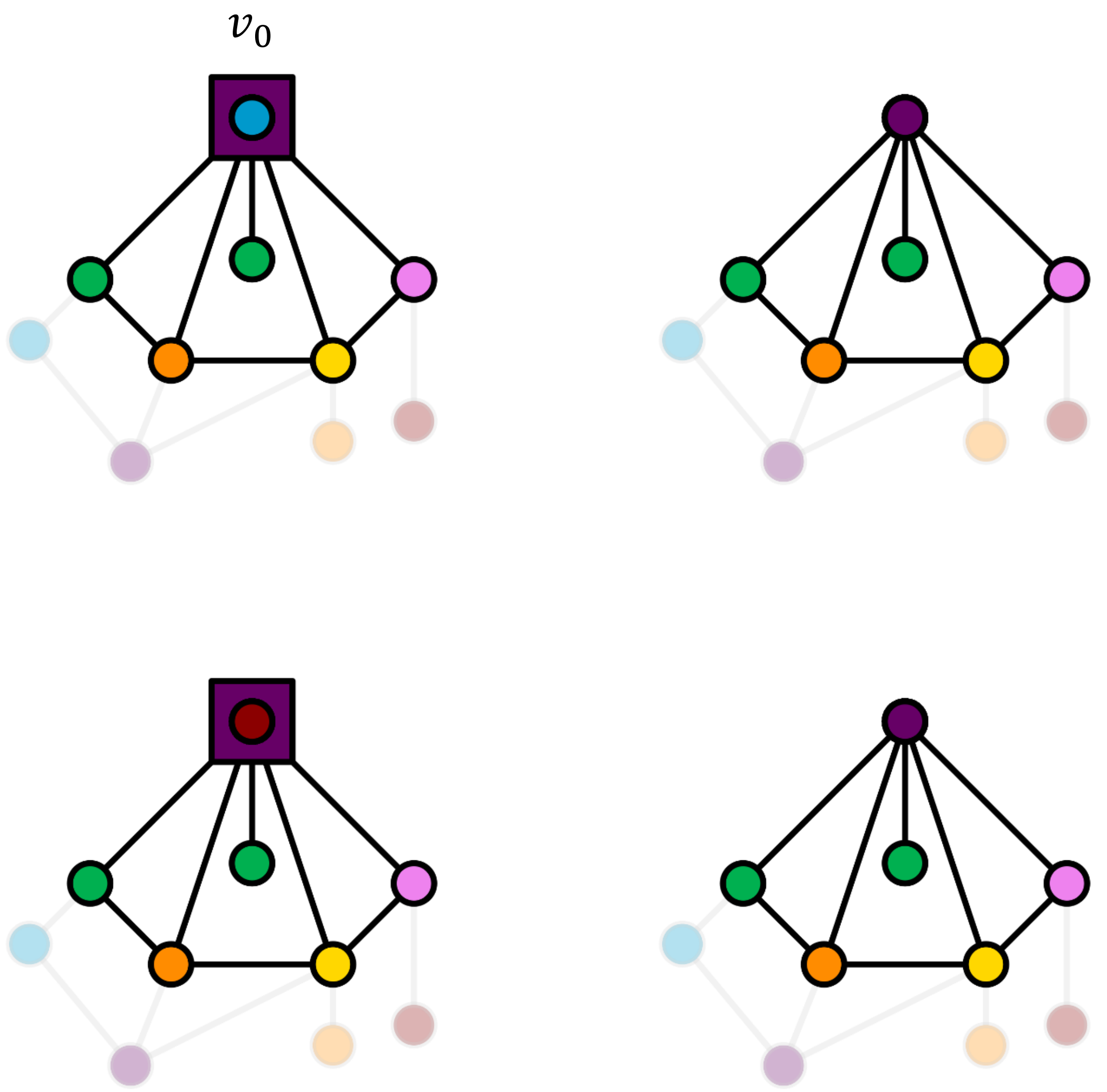

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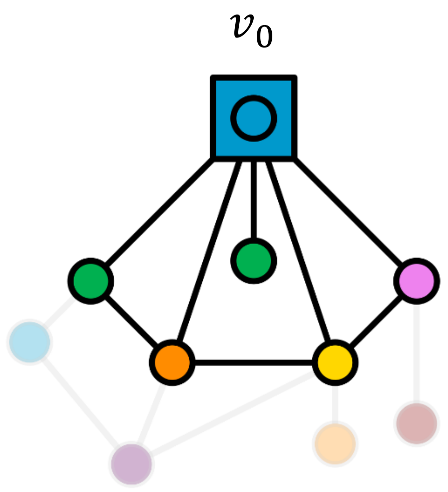

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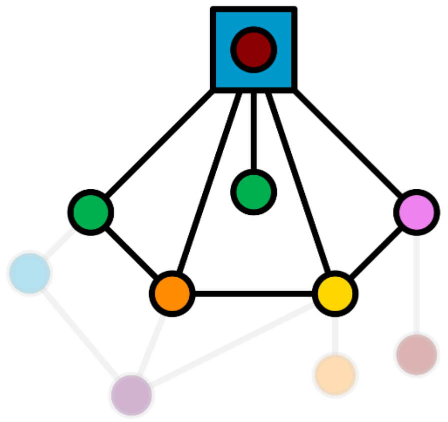

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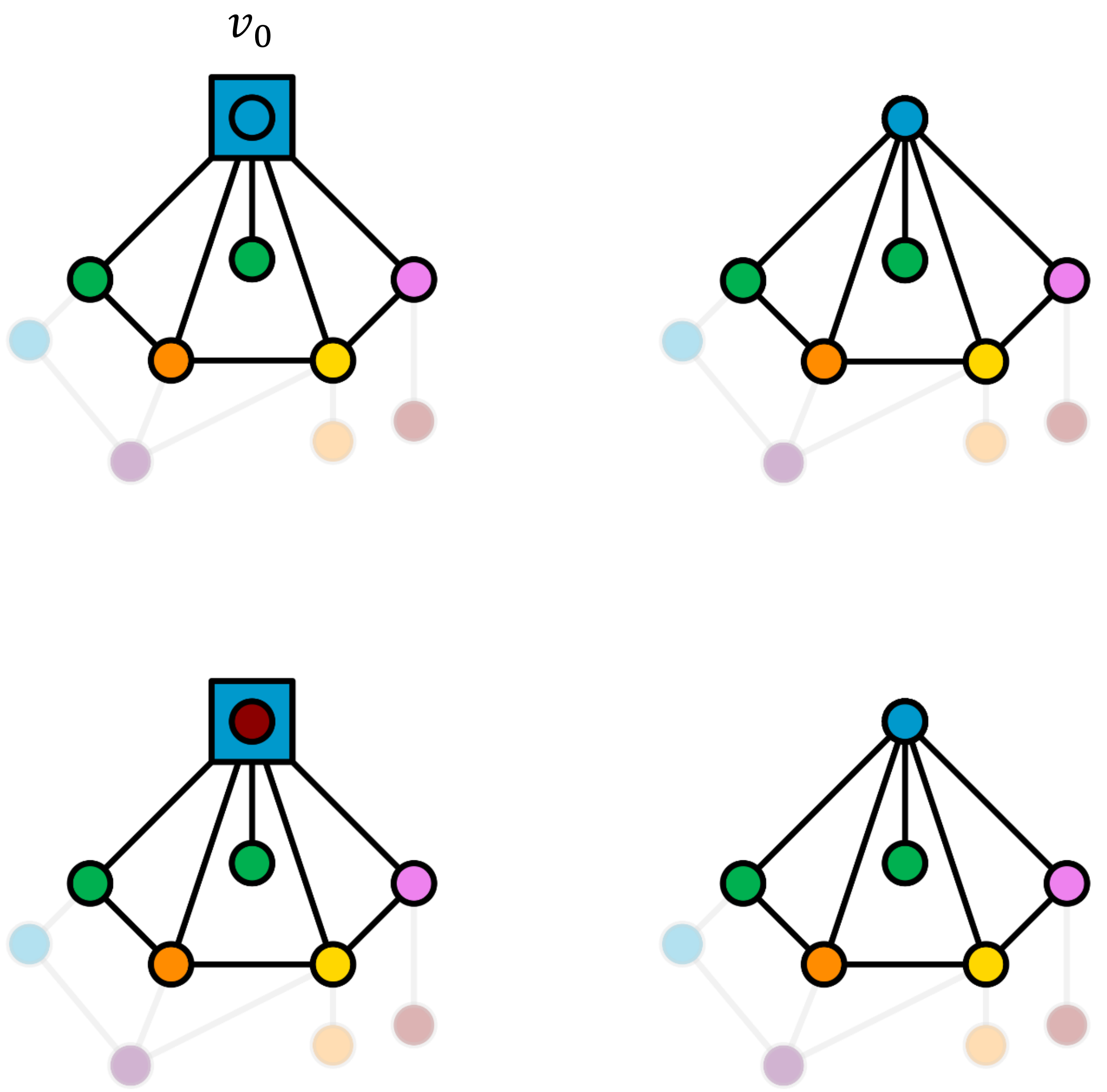

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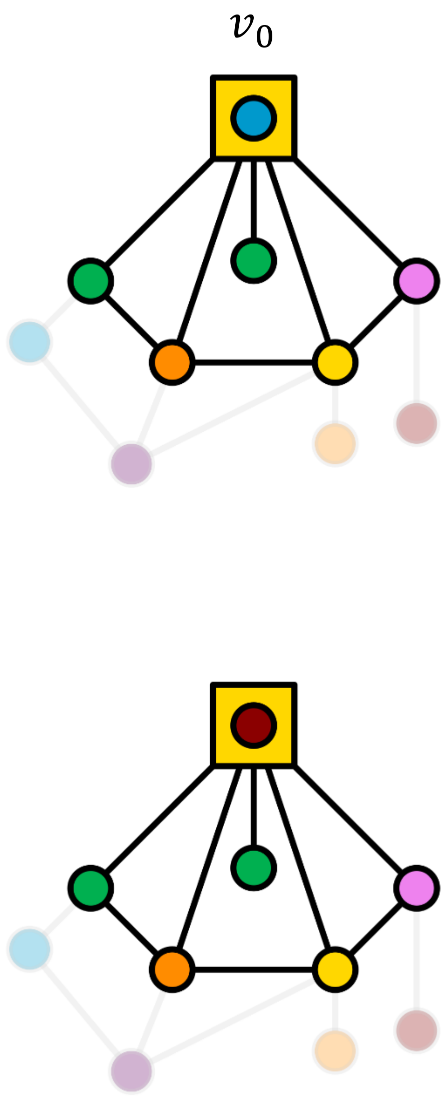

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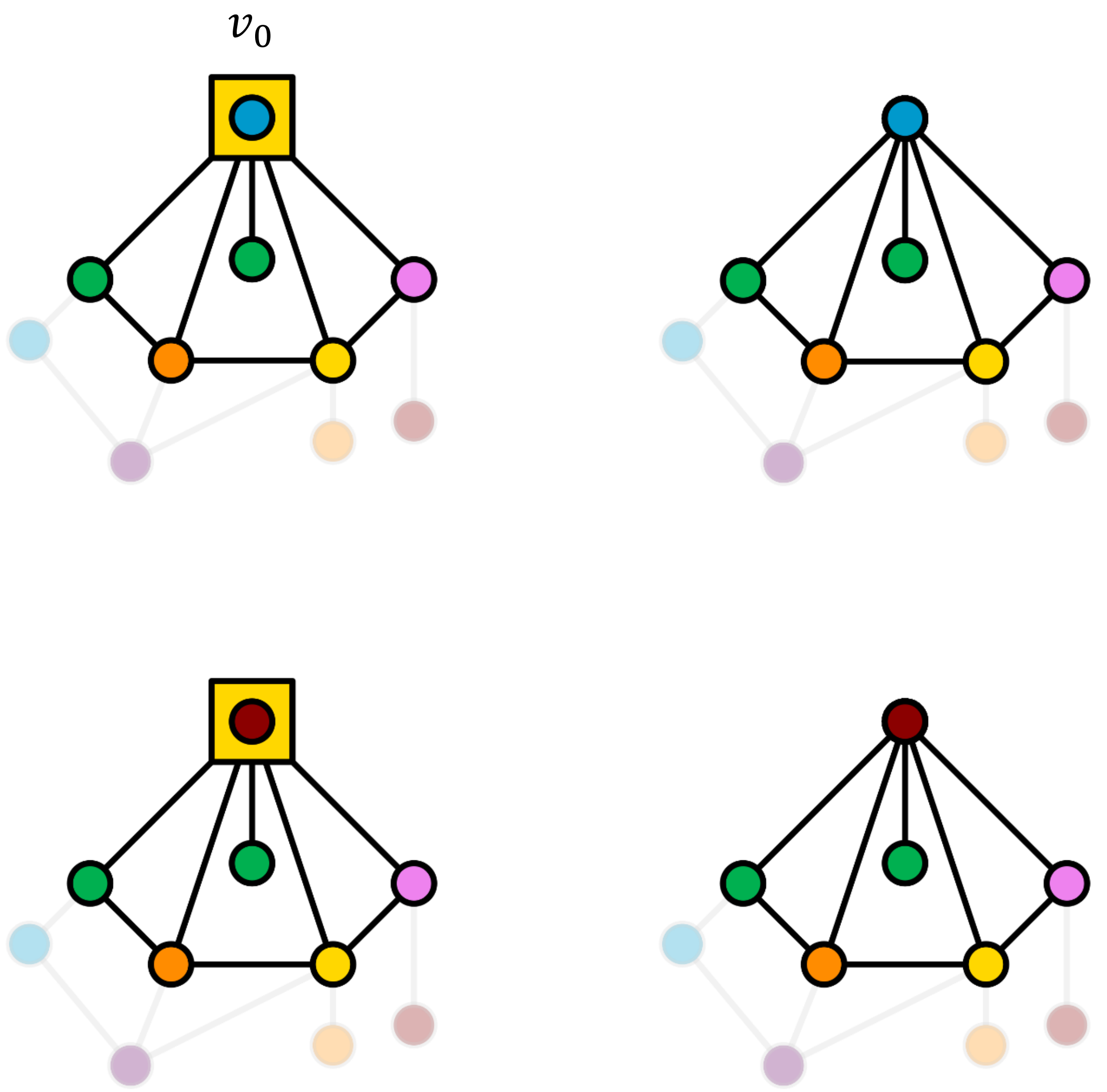

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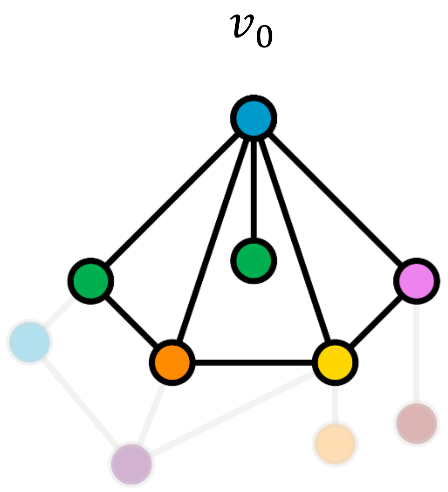

expected number of differing nodes:

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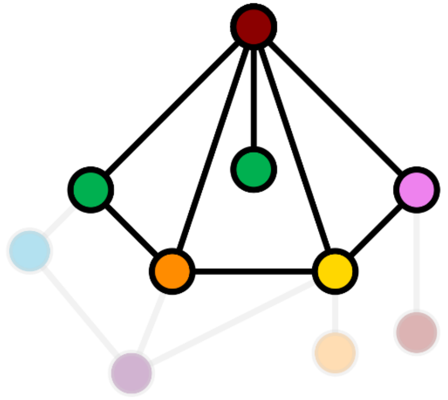
$$+\Delta\cdot\frac{1}{n}\cdot\frac{2}{q} \qquad \textit{[neighbors of } v_0\textit{]}$$



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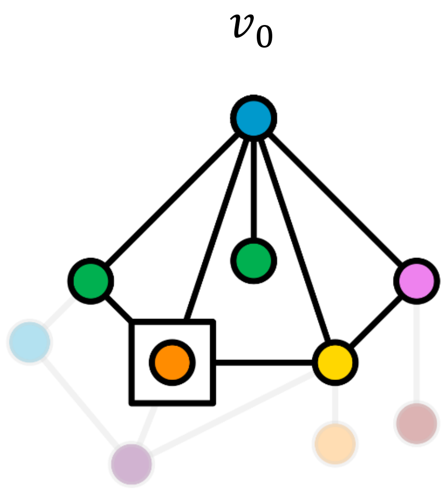

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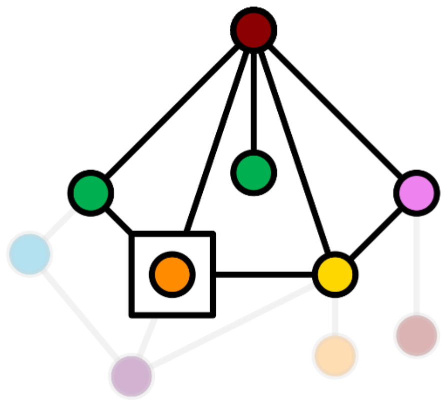

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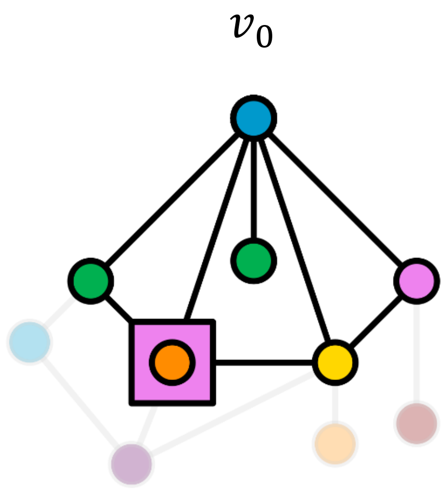

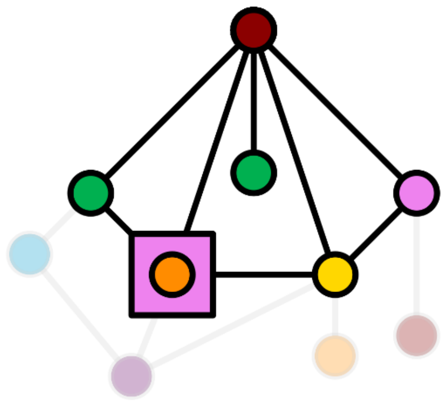

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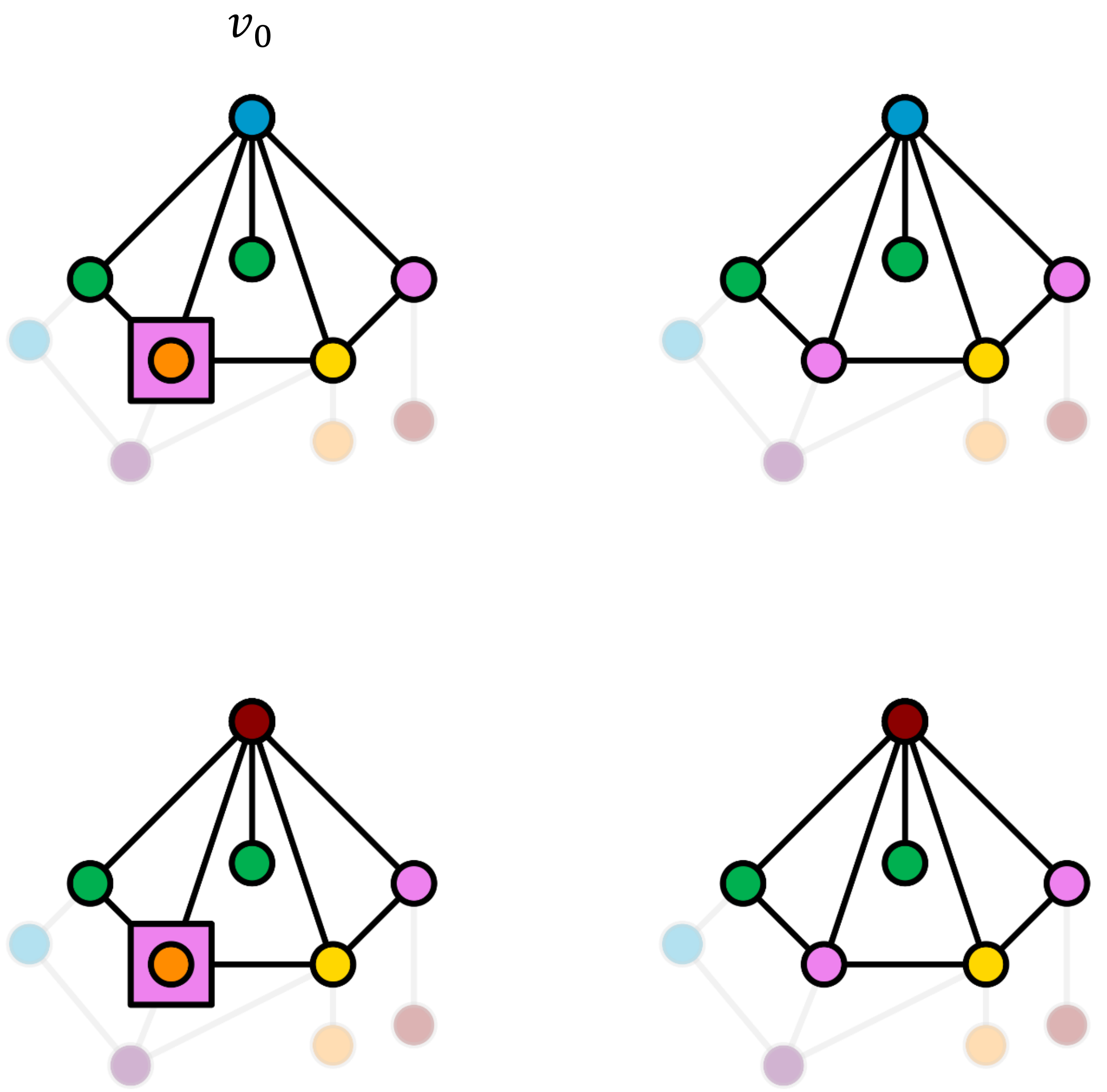

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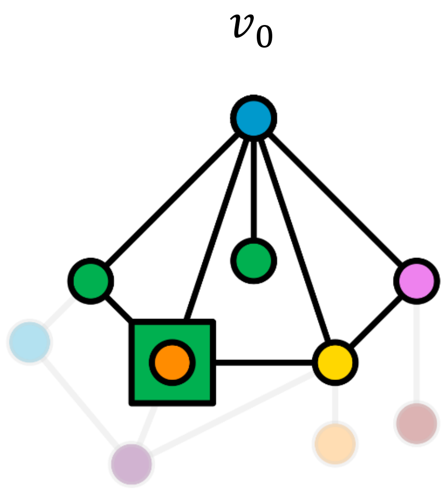

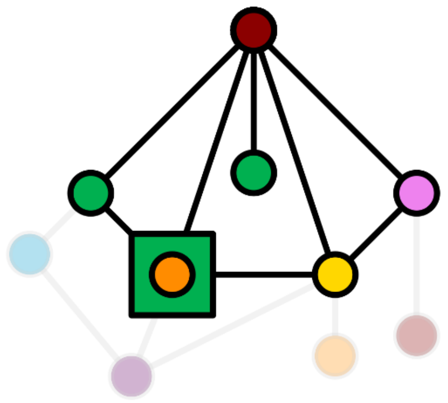

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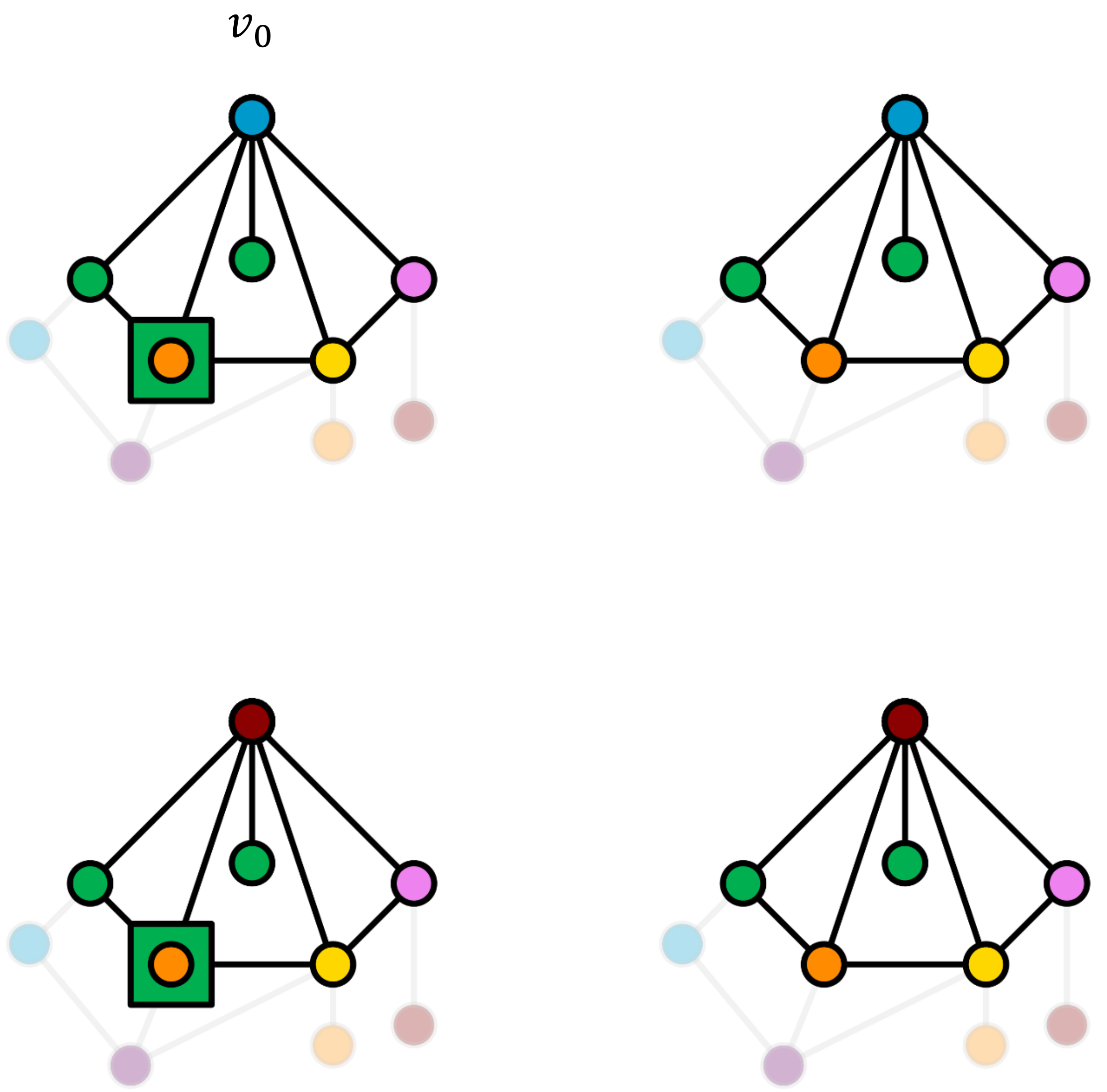

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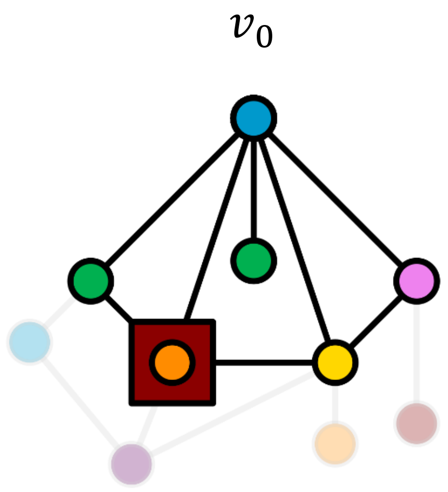

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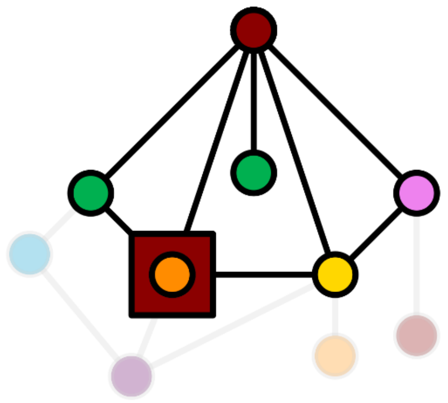

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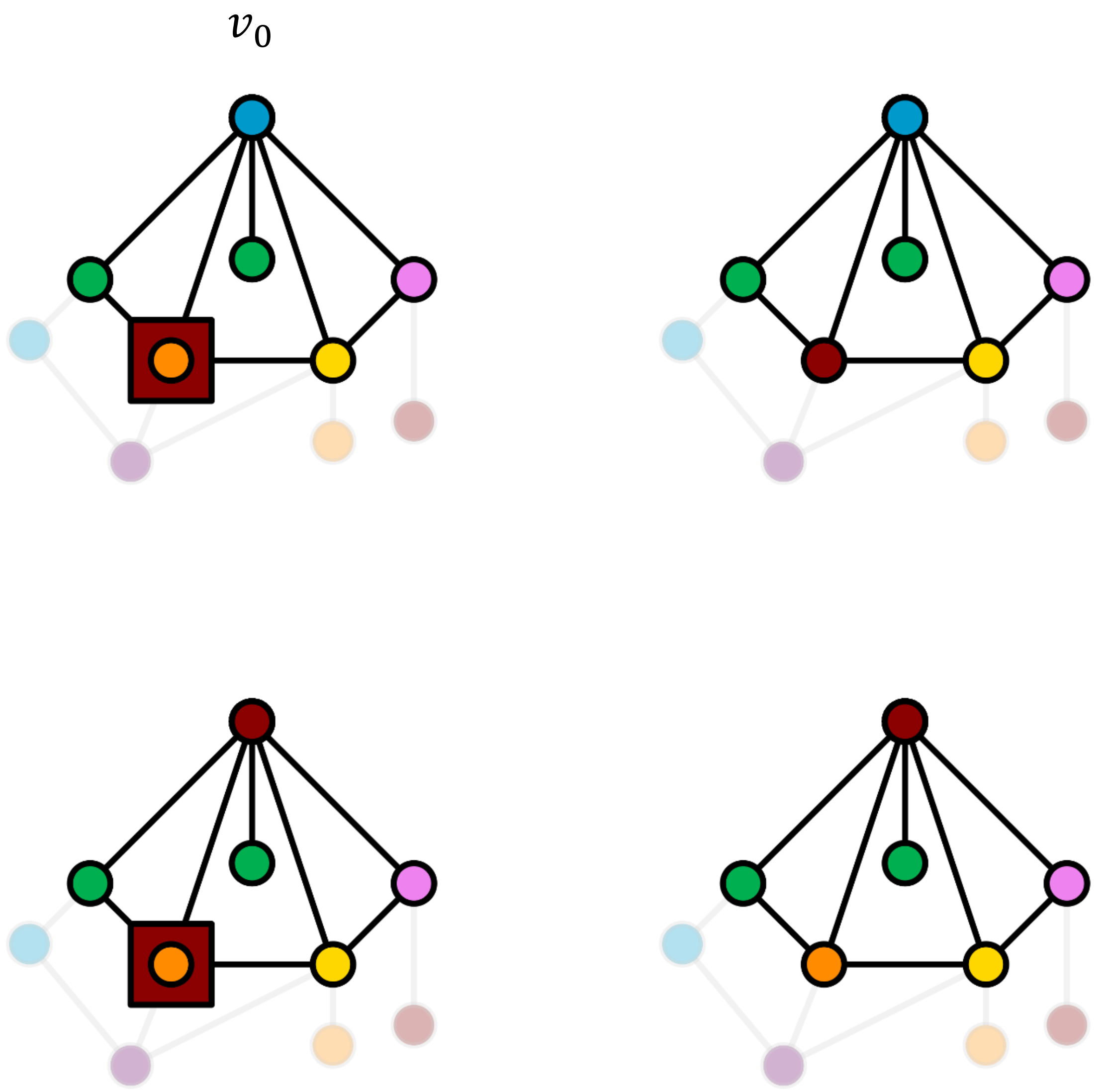

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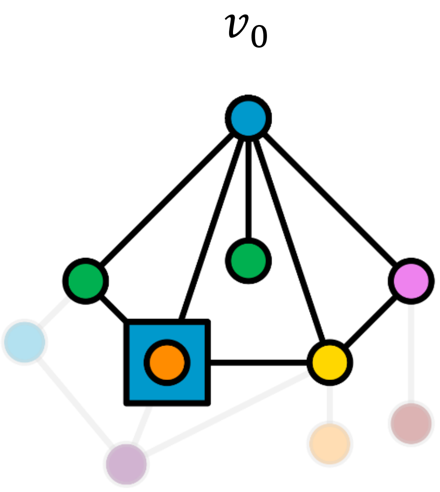

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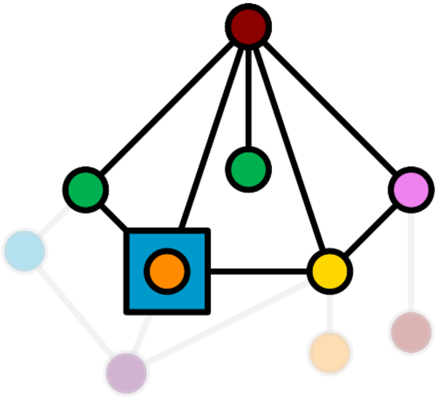

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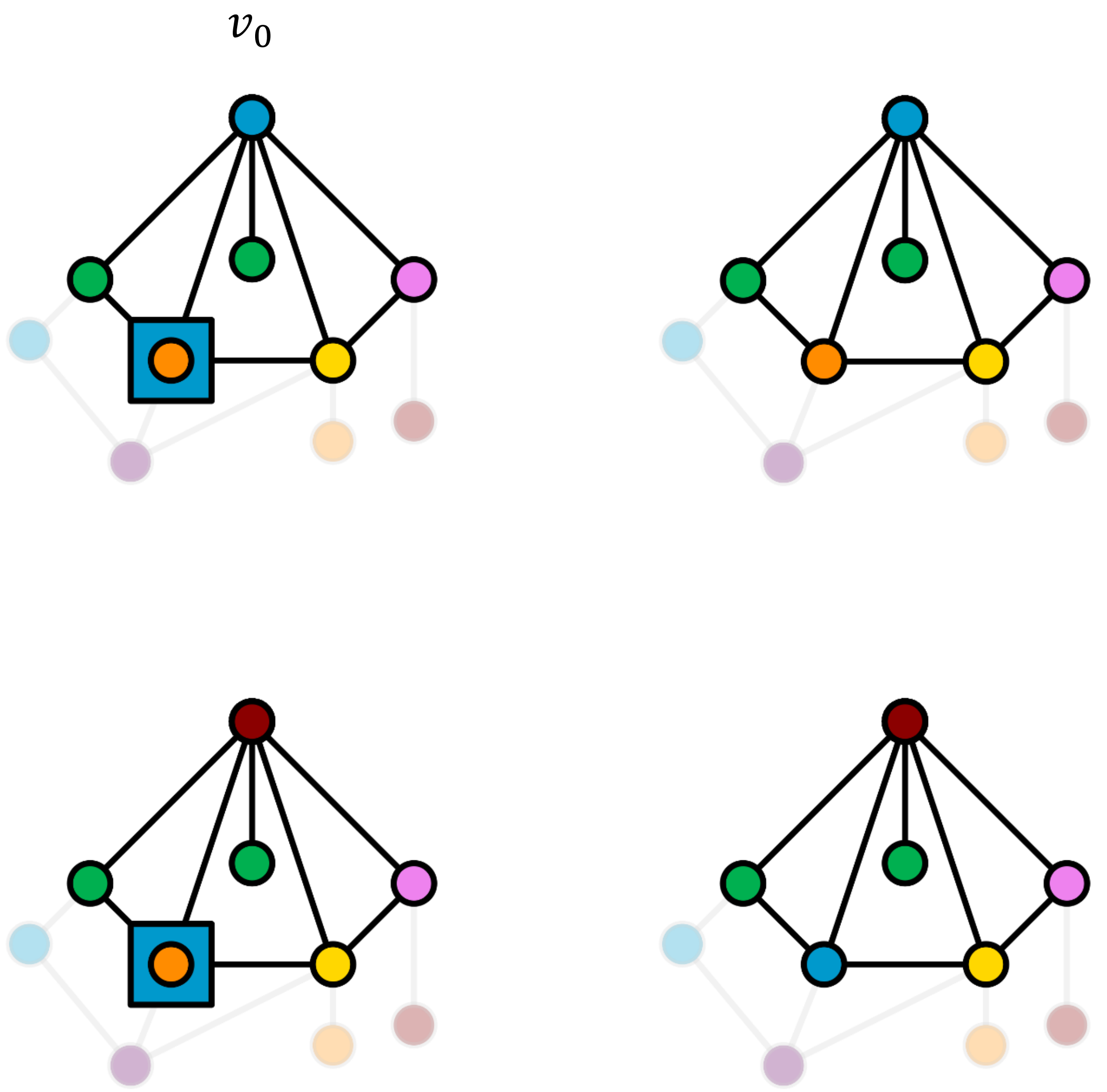

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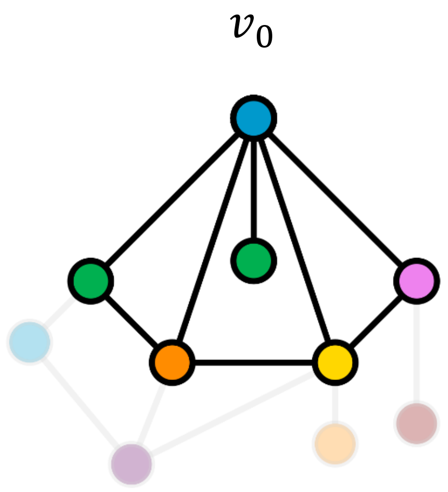

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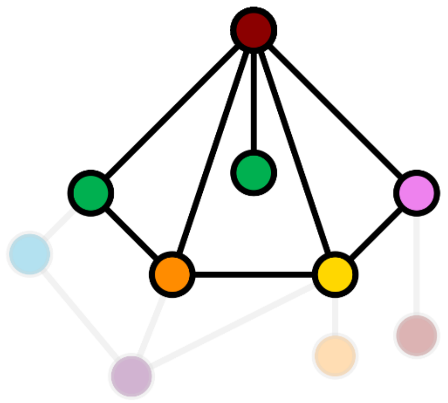

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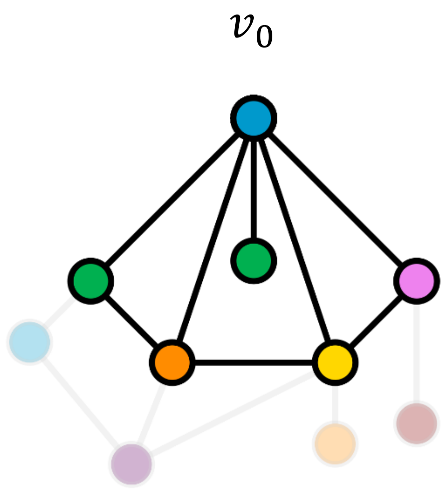

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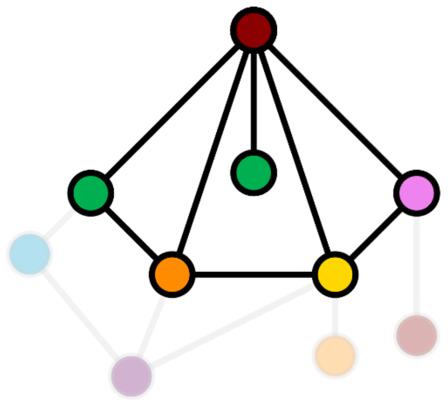
$$= 1 - \frac{1}{n}\left(1 - \frac{3\Delta}{q}\right)$$



Improved Path Coupling for Single-Site Glauber Dynamics

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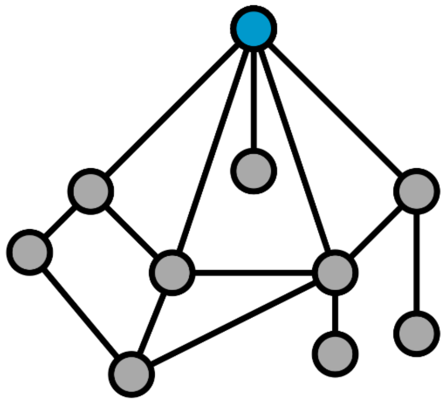
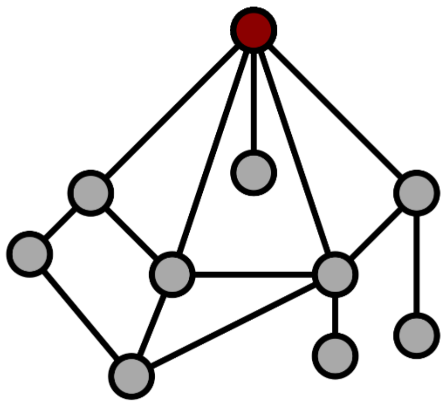
v_0





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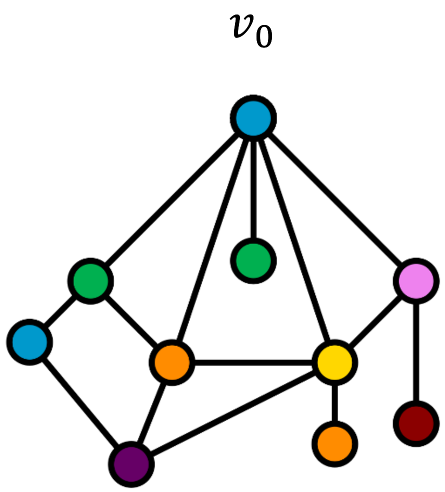
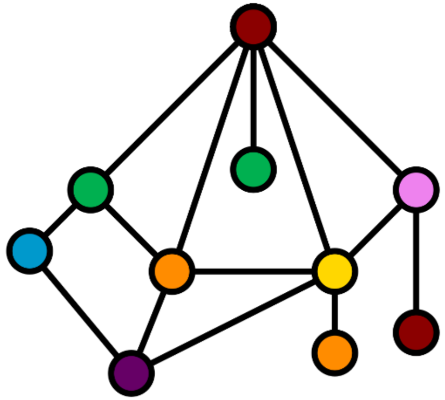
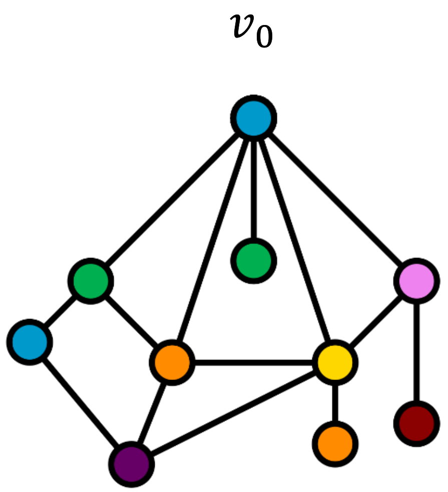


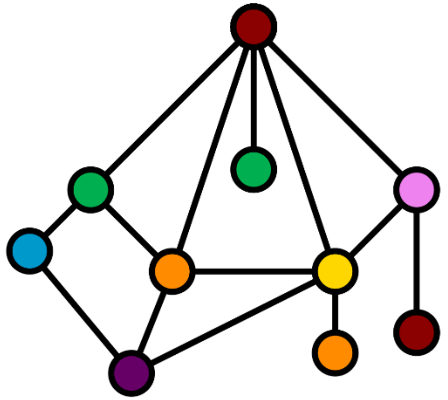


Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:

- same node v
- same color for $v \notin N(v_0)$
- flipped color for $v \in N(v_0)$





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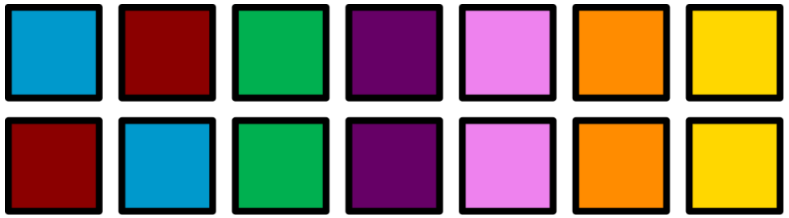

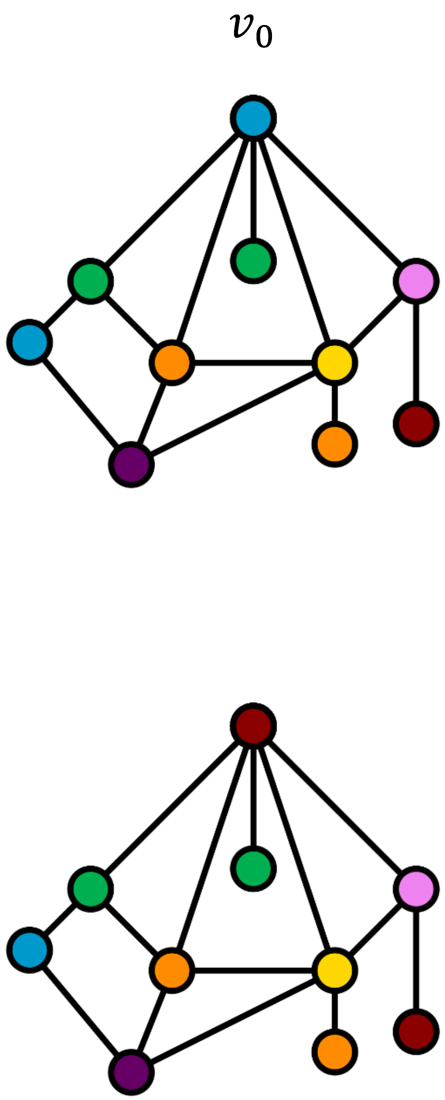

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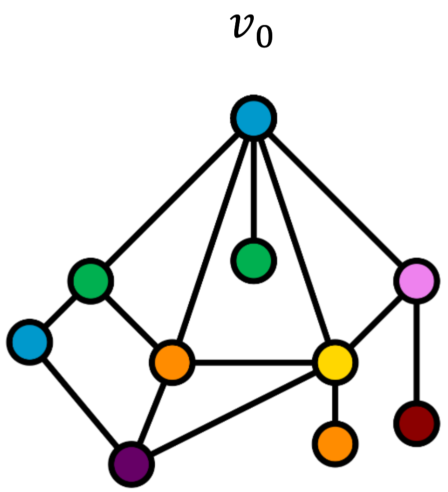
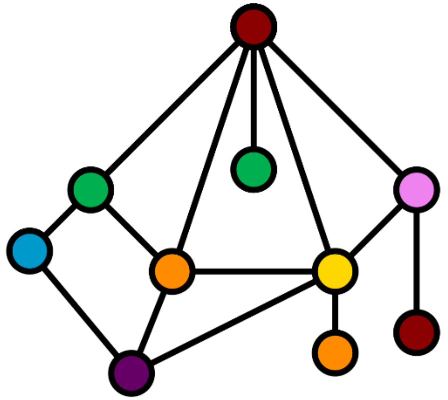
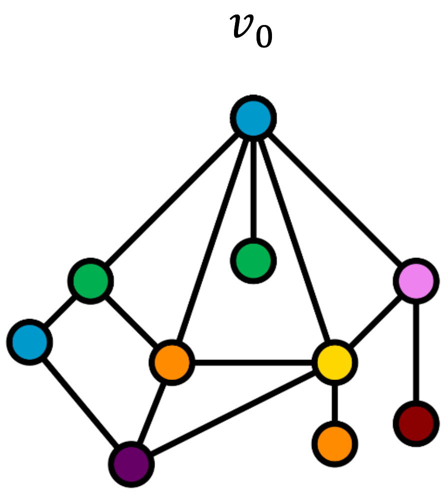




Improved Path Coupling for Single-Site Glauber Dynamics



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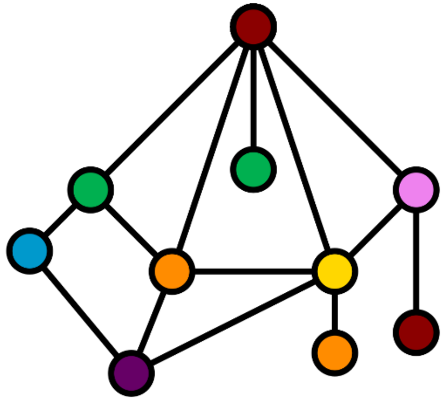

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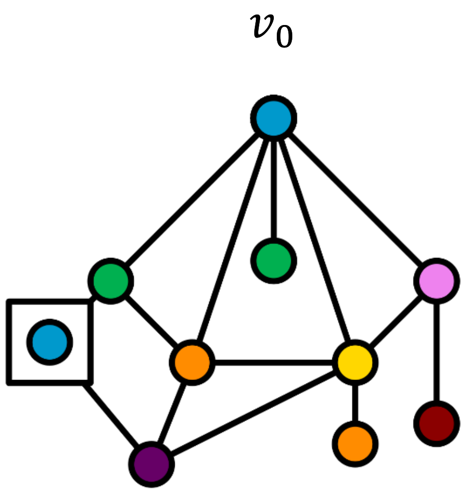

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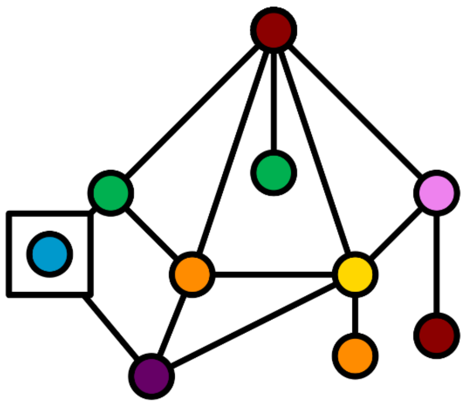

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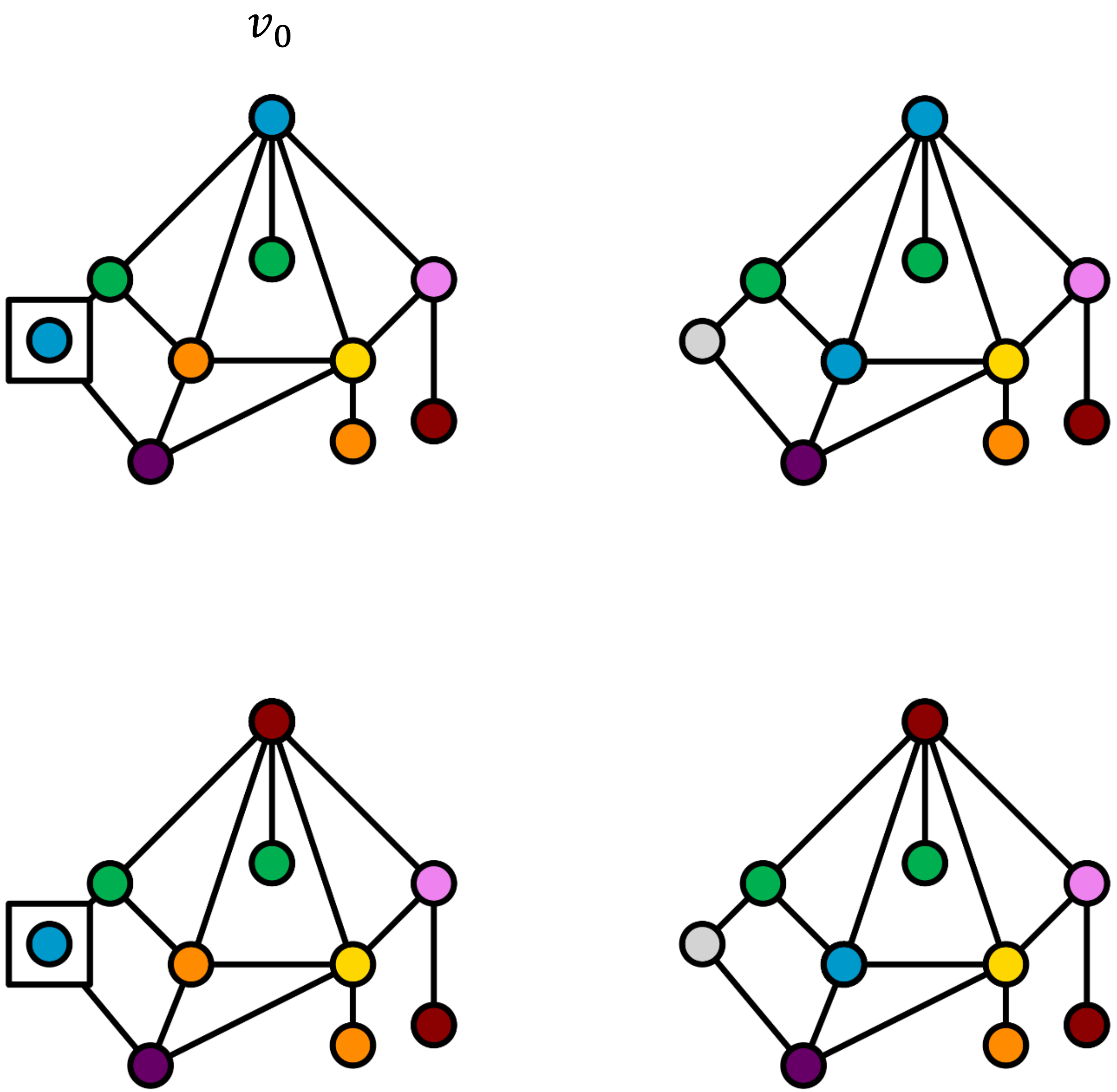

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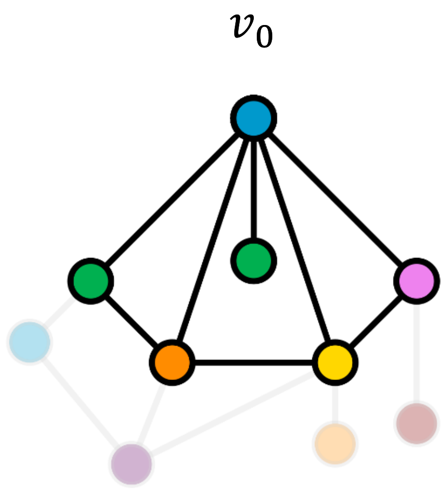

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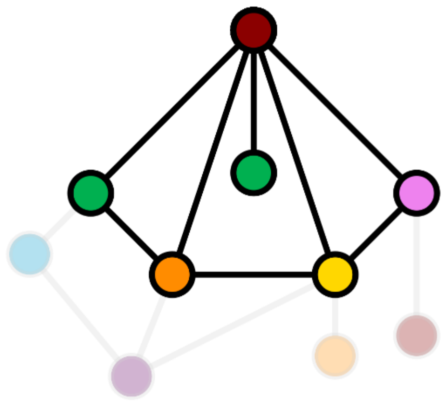

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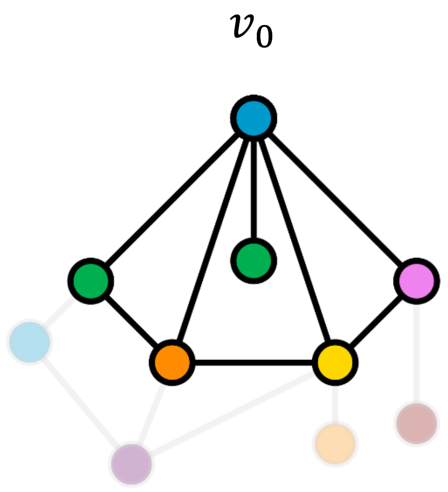

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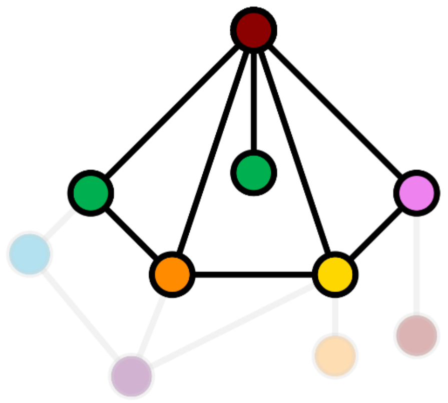
$1 - \frac{1}{n}\left(1 - \frac{\Delta}{q}\right)$ *[node v_0]*



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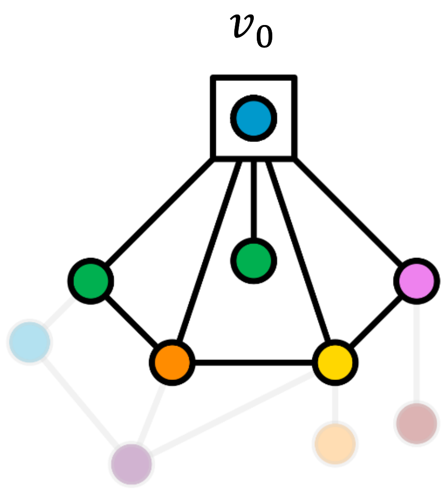

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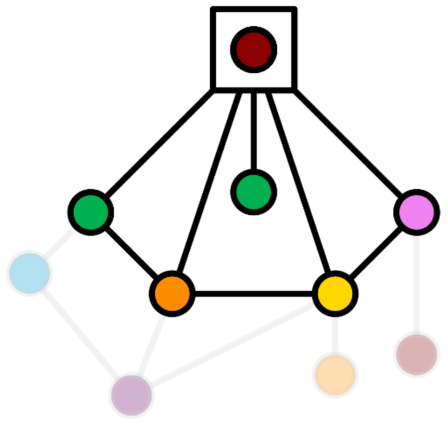

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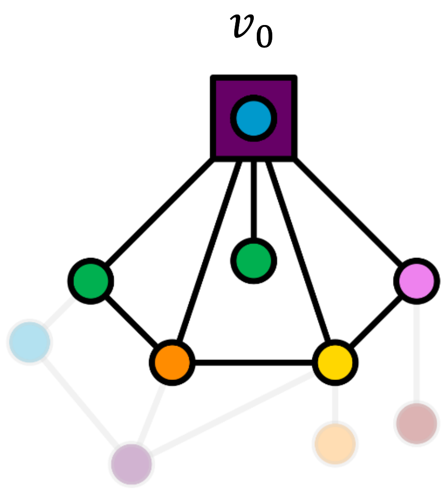

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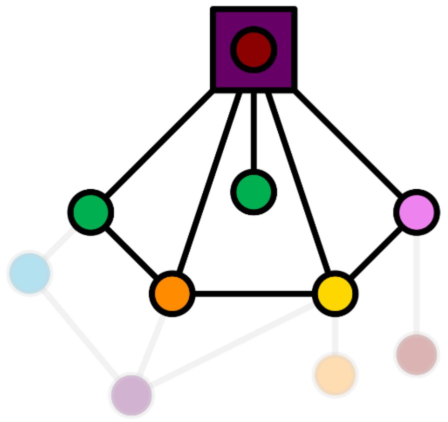

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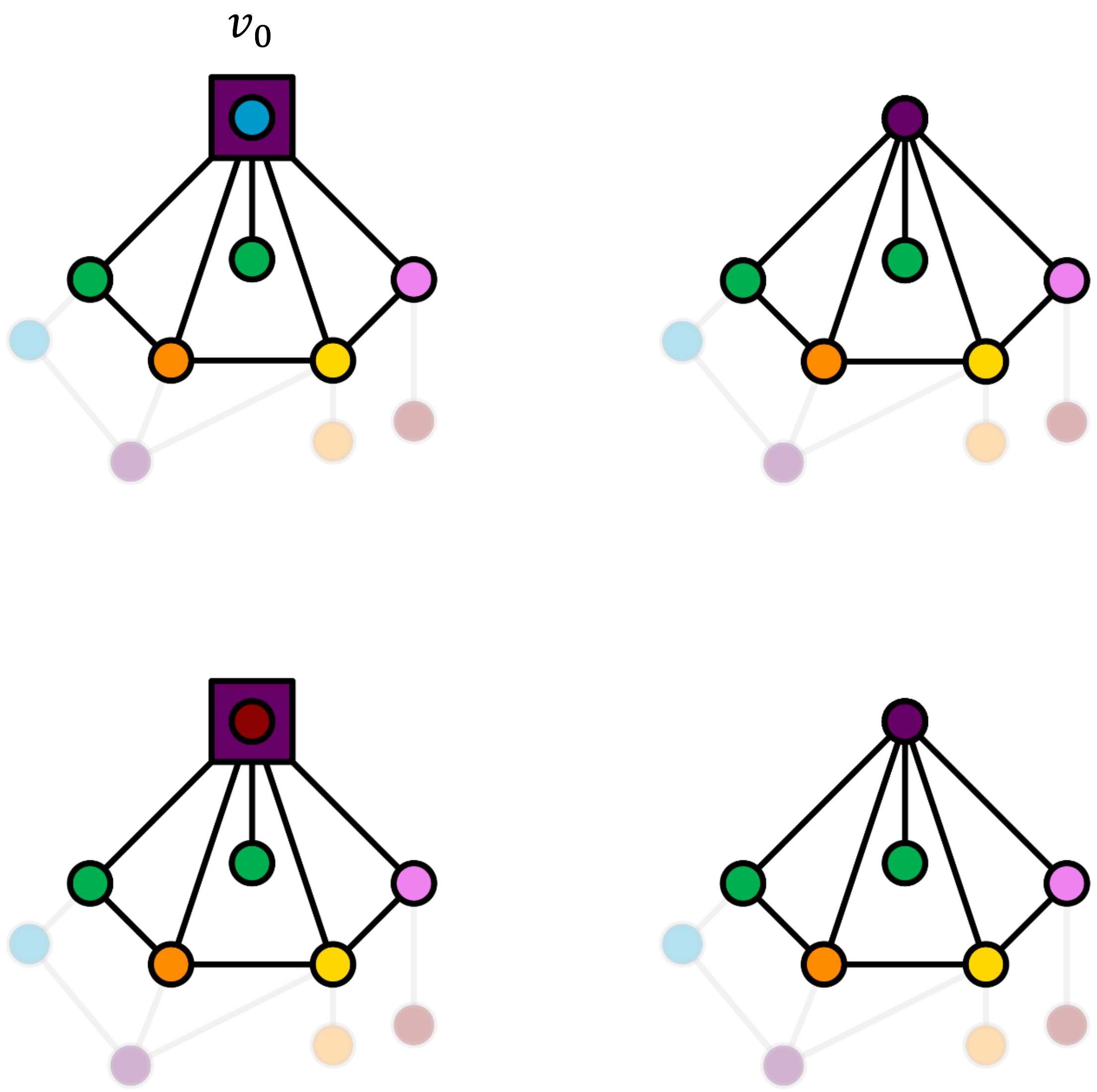
[node v_0]



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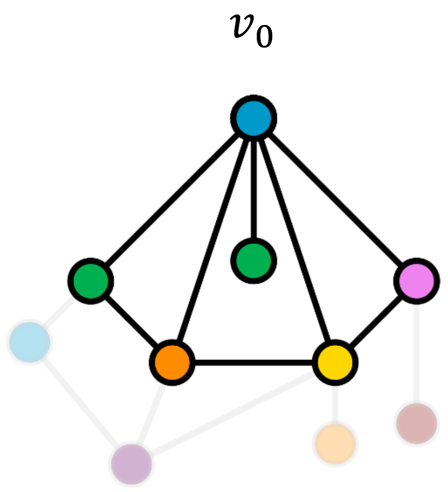

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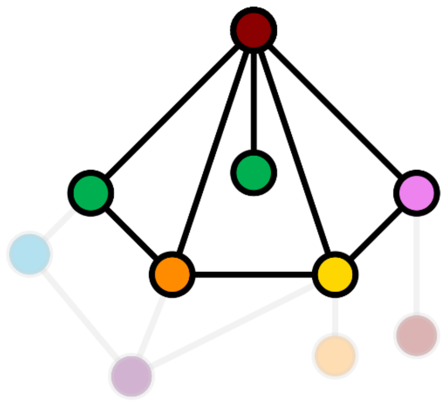

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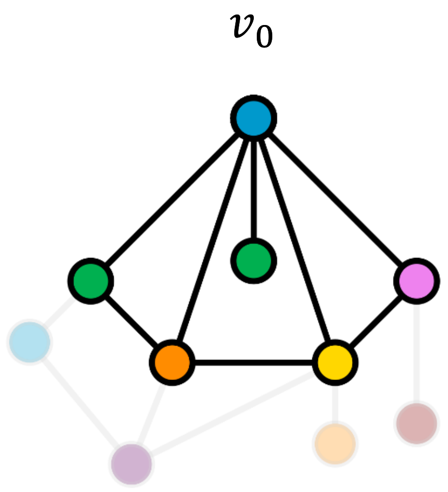

expected number of differing nodes:

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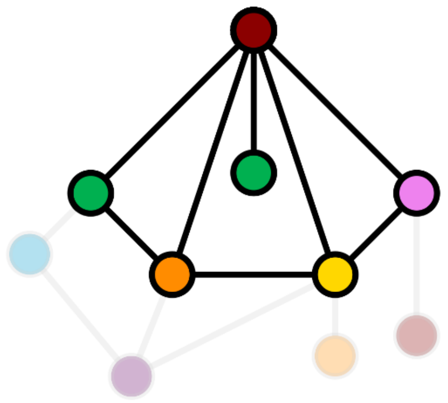
$$+ \Delta \cdot \frac{1}{n} \cdot \frac{1}{q}$$ *[neighbors of v_0]*



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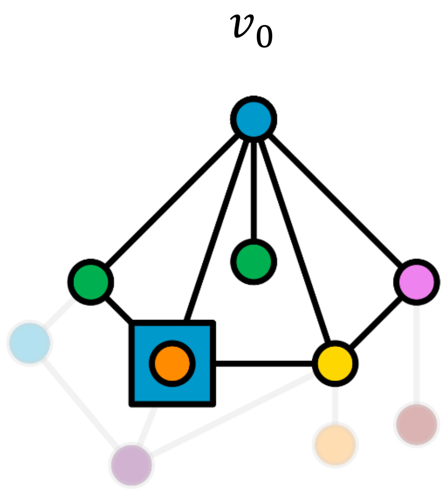

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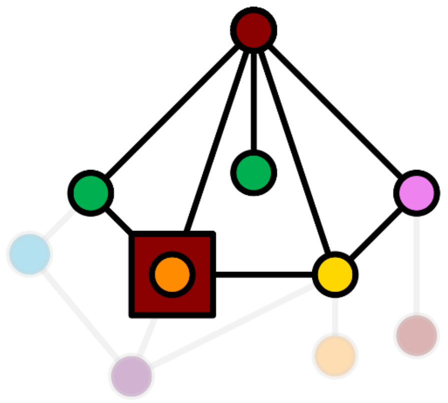

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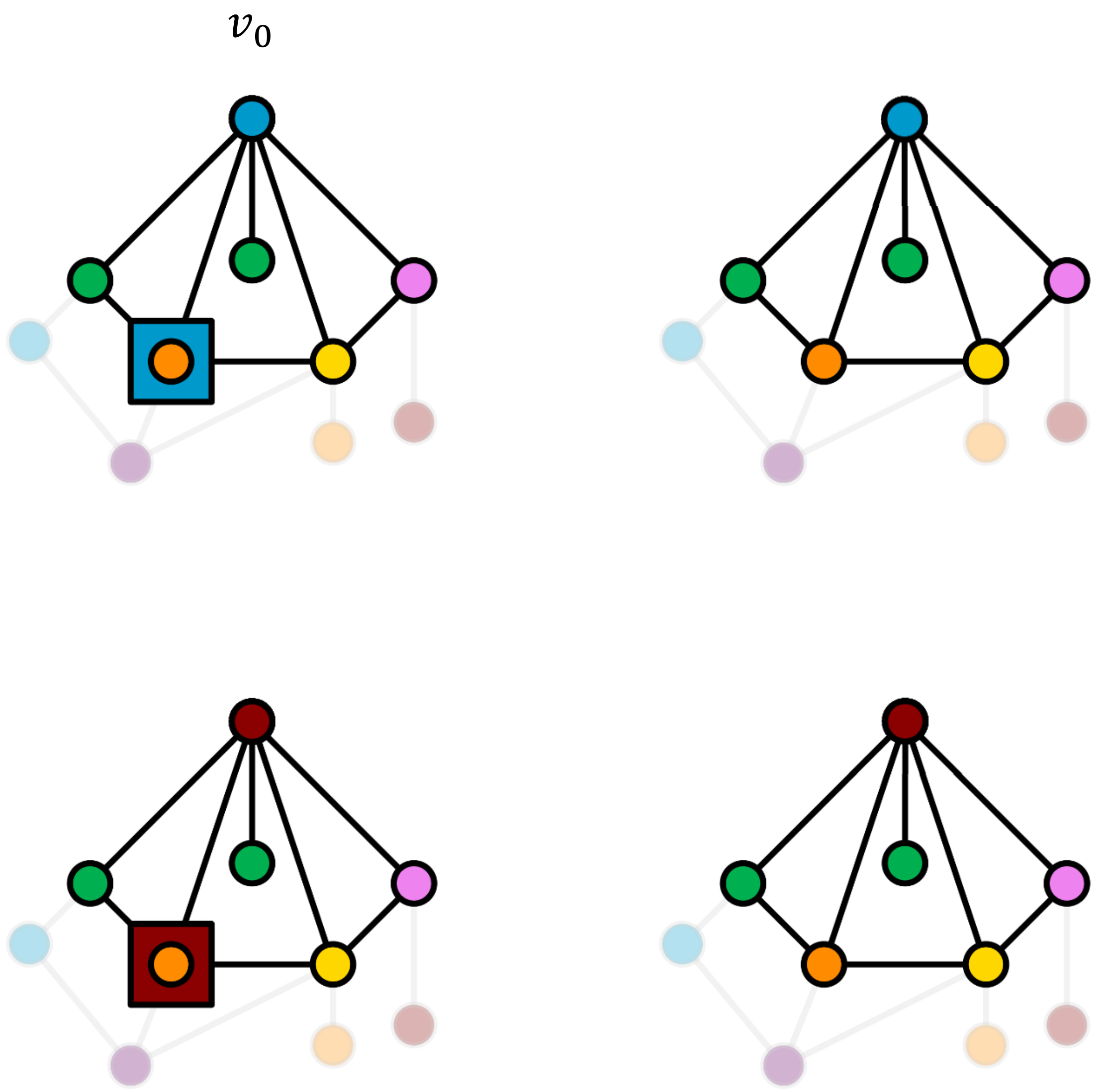

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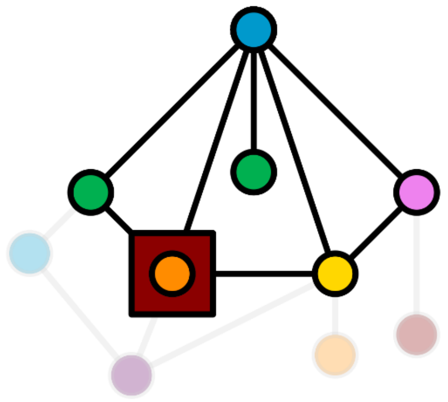

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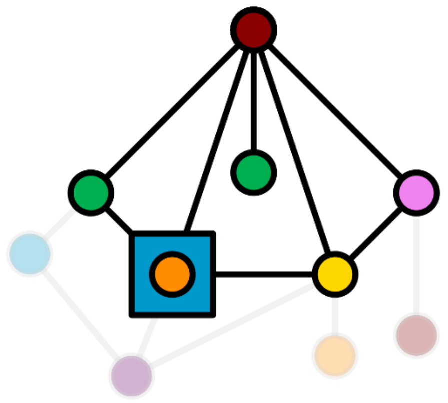

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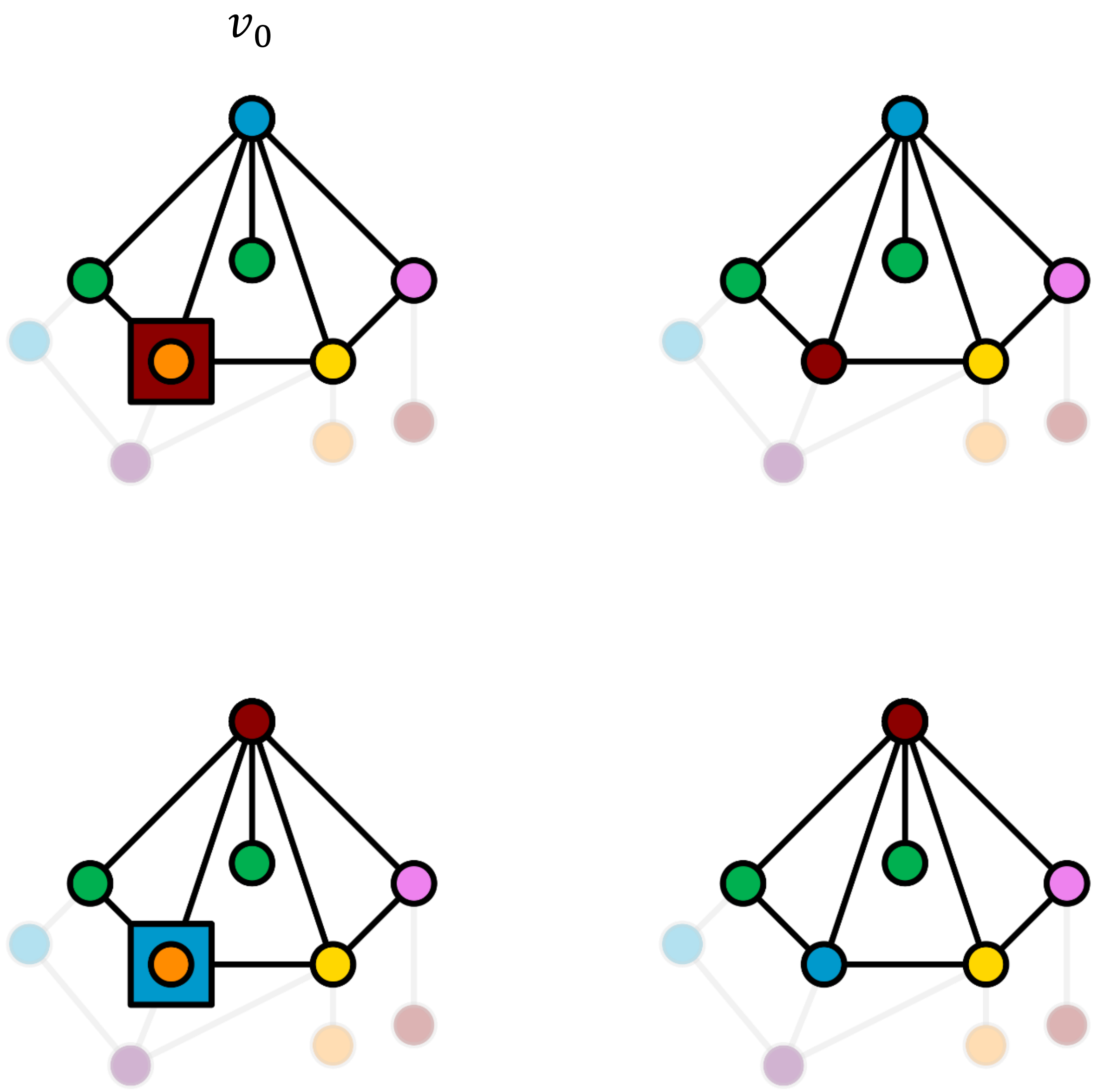

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Sketch of
Path Coupling for
Local Glauber Dynamics

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update colors of a **small-degree node set** to **random colors**, if still proper

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each node v

independently marks itself with probability $\gamma \in (0,1)$

if marked, picks a proposal $\mathrm{c_v}$ uniformly at random

if $\mathrm{c_v}$ does not lead to (potential) conflicts, updates color

Path Coupling for Local Glauber Dynamics

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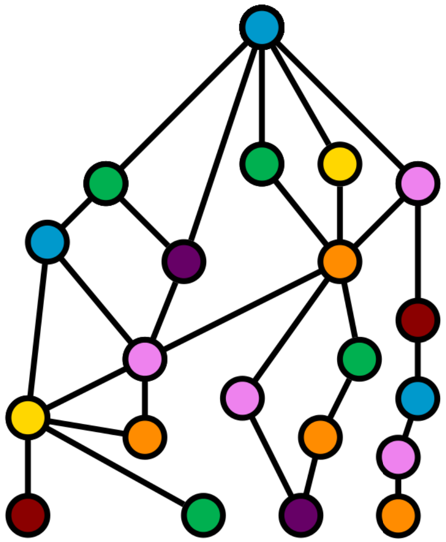

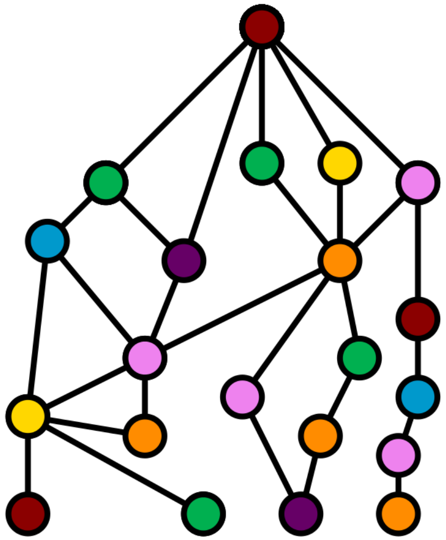

Path Coupling for Local Glauber Dynamics

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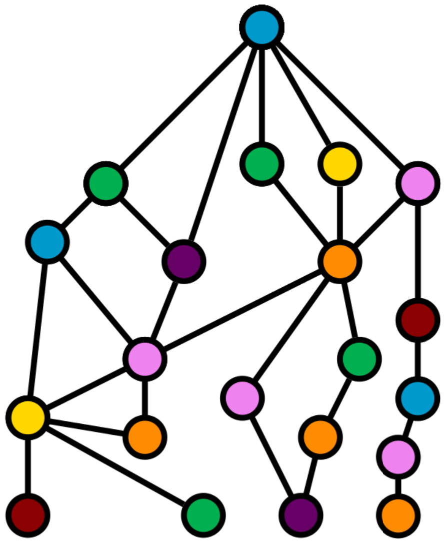

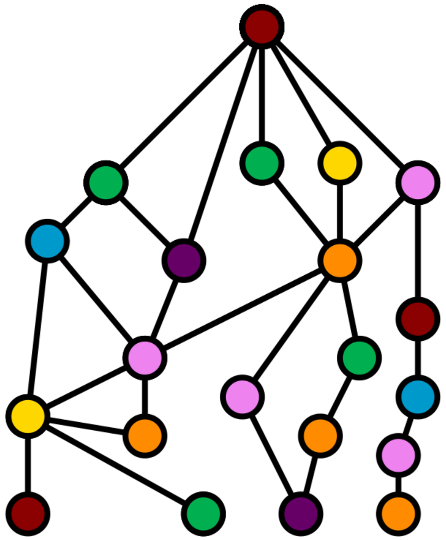

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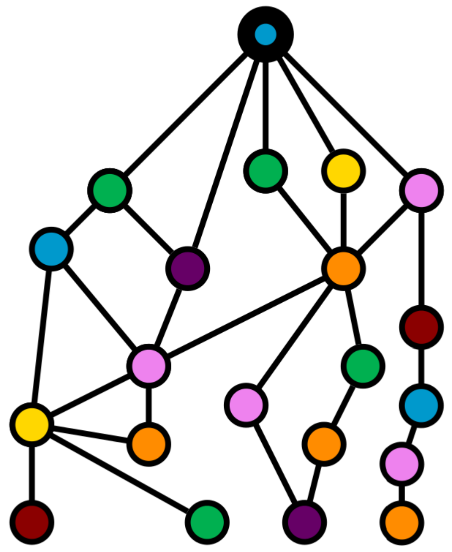

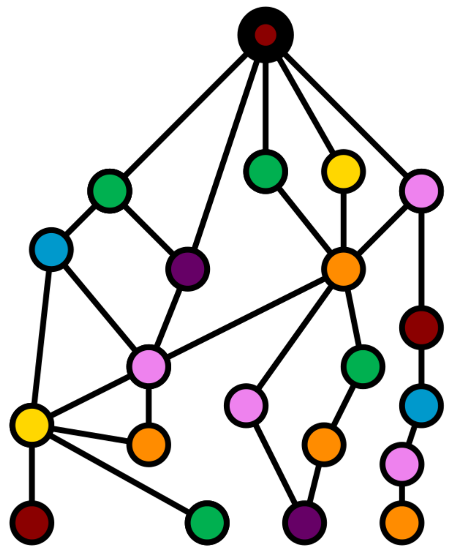

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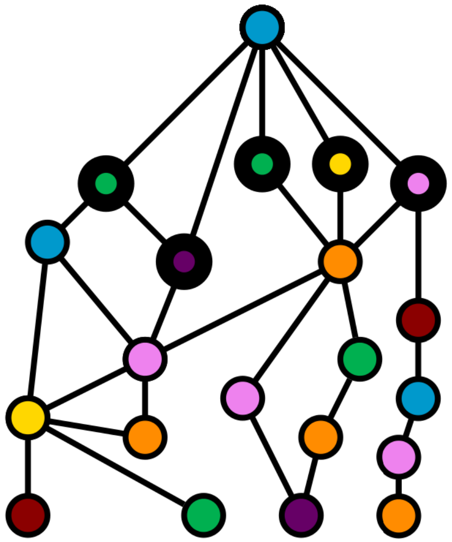

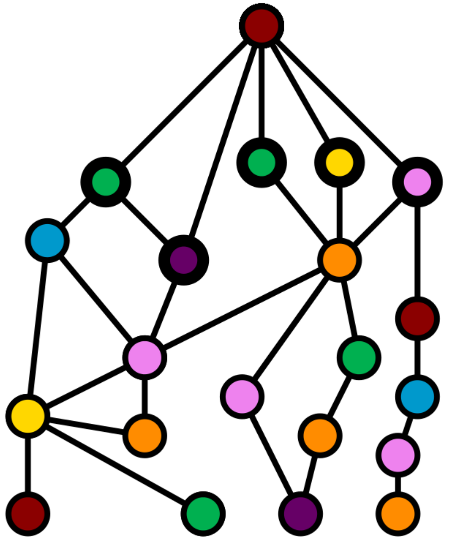

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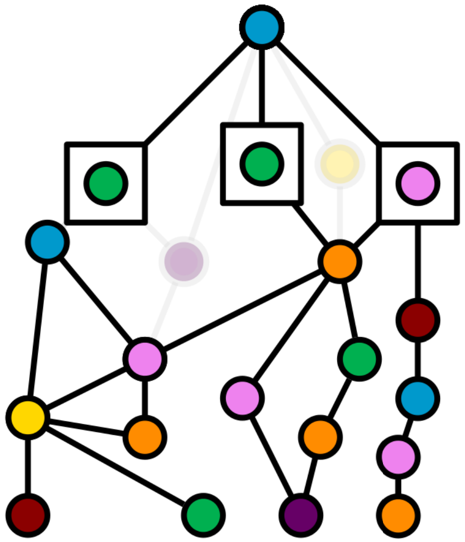

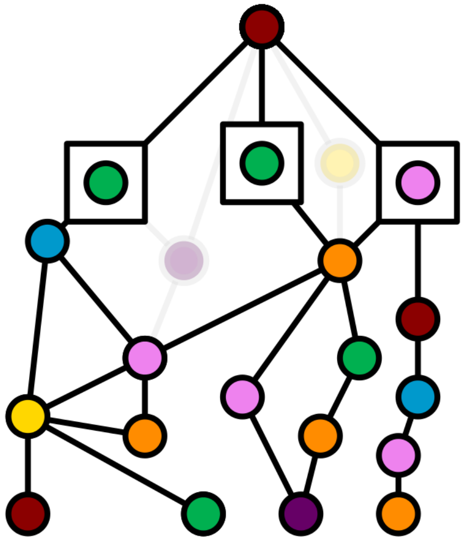

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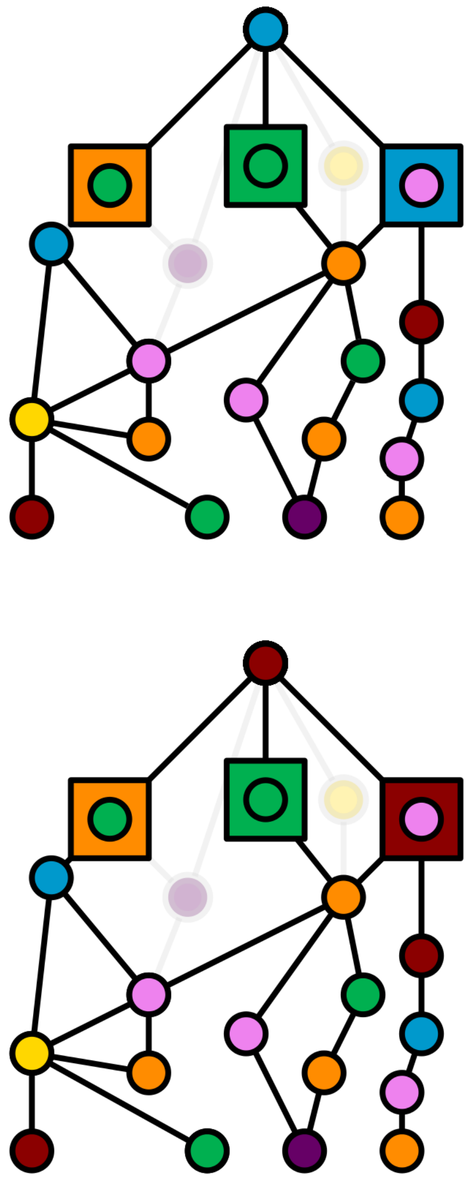

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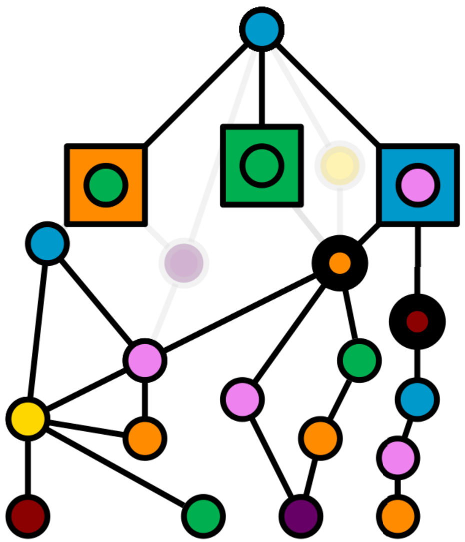

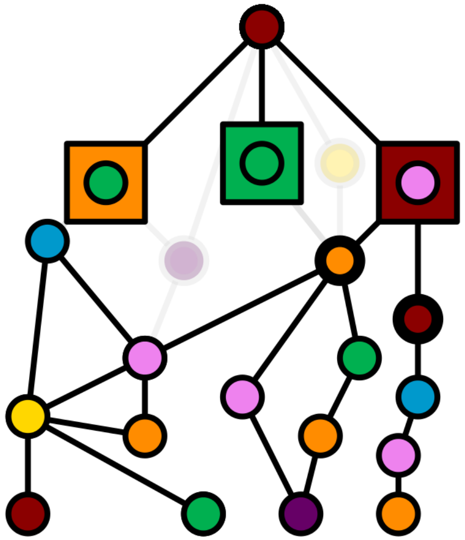

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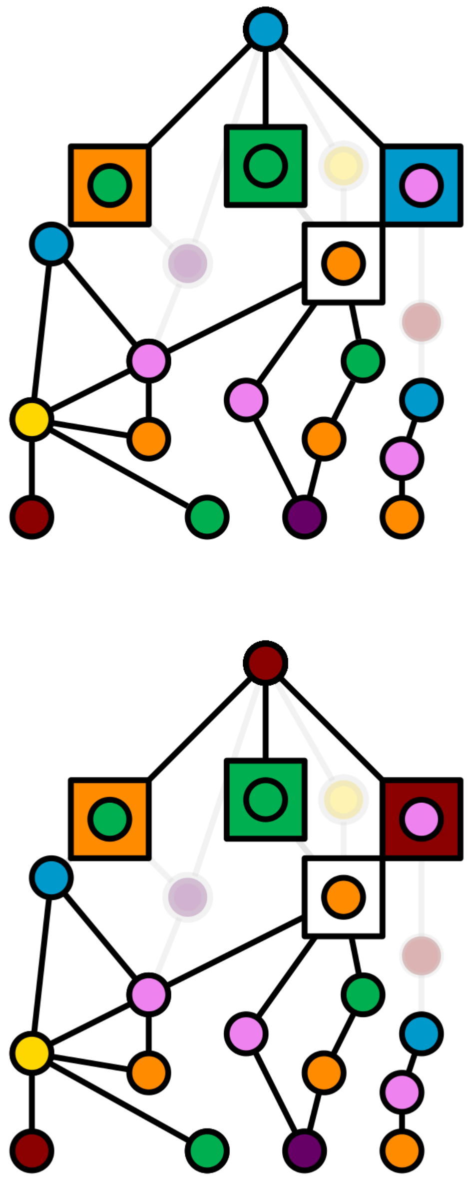

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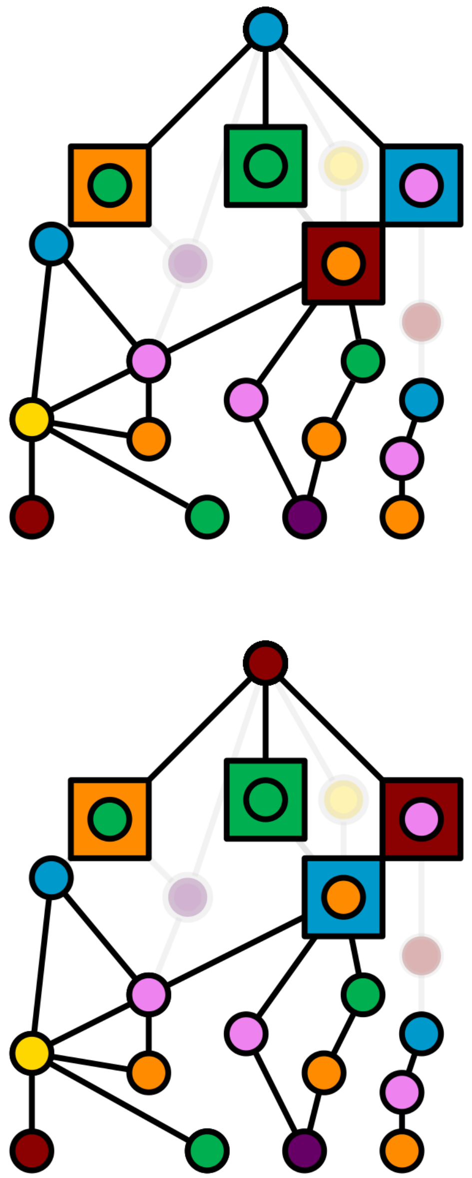

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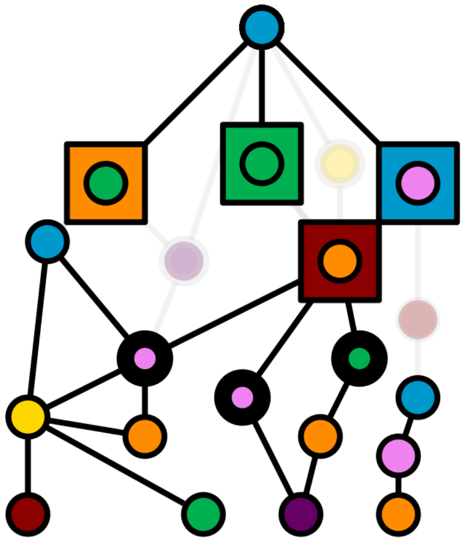

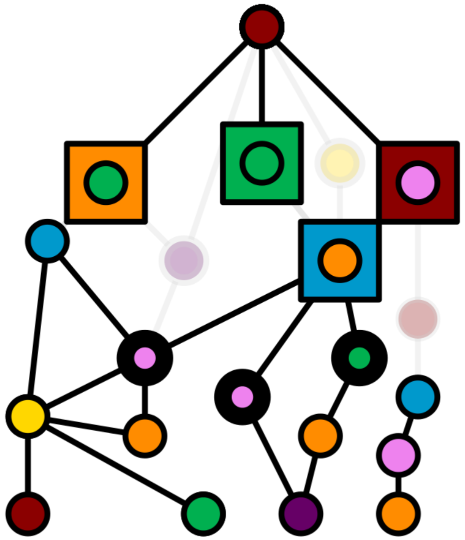

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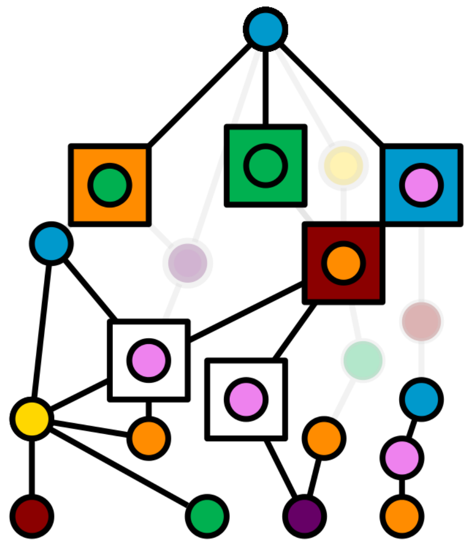

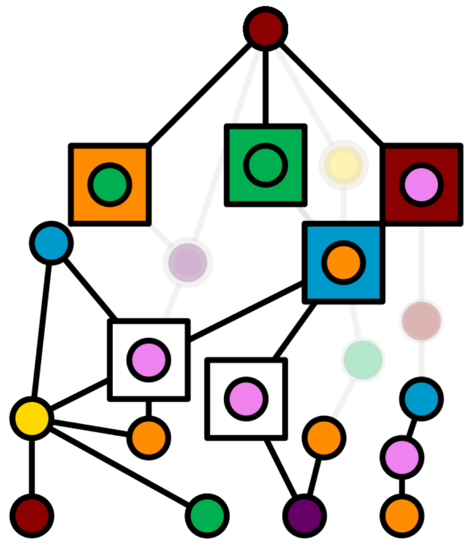

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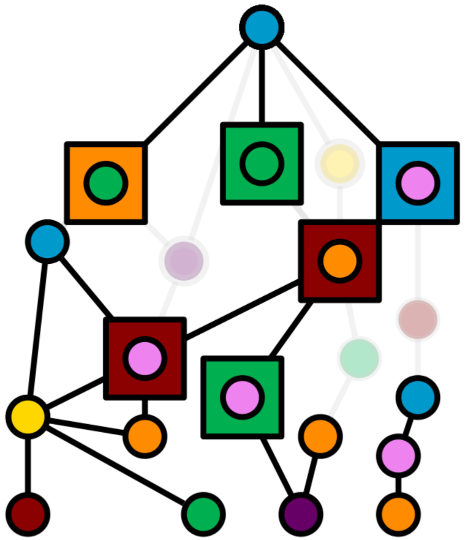

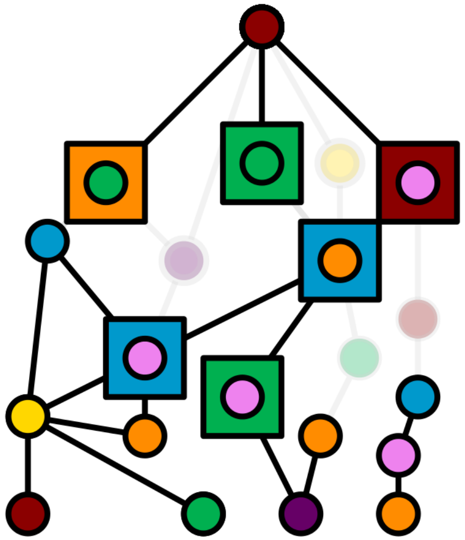

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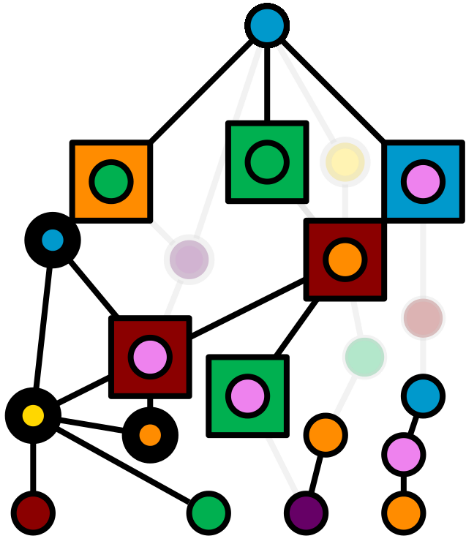

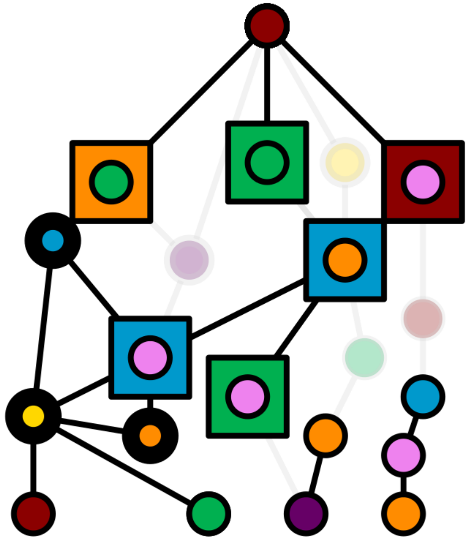

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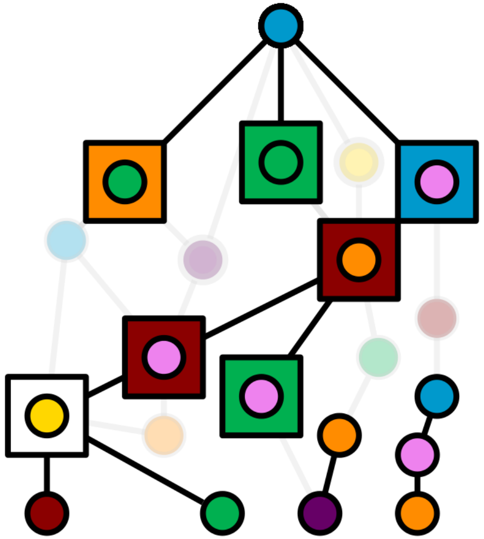

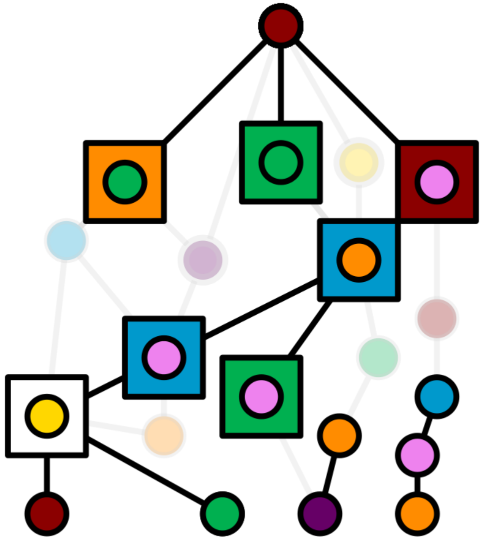

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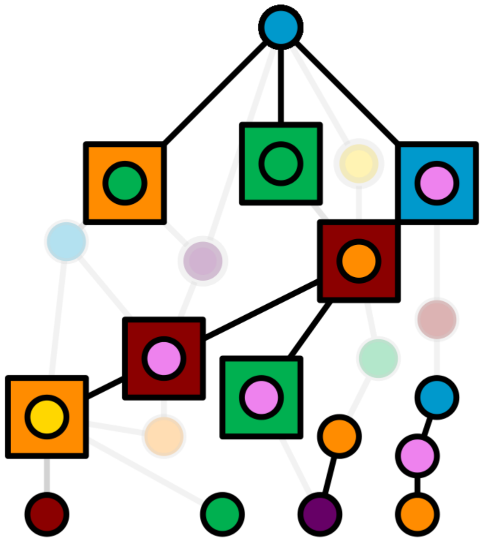

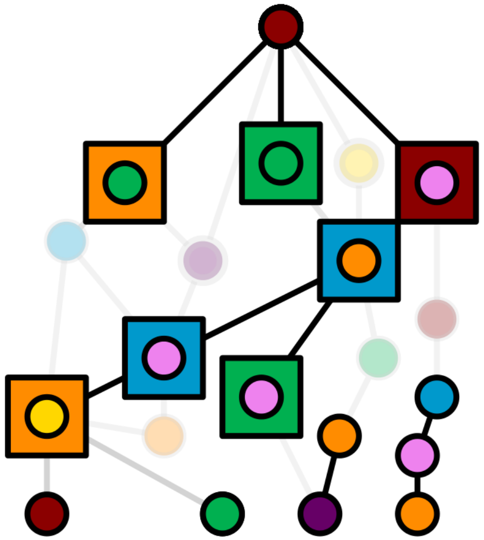

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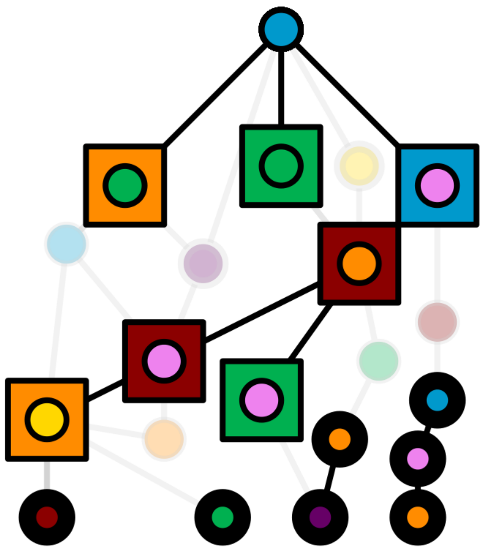

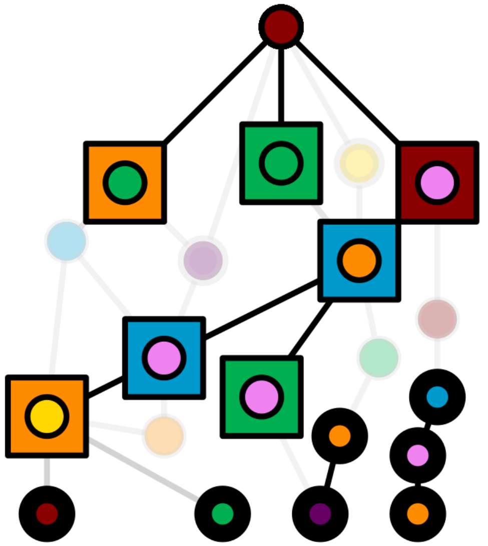

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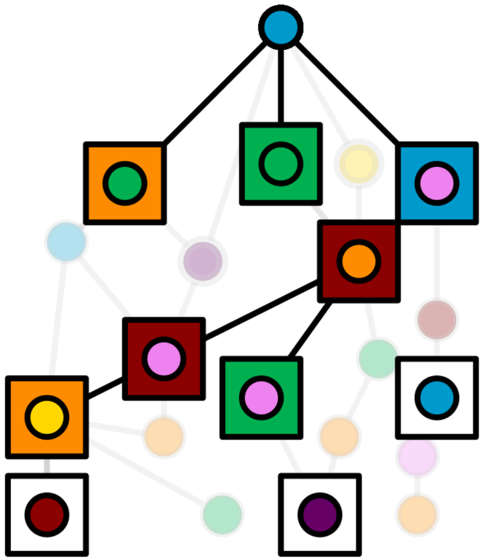

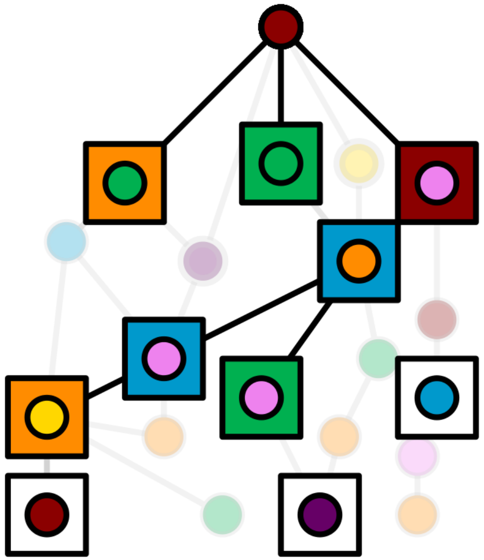

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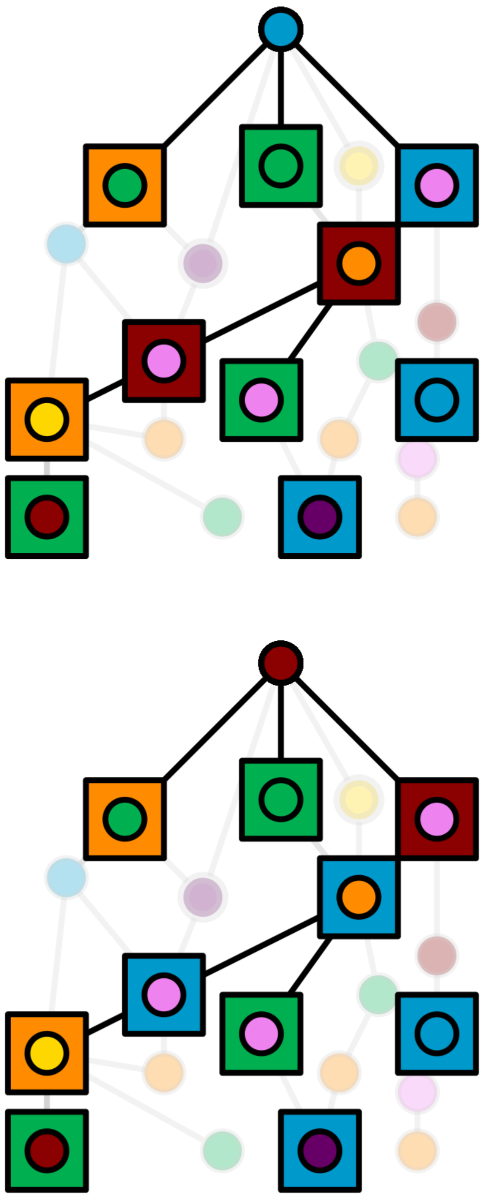

Path Coupling for Local Glauber Dynamics

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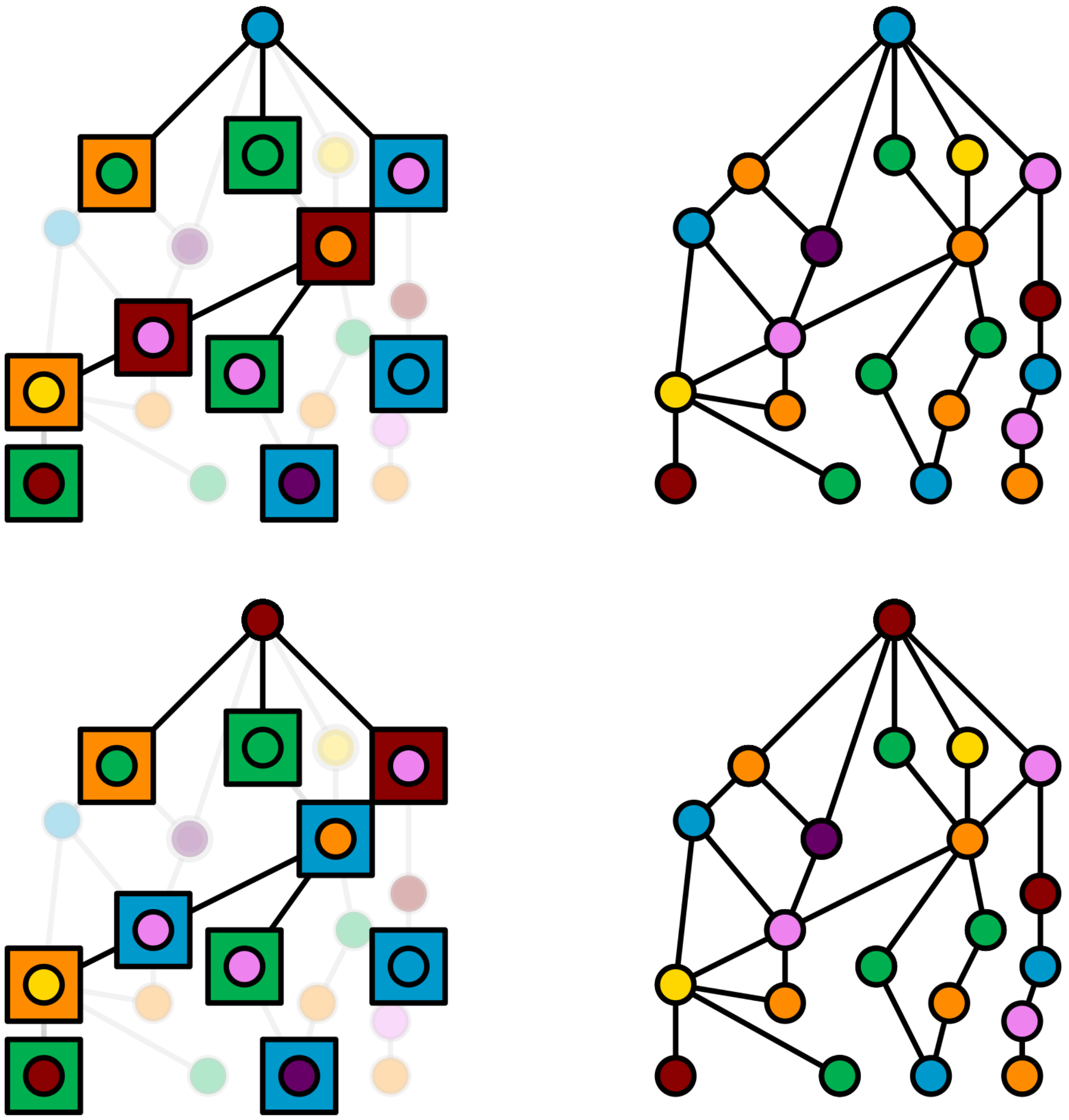

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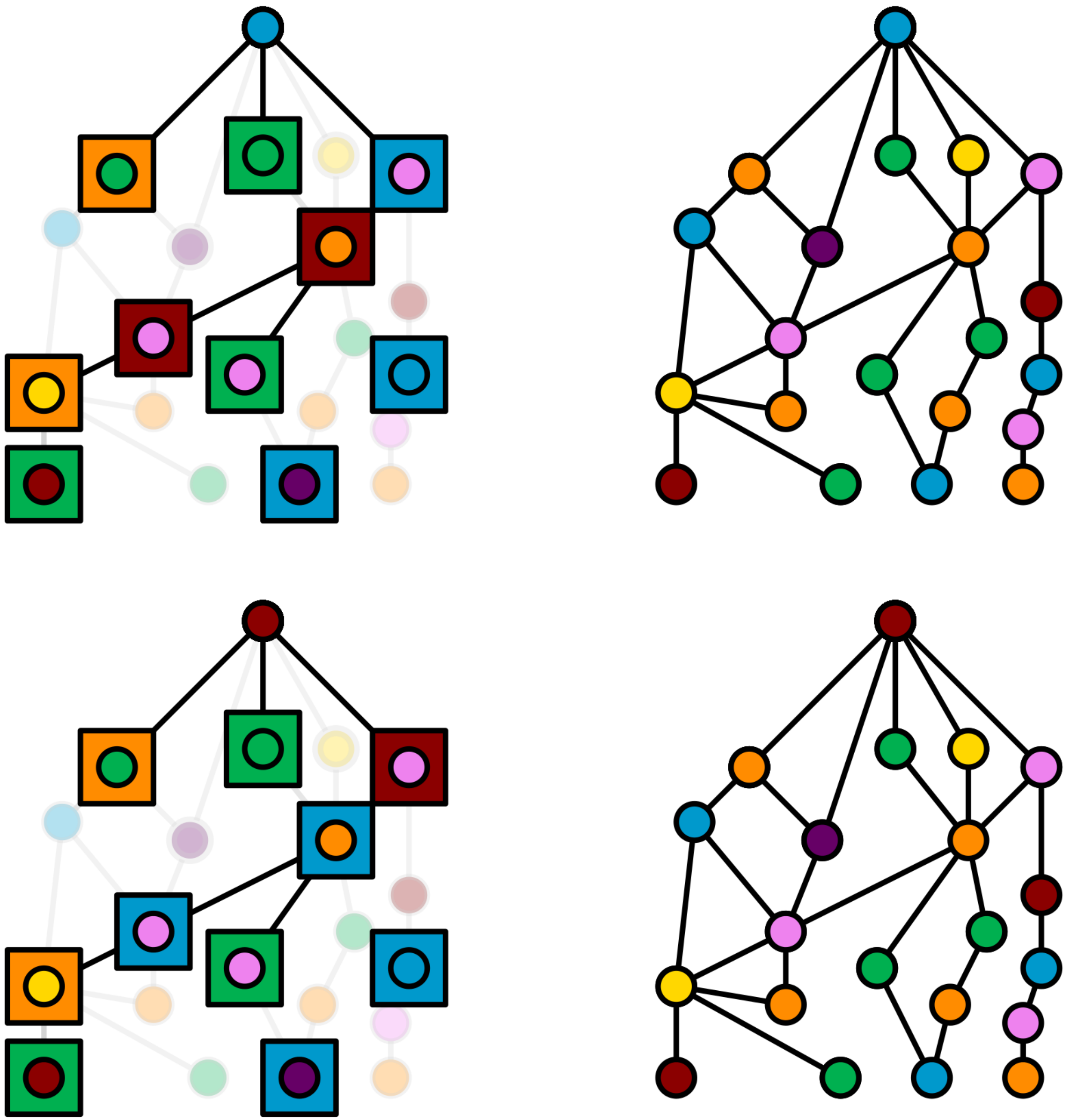

Path Coupling for Local Glauber Dynamics

flipped coupling:

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expected number of differing nodes:

$$1 - \gamma\left(1 - \frac{\Delta}{q}\right)\left(1 - \frac{3\gamma}{q}\right)^{\Delta} \qquad \textit{[node } v_0\textit{]}$$

Path Coupling for Local Glauber Dynamics

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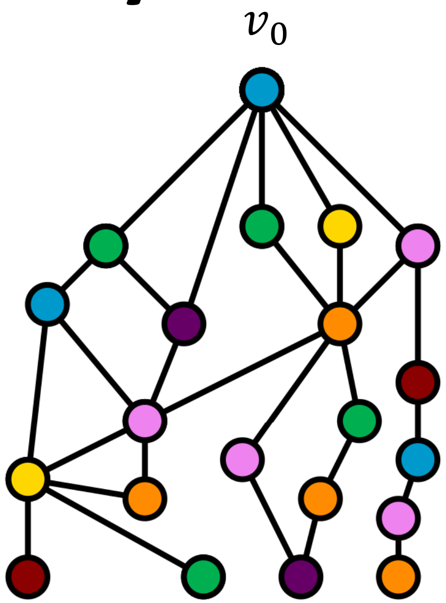

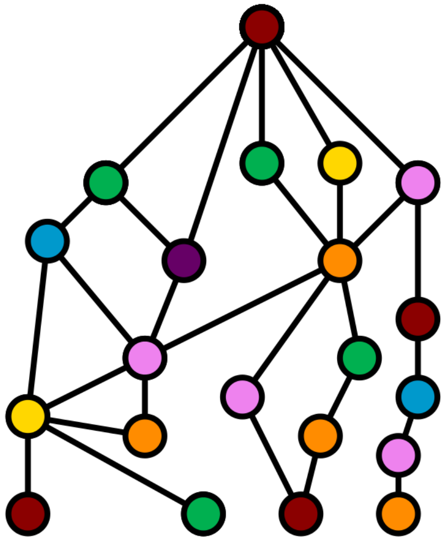

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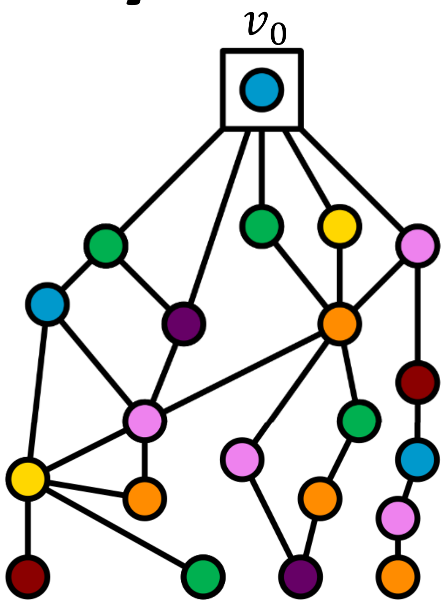

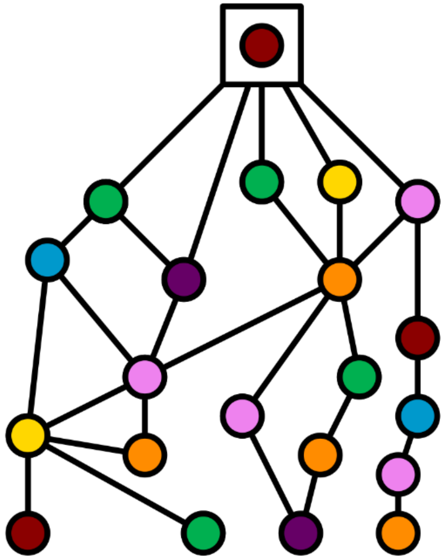

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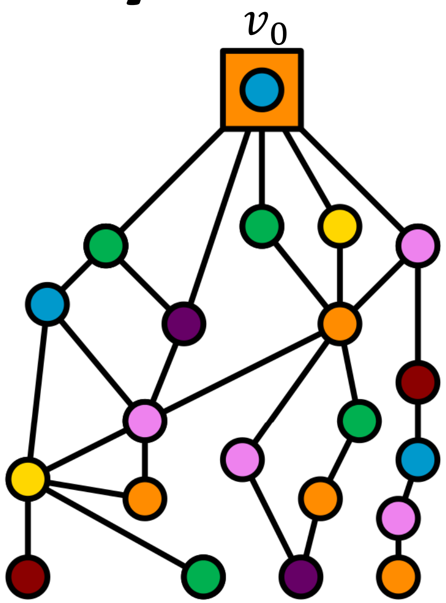
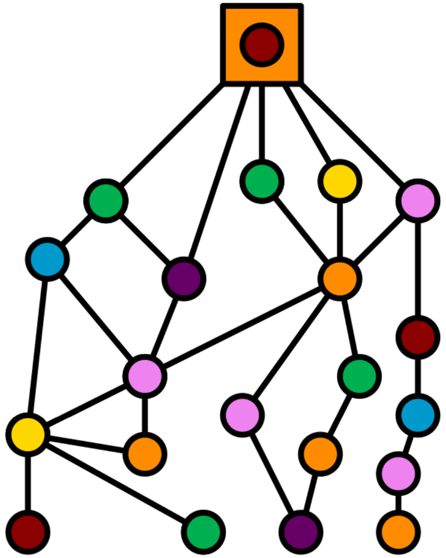
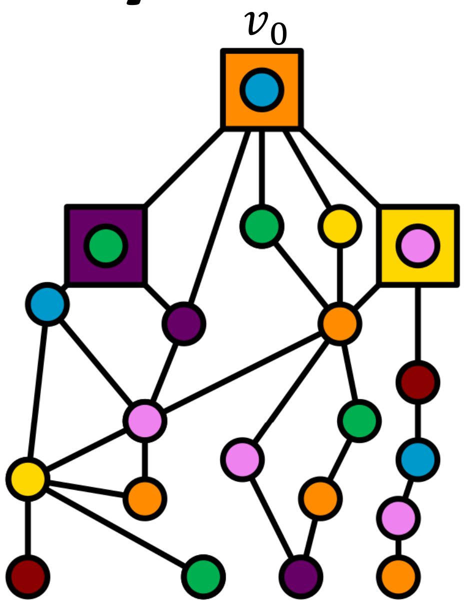




Path Coupling for Local Glauber Dynamics



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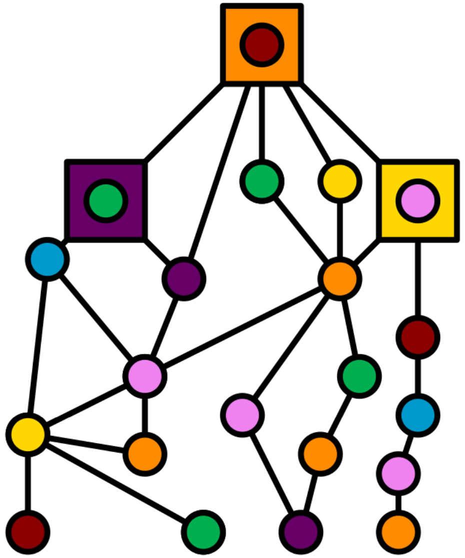

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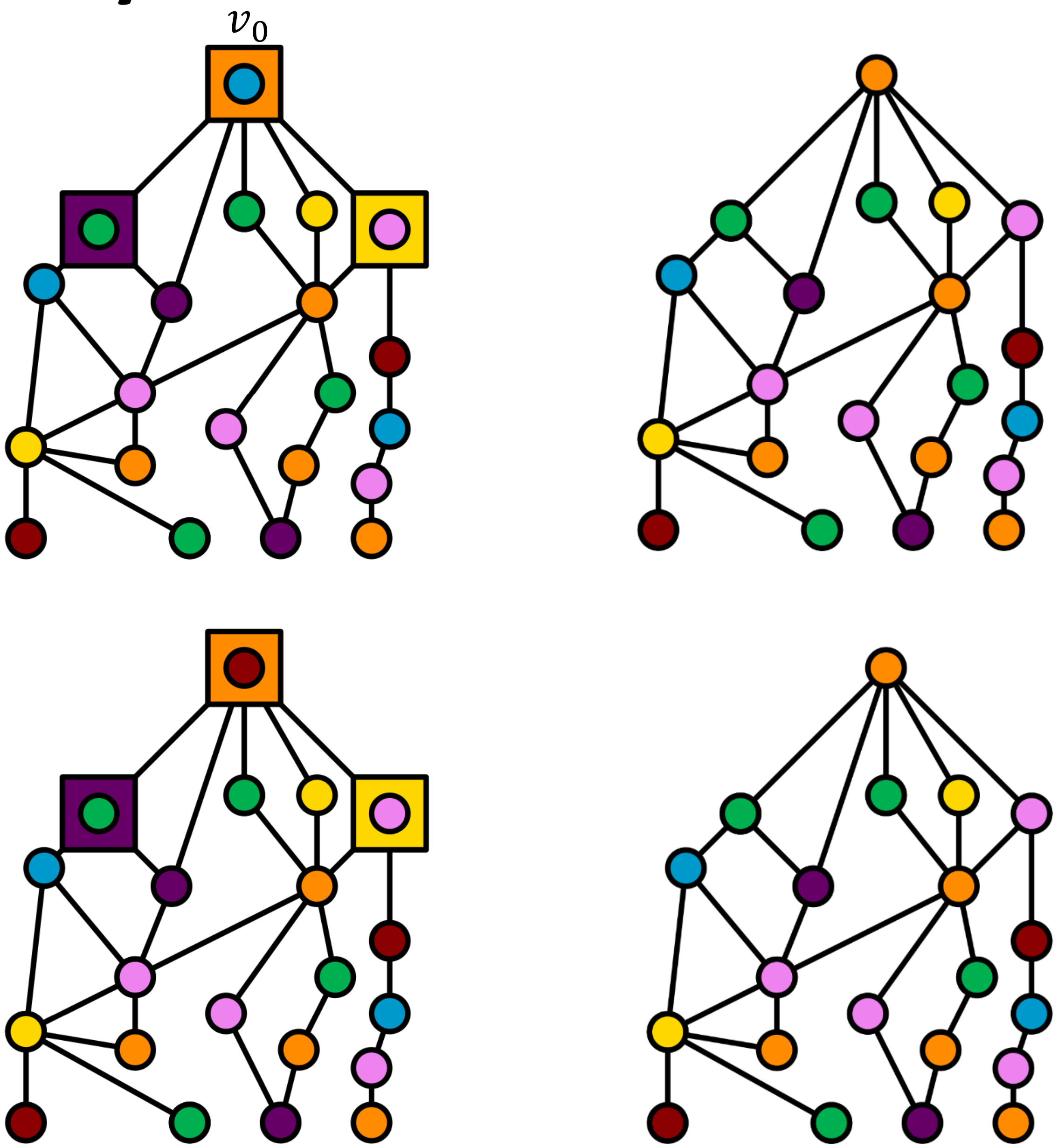

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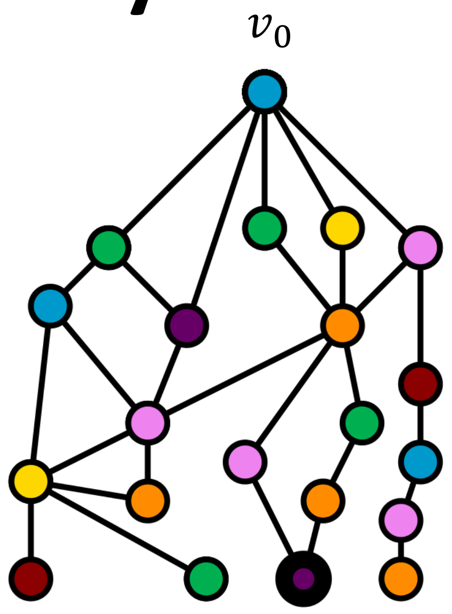
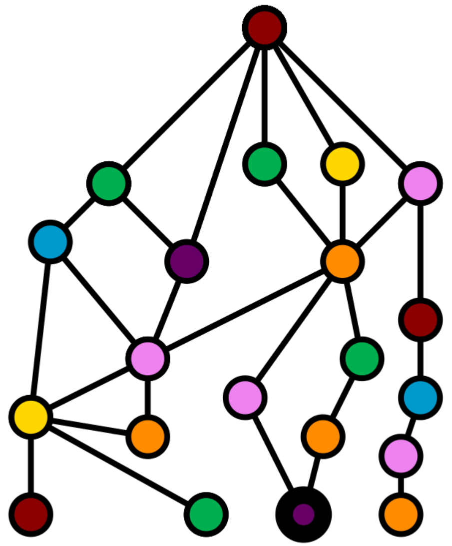
[node v_0]





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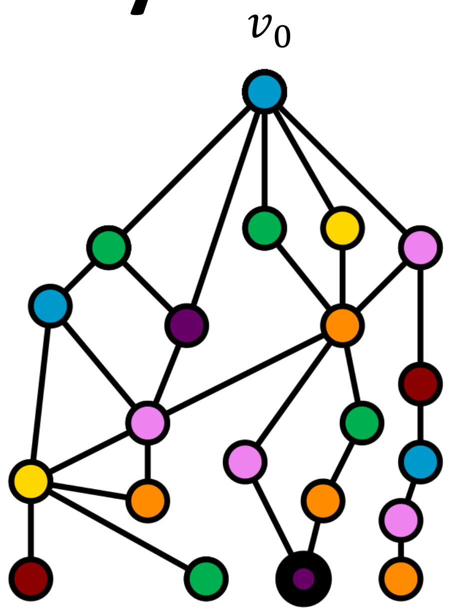

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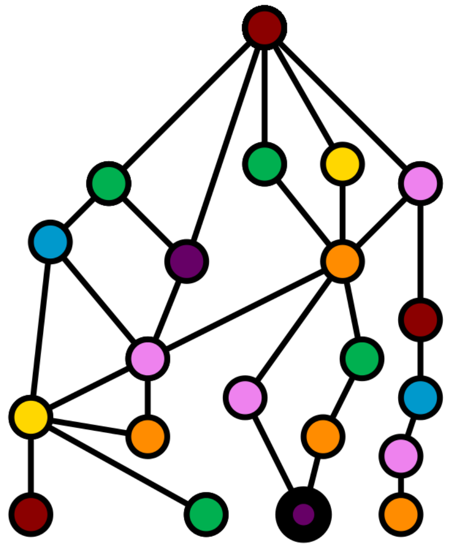

Main Observation:
to have different colors,
node needs to be on a **red**-**blue**-path
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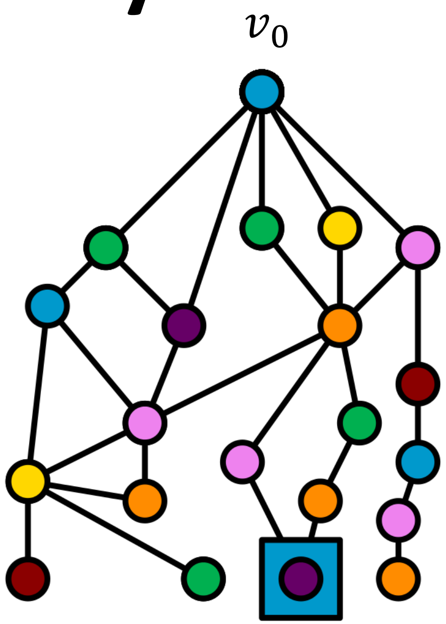

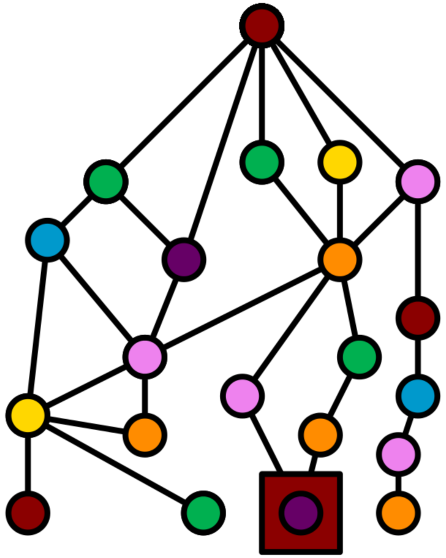

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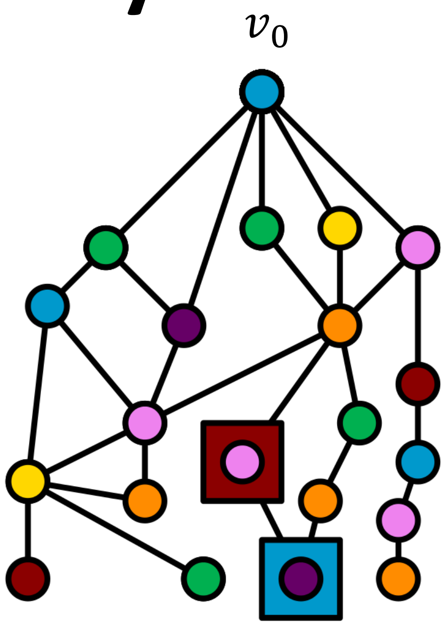

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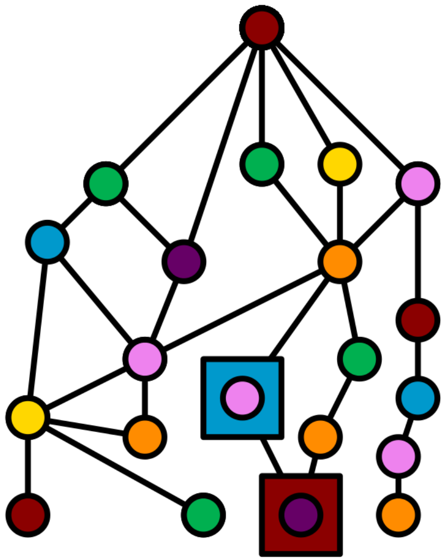

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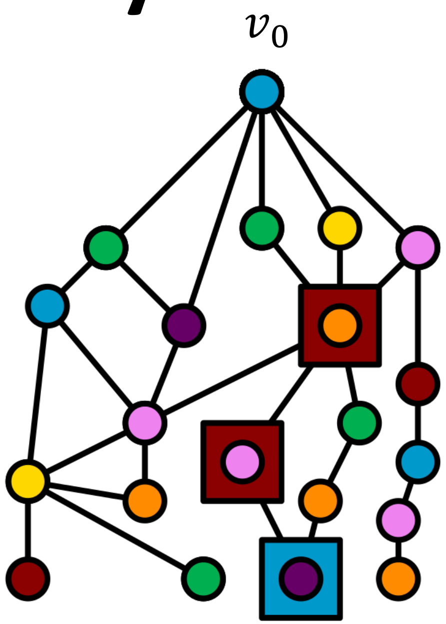
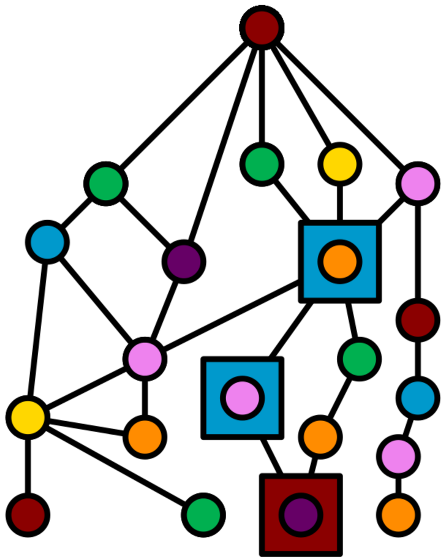
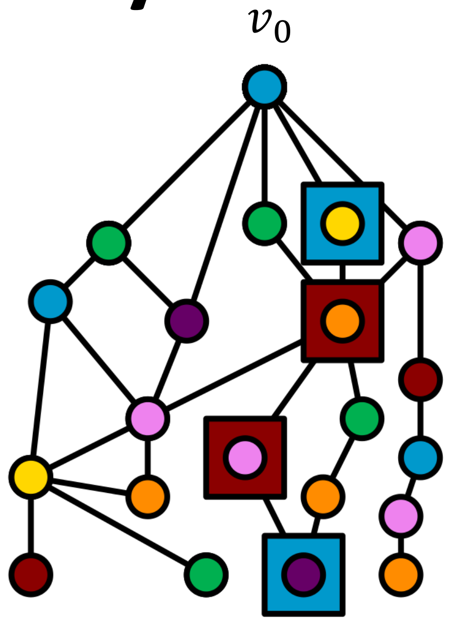




Path Coupling for Local Glauber Dynamics



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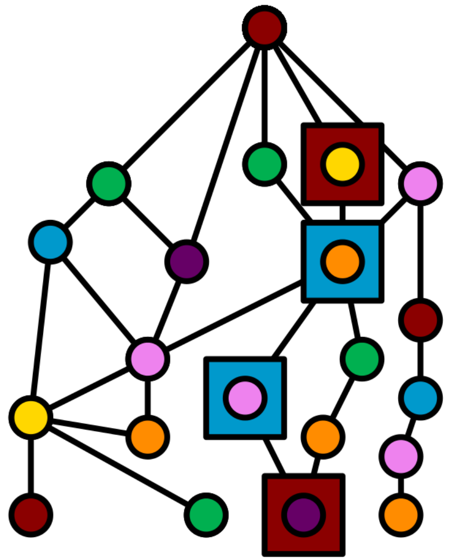

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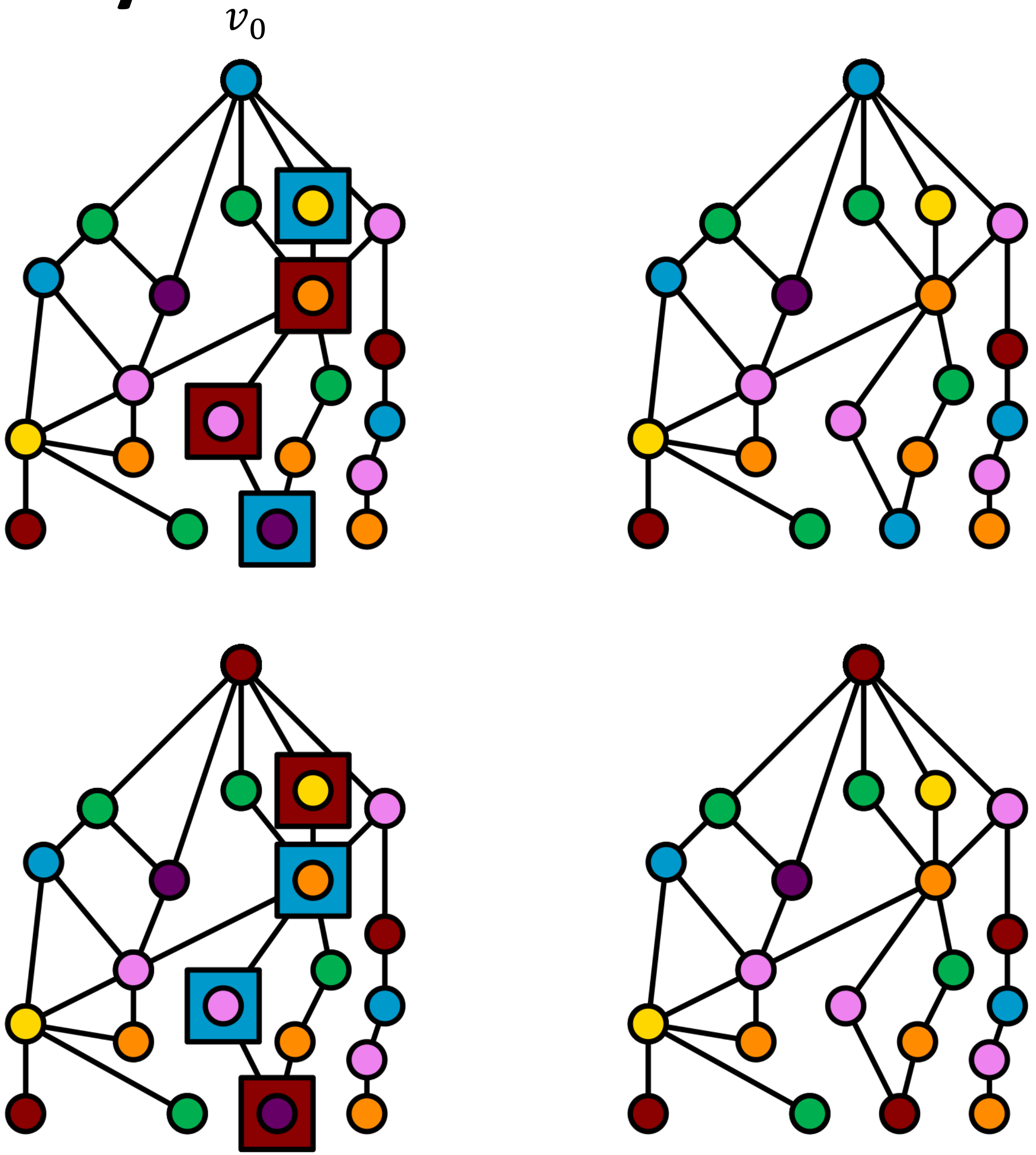

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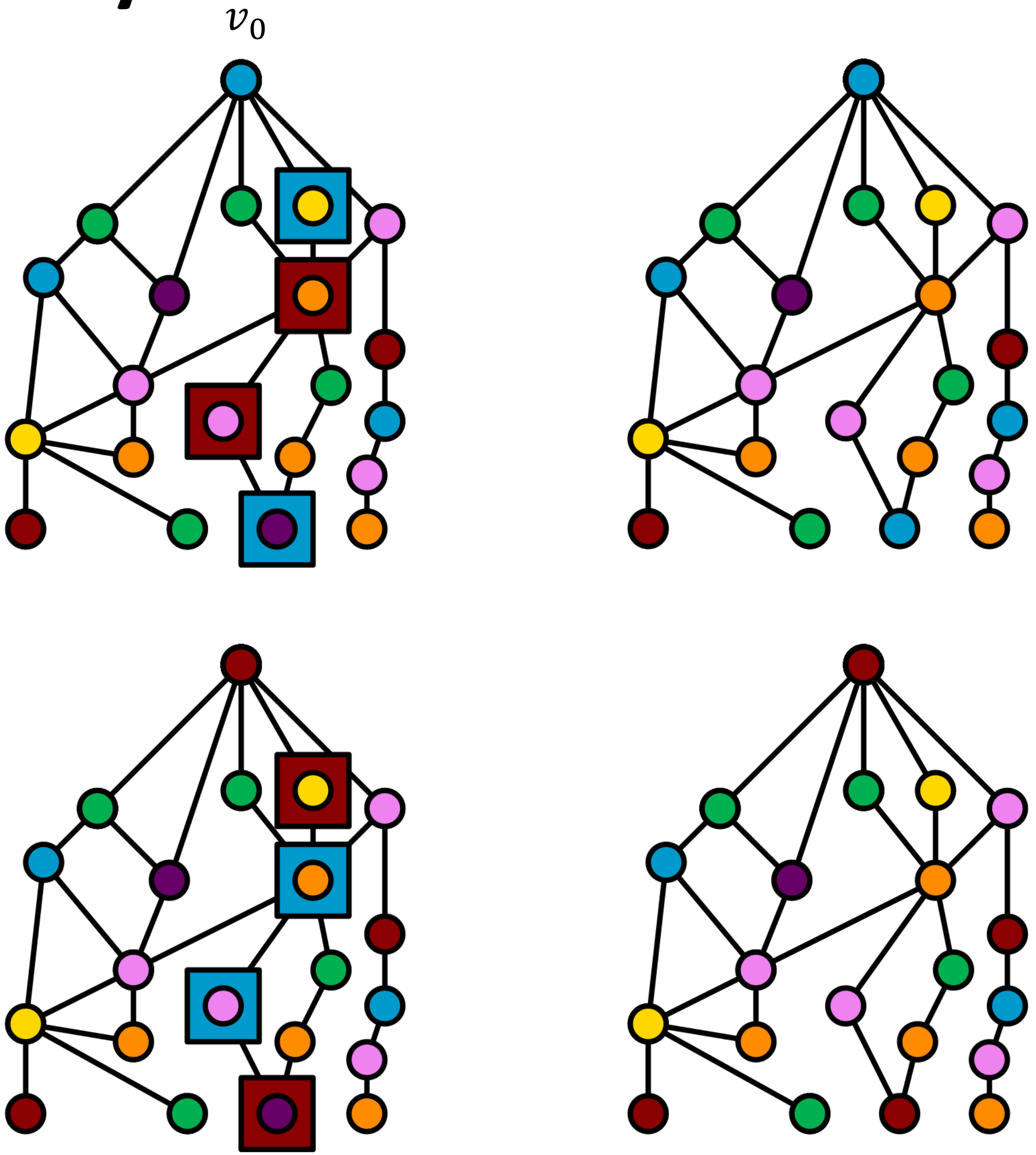

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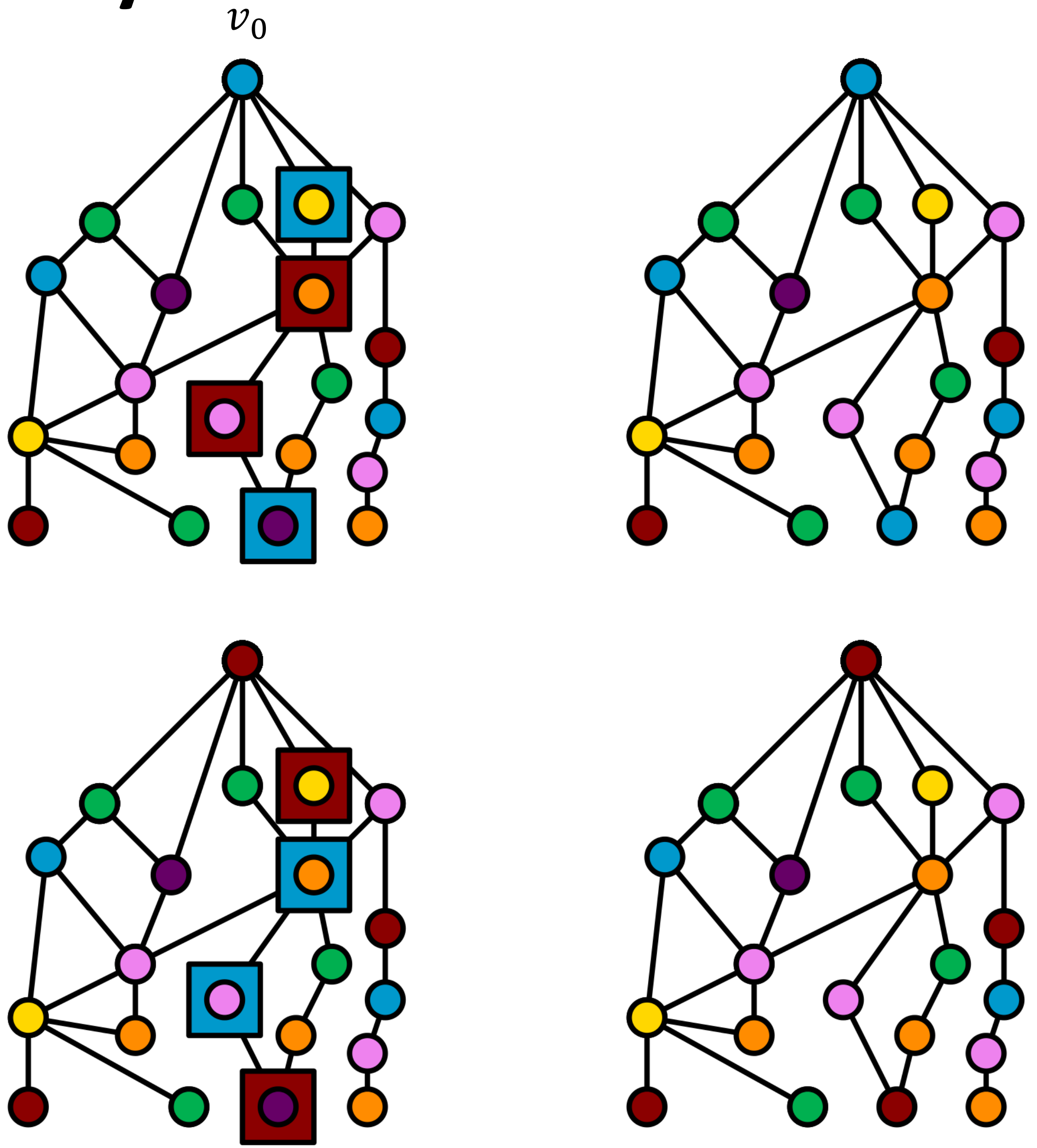
$$< 1$$



Centralized

[*Jerrum 1995*]
Single-Site Glauber $\boldsymbol{O(n \log n)}$ steps $\boldsymbol{q \geq 2\Delta + 1}$

Decentralized

[*Feng, Sun, Yi 2017*]
LubyGlauber $\boldsymbol{O(\Delta \log n)}$ steps $\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2}$

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LocalMetropolis $\boldsymbol{O(\log n)}$ steps $\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2 + \sqrt{2}}$

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almost independent set suffices
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[Vigoda 2000],
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$\mathbf{poly}(\boldsymbol{n})$ steps $\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > \frac{11}{6}}$

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LubyGlauber $\boldsymbol{O(\Delta \log n)}$ steps

fewer colors?

$\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2}$

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$\boldsymbol{O(\log n)}$ steps $\boldsymbol{q \geq \alpha\Delta}$ for $\boldsymbol{\alpha > 2}$

Centralized

Thank you!

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