The seminar and consequently this handout and the notation closely follow the PhD thesis of Lukas Finschi [1].

**Notation 1 (Ordered sets)** Let $S$ be a finite set. We write $(S)$ for some fixed (linear) order of the elements in $S$. If $\pi$ is a permutation of the elements in $S$ then we will denote by $\pi(S)$ the ordered set obtained from $(S)$ by reordering the elements according to $\pi$. For $e \in S$, $f \notin S$ we denote by $(S : e \rightarrow f)$ the ordered set obtained from $(S)$ when $e$ is replaced by $f$ at the same position while keeping the relative ordering of the other elements.

**Motivation**

We now want to generalise the concept of determinants that is known from linear algebra to oriented matroids. We know that the determinant satisfies some elementary properties. Since matrices give rise to oriented matroids it is natural to ask for similar properties to hold when defining basis orientations as generalisations of the determinant. Consider a $n \times m$-matrix such that the rank of $A$ is equal to $n$ and suppose that $B$, consisting of a subset of columns of $A$, is a basis of $A$. In this case we have $\det(B) \neq 0$ so the determinant has a sign. Moreover it is defined for a specific ordering of basis vectors and it is alternating, that is, a permutation $\pi$ multiplies the sign of the determinant by $\text{sign}(\pi)$.

**Fundamental cocircuits**

The first lemma will allow us to define fundamental cocircuits which are in turn needed for the definition of basis orientations.

**Lemma 1** Let $\mathcal{M}(E,F)$ be an oriented matroid and $\mathcal{D}$ the set of its cocircuits. For a basis $B \in \mathcal{B}$ (of the underlying matroid $\mathcal{M}$) and every $e \in B$ there exist exactly two cocircuits $X,-X \in \mathcal{D}$ such that $B \setminus e \subseteq X_0$. It then holds that $X_e \neq 0$.

**Proof** Let $B$ be a basis of the underlying matroid $\mathcal{M} = (E,\mathcal{A})$. and let $e \in B$. Then $\mathcal{A}$ contains the flat $\overline{B \setminus e}$. This means that there exists $Y \in \mathcal{F}$ such that $\overline{B \setminus e} = Y^0$ and one has $B \setminus e \subseteq Y^0$. Now it follows $Y_e \neq 0$, since otherwise $B \subseteq Y^0$ and thus $\overline{Y^0} = \overline{B \setminus e} = B$, a contradiction. By [1], Lemma 0.6.2 there exists $X \in \mathcal{D}$ such that $X \preceq Y$ and $X_e = Y_e$. We also have $B \setminus e \subseteq X^0$ which gives the existence. It remains to show that for any other cocircuit $Z$ satisfying this property we have $X = Z$ or $X = -Z$. Assume $Z$ is another cocircuit satisfying $B \setminus e \subseteq Z^0$ such that $X \neq Z$. The above consideration implies $Z_e \neq 0$ as well. Applying cocircuit elimination (C3) to $X, -Z$ and $e$ yields a cocircuit $\tilde{X}$ that satisfies $B \setminus e \subseteq \tilde{X}^0$ and $\tilde{X}_e = 0$, a contradiction. Hence, either $Z = X$ or $Z = -X$. □

This lemma ensures that the following definition makes sense.
**Definition 1 (Fundamental cocircuit)** Let $\mathcal{M} = (E, F)$ be an oriented matroid, $\mathcal{D}$ the set of its cocircuits. For a basis $B \in \mathcal{B}$ and $e \in B$ we call the cocircuit $X \in \mathcal{D}$ determined by $B \setminus e \subseteq X^0$ and $X_e = +$ the **fundamental cocircuit of $\mathcal{M}$ with respect to $B$ and $e$** and denote it by $X(B, e)$.

**Basis orientations**

**Definition 2 (Basis orientation)** Let $\mathcal{M} = (E, F)$ be an oriented matroid, $\mathcal{D}$ the set of its cocircuits. Let $\mathcal{B}$ be the set of bases of the underlying matroid $\mathcal{M}$. A map $\chi : (\mathcal{B}) \rightarrow \{+,-\}$ is called a **basis orientation** of $\mathcal{M}$ if

(B0) $\chi$ is alternating, i.e., $\chi(B) = \text{sign}(\pi) \cdot \chi(\pi(B))$ for all $(B) \in (\mathcal{B})$ and all permutations $\pi$ of $B$,

(B1) for all $(B) \in (\mathcal{B}), e \in B, f \notin B$ such that $B \setminus e \cup f \in B$ it holds

$$\chi(B : e \rightarrow f) = X_e \cdot X_f \cdot \chi(B)$$

where $X = X(B, e) \in \mathcal{D}$ is the fundamental cocircuit with respect to $B$ and $e$.

**Remark 1** As was seen in the beginning, in the case where the matroid is defined given a $n \times m$ matrix $A$ with rank $n$ and we have a basis $B$ consisting of columns of $A$ we can obtain a basis orientation by looking at the sign of the determinant. As is known from linear algebra, the determinant is alternating, that is, if two columns in $B$ are interchanged, the sign of the determinant is inverted. Moreover, we also have the feature (B2) appearing in the context of matrices. Indeed, if $B$ is a basis, $e \in B$ and $f \notin B$ then there exists exactly one $y \in \mathbb{R}^n$ such that $A_{B \setminus e} y = 0$ and $(A^T y)_e = 1$ where $A_{B \setminus e}$ denotes the matrix formed by the subcolumns $B \setminus e$ of $A$ and we abuse the notation by writing $y_e$ instead of $y_i$ if $e$ is the $i$-th column in $A$. The covector $x := A^T y$ is nothing else than the fundamental cocircuit with respect to $B$ and $e$. By Cramer’s rule we know that the unique solution $y$ can be expressed by

$$y_f = \frac{\det(A_{B \setminus e \rightarrow f})}{\det A}$$

For an arbitrary value of $y_e$ this gives

$$\det(A_{B \setminus e \rightarrow f}) = y_e \cdot y_f \cdot \det(A)$$

and if we pass to the sign (recalling that sign$(y)$ is defined to be sign$(A^T y)$) we obtain the desired result.

The following theorem due to Las Vergnas [2] constitutes the main result of this handout.
Theorem 1 (Las Vergnas) Every oriented matroid $\mathcal{M} = (E,F)$ has exactly two basis orientations $\chi$ and $-\chi$.

Proof We show the result using induction on the cardinality $|E|$ and cocircuit elimination. We denote by $r$ the rank of $\mathcal{M}$. In the base case where $|E| = r$ there exists only one basis, namely $E$. Consequently one either has $\chi(B) = +$ or $\chi(B) = -$ which determines $\chi$ completely and the assumptions for (B2) do not come into play since there is no element $e \notin f$. Hence assume that $r \leq |E|$. Let $B \in \mathcal{B}$. Define $\chi(B) = +$. We need to show that this determines $\chi$ uniquely and consistently. Choose $a \in E \setminus B$ and consider the deletion minor $a \in M\setminus a$. It holds that $a \notin B$. Hence, $a$ is not a coloop. Indeed: Assume that $a$ is a coloop. This means that $E \setminus a \in M$ and $B \subseteq E \setminus a$. This now implies $B \setminus a \cup e$ is a basis since $B \setminus a \cup e = B \cup e \setminus a = B \cup e = E$ and $\chi(B)$ is independent of the choice of $a$ since $\chi$ satisfies the property (B1) for all ordered bases of $E$ which do not contain $a$. Moreover $\hat{X} \setminus a$ is nonempty since $a$ is not a coloop. At this point $\chi$ is defined for all ordered sets and satisfies the property (B0) by induction and the above definition. It therefore remains to prove (B1) for $e,f \in E$ and $e \neq f$. We assume $e \neq a$ and $f \neq a$ since the other cases are trivial. We consider a basis $B \in \mathcal{B}$ such that $a \in B,e \in B,f \notin B$ such that $B \setminus e \cup f$ is a basis. We need to show that $\chi(B : e \rightarrow f) = X_e \cdot X_f \cdot \chi(B)$ where $X = X(B,e)$.

We now distinguish between two cases. In the first case, we assume $f \notin \text{span}_M(B \setminus a)$.

This means that $B \setminus a$ is a basis and hence by definition we have

$$
\chi(B) = X^a_{f} \cdot X^a_{e} \cdot \chi(B : a \rightarrow f)
$$

$$
\chi(B : a \rightarrow f, e \rightarrow a) = X^a_{e} \cdot X^a_{e} \cdot \chi(B : a \rightarrow f)
$$
where $X^{af} = X(B,a)$ and $X^{ae} = X(B\backslash a \cup f,e)$. The next step is to apply cocircuit elimination to $X^{af}$, $-X^{ae}$ and $a$ to obtain a cocircuit $X^{ef}$ such that $X^{ef}_a = 0$ and $X^{ef}_h \in \{X^{af}_h, -X^{ae}_h, 0\}$ for $h \in E \backslash a$. This means that $B \setminus e \subseteq X^{ef}$ and by uniqueness it must hold that $X^{af} \pm X(B,e)$ and $X^{ef}_e = -X^{ae}_e$ as well as $X^{ef}_f = X^{af}_f$. In combination we have

$$\chi(B : e \rightarrow f) = -\chi(B : a \rightarrow f, e \rightarrow a)$$

$$= -X^{af}_a \cdot X^{af}_f \cdot X^{ae}_e \cdot \chi(B).$$

$$= X^{ef}_e \cdot X^{ef}_f \cdot \chi(B)$$

For the second case where $f \in \text{span}_M(B \backslash a)$ one proceeds analogously. We put $Y := X(B,a) \in D$. Hence, $Y_f = 0$. Then choose $g \in Y \backslash a$ and note that $B \backslash a \cup g \in B$ as well as $B \{a,e\} \cup \{f,g\} \in B$. Like in the first case one uses a decomposition

$$(B : e \rightarrow f) = (B : a \rightarrow e \rightarrow f : g \rightarrow a)$$

and cocircuit elimination twice to obtain the result. □

**Chirotopes**

**Definition 3 (Chirotepe)** Let $M = (E,F)$ be an oriented matroid. Set $n := |E|$ and $r := \text{rank}(M)$. We call $\{\chi, -\chi\}$ the chirotepe of $M$ if $\chi$ is a map defined on all ordered subsets $(S)$ of $E$ with cardinality $r$ such that $\chi$, restricted to the set of ordered bases of $M$, is a basis orientation of $M$ and $\chi(S) = 0$ if $S \notin B$.

**Proposition 1** Let $M = (E,F)$ be an oriented matroid of rank $r$. The chirotepe of $M$, together with $E$ and $r$ determines $M$.

**Proof** Let $\chi$ be one of the maps in the chirotepe of $M$. The set of bases $B$ of $M$ is determined as the set of subsets of cardinality $r$ of $E$ for which $\chi(B) \neq 0$. A sign vector $X \in \{-,+,0\}^E$ is a cocircuit of $M$ if and only if there exists $B \in B$ and $e \in B$ such that $(B \setminus e) \subseteq X^0$, $X_e \neq 0$ and $X_f = X_e \cdot \chi(B) \cdot \chi(B : e \rightarrow f)$ for all $f \notin B$. As the set of cocircuits determines the oriented matroid, the result follows. □

**Chirotopes of the dual matroid**

In this last part we investigate the relationship between the chirotepe of a oriented matroid and the chirotepe of its dual. Recall the fact that if $M = (E,F)$ is an oriented matroid and $M^*$ denotes its dual then if $B$ is a basis of $M$ we have that $N := E \setminus B$ is a basis of $M^*$. In a similar vein as before we
can investigate the cocircuits of $\mathcal{M}$, also called the circuits of $\mathcal{M}$. A natural question is, how circuits and cocircuits are related. To this end, consider the fundamental cocircuit $X = X(B, e) \in \mathcal{F}$ and the fundamental circuit $Y = Y(N, f)$ characterised by $N \setminus f \subseteq Y^0$ and $Y_f = +$ obtained by using lemma\[1\]. By definition we have that $X$ is orthogonal to $Y$, in symbols $X \ast Y$. Moreover we have $X \cap Y = \{e, f\}$. In addition, $X_e = Y_f = +$ whence $X_f = -Y_e$ by duality. Now let $\chi$ and $\chi^*$ be basis orientations of $\mathcal{M}$ and $\mathcal{M}^*$ respectively. Then

\[\chi^*(N : f \to e) = Y_f \cdot Y_e \cdot \chi^*(N) = -X_e \cdot X_f \cdot \chi^*(N).\]

This property can be seen as the dual form of (B1) appearing in\[2\]. These considerations lead to the following result which describes how the chirotope of the dual is obtained from the chirotope of the predual.

**Lemma 2** Let $\mathcal{M} = (E, \mathcal{F})$ be an oriented matroid and $\chi$ one of the two maps in the chirotope of $\mathcal{M}$. Consider a fixed order of $E$. Then, one of the two maps in the chirotope of $\mathcal{M}^*$ is determined by

\[\chi^*(N) = \text{sign}(\pi(B, N)) \cdot \chi(B)\]

where $(B) = (b_1, \ldots, b_r)$ and $(N) = (b_{r+1}, \ldots, b_n)$ are ordered bases of $\mathcal{M}$ and $\mathcal{M}^*$ respectively where $N = E \setminus B$ and $\pi(B, N)$ is the permutation which sorts $(b_1, \ldots, b_n)$ according to the fixed order of $E$.

**Proof** First we note that $X^*(N) \neq 0$ if and only if $N$ is a basis of $\mathcal{M}^*$. This in turn is the case if and only if $B = E \setminus N$ is a basis of $\mathcal{M}$ which is equivalent to $\chi(B) \neq 0$ for all $(B)$.

Now let $\chi^*$ be such that $\chi^*(N) = \text{sign}(\pi(B, N)) \cdot \chi(B)$ for all ordered bases $(B)$ and $(N)$ of $\mathcal{M}$ and $\mathcal{M}^*$ respectively, where $N = E \setminus B$. We have to show that $\chi^*$ restricted to the set of ordered bases of $\mathcal{M}^*$ is a basis orientation of $\mathcal{M}^*$. Let $(B) = (b_1, \ldots, b_r)$ and $(N) = (b_{r+1}, \ldots, b_n)$ be ordered bases of $\mathcal{M}$ and $\mathcal{M}$ respectively. Now let $e \in B, f \in N$. By assumption on $\chi^*$ and the property (B1) of basis orientations we obtain

\[\chi^*(N : f \to e) = -\text{sign}(\pi(B, N)) \cdot \chi(B : e \to f)\]

\[= -\text{sign}(\pi(B, N)) \cdot X_e \cdot X_f \cdot \chi(B)\]

\[= -X_e \cdot X_f \cdot \chi^*(N)\]

where $X = X(B, e)$ is the fundamental cocircuit with respect to $B$ and $e$. This determines $\chi^*$ up to a sign change.\[\square\]