The goal of this chapter is to solve the isomorphism class generation problem. This is to generate for given integers $n, r$ all oriented matroids $\mathcal{M} = (E, F)$ with $n = |E| = \text{size of the ground set and rank}(\mathcal{M}) = r$ up to isomorphism. As described in the previous chapter this can be done by an incremental method: Starting from well-characterized ones ($\text{rank} \leq 2$ or $\text{rank} = |E|$) and then adding single elements.

So it is enough to find an algorithm which finds for a given oriented matroid all single element extensions, up to isomorphism. In this chapter this will be done using tope graphs.

By Corollary 1.4.2, the tope graph of an oriented matroid determines the oriented matroid up to isomorphism. Which means the discussion can be restricted to simple oriented matroids.

An isomorphism of oriented matroids is a map which first relabels and then reorients the elements. An automorphism is a self-isomorphism, which means the relabeling will not delete or create any elements, hence it is only a permutation on the ground set:

4.1.1 Definition For a simple oriented matroid $\mathcal{M} = (E, F)$ is $\text{Aut}(\mathcal{M})$ the set of automorphisms of $\mathcal{M}$.

A map $\phi = \rho \pi$ with $\rho$ a reorientation and $\pi$ a permutation on $E$ belongs to Aut($\mathcal{M}$) if and only if $F = \{\phi(X) | X \in F\}$.

4.1.2 Proposition Let $\mathcal{M}$ be a simple oriented matroid with tope graph $G$. Then Aut($G$) and Aut($\mathcal{M}$) are isomorphic groups.

Proof Let $\mathcal{M}$ be a simple oriented matroid with tope graph $G$ and associating bijection $L : V(G) \to T$. 

• \( \phi_g := LgL^{-1} \in \text{Aut}(\mathcal{M}) \) for every \( g \in \text{Aut}(G) \): By Corollary 1.4.2, \( \mathcal{M} \), and hence \( T \), is determined by \( G \) up to isomorphism which means there exists \( \phi \in \text{Aut}(\mathcal{M}) \) such that \( Lg = \phi L \), since \( g \) is a graph automorphism. Hence \( \phi_g = LgL^{-1} = \phi \in \text{Aut}(\mathcal{M}) \).

• \( g_\phi := L^{-1}\phi L \in \text{Aut}(G) \) for every \( \phi \in \text{Aut}(\mathcal{M}) \): for all \( v, w \in V(G) \) and every \( \phi \in \text{Aut}(\mathcal{M}) \), \( |D(L(v), L(w))| = |D(\phi L(v), \phi L(w))| \) because \( \phi \) is an automorphism. By Proposition 1.2.4 \( d_G(x, y) = |D(L(v), L(w))| \), this means that all distances are preserved under \( g_\phi := L^{-1}\phi L \) and hence \( g_\phi \in \text{Aut}(G) \).

The two maps \( g \mapsto \phi_g \) and \( \phi \mapsto g_\phi \) are obviously inverse to each other. And they are homomorphisms:

\[
\phi_{gh} = LghL^{-1} = LgL^{-1}LhL^{-1} = \phi_g \phi_h \quad \forall g, h \in \text{Aut}(G)
\]

\[
g_{\phi \psi} = L^{-1}\phi L \psi L = L^{-1}\phi L \psi L^{-1}L = g_\phi g_\psi \quad \forall \phi, \psi \in \text{Aut}(\mathcal{M}).
\]

Maybe more interesting is the question, when two tope graphs are isomorphic. The algorithm \textsc{AcycloidOrientationReconstruction} in section 1.4 chooses a pair of antipodal vertices and builds then the set of covectors while going along a shortest path between them. With this two tope graphs of oriented matroids \( G, G' \) can be tested for being isomorphic as follows:

• Check if they have the same diameter.

• For a pair of antipodal vertices in \( G \) and a shortest path between them find by the algorithm the set of covectors.

• Check for all pairs of antipodal vertices in \( G' \) and all shortest paths between them if the algorithm yields the same set of covectors.

To be able to work with tope graphs instead of oriented matroids, the notion of localization is needed. A localization of tope graphs corresponds to a single element extension of oriented matroids.

**Definition** A signature of a graph \( G \) is a map \( \sigma : V(G) \to \{-, +, 0\} \).

Every signature \( \sigma \) defines a partition on the vertex set \( V(G) \) by \( V^s := \{ v \in V(G) | \sigma(v) = s \} \) for \( s \in \{-, +, 0\} \). In addition \( V^\oplus := V^- \cup V^0 \) and \( V^\ominus := V^+ \cup V^0 \). This sets induce the subgraphs \( G^-, G^+, G^0, G^\oplus \) and \( G^\ominus \) of \( G \).
**Definition** The localization of \( G \) w.r.t. \( L \) and the single element extension \( M \to M' \) is the signature \( \sigma : V(G) \to \{-, +, 0\} \) defined by

\[
\sigma(v) := \begin{cases} 
+ & \text{if } T_E = L(v) \text{ implies } T_f = + \text{ for } T \in T' \\
- & \text{if } T_E = L(v) \text{ implies } T_f = - \text{ for } T \in T' \\
0 & \text{otherwise}
\end{cases}
\]

where \( M' \) is the single element extension of \( M \), \( M = M' \setminus f \), \( T \) and \( T' \) their tope sets, and \( G \) the tope graph of \( M \) with associating bijection \( L \).

\( \sigma \) determines the extended tope set \( T' \) by \( T' = \{ T \in \{ +, - \}^{E \cup f} | \text{there exists } v \in V(G) \text{ s.t. } T_E = L(v) \text{ and } \sigma(v) \in \{ T_f, 0 \} \} \).

In the pseudosphere arrangement, \( \sigma(v) = + \) means that the corresponding tope \( T = L(v) \) lies completely on the + side of the new sphere \( S_f \), and simply gets a + in the component \( f \). Analog for \( - \). \( \sigma(v) = 0 \) is the case that \( S_f \) cuts the tope into two new ones. So if \( \sigma(v) = 0 \) then there are topes \( T^+, T^- \in T' \) with \( T^+_E = T^-_E = T \) and \( T^+_f = s \) for \( s \in \{ +, - \} \). Topes can not have 0 entries since the oriented matroids are assumed to be simple, which means they have no loops or parallel elements.

**4.2.1 Definition** A signature \( \sigma : V(G) \to \{-, +, 0\} \) is a localization of \( G \), for \( G \) the tope graph of some oriented matroid, if there exist \( \mathcal{M}, L \) and \( \mathcal{M}' \) such that \( \sigma \) is the localization of \( G \) w.r.t. \( L \) and the single element extension \( \mathcal{M} \to \mathcal{M}' \).

The following proposition shows how to determine the tope graph of the single element extension out of the tope graph and the corresponding localization. This allows to make an algorithm which for a given tope graphs gives all localizations, since the corresponding oriented matroids can then easily be determined out of them.

**4.2.2 Proposition** The tope graph of the single element extension \( \mathcal{M}' \) determined by \( G \) and a localization \( \sigma \) of \( G \) is a graph \( G' \) with vertex set \( \{ v^-|v \in V^\oplus \} \cup \{ v^+|v \in V^\oplus \} \) and edge set \( \{\{v^-, v^+\}|v \in V^\oplus\} \cup \{\{v^-, w^-\}|\{v, w\} \in E(G^\ominus)\} \cup \{\{v^+, w^+\}|\{v, w\} \in E(G^\oplus)\} \)

**Proof** Let \( G \) be the tope graph of an oriented matroid \( \mathcal{M} = (E, \mathcal{F}) \) with associating bijection \( L \) and tope set \( T \), furthermore let \( \sigma \) be a localization of \( G \) defining a single element extension \( \mathcal{M}' = (E \cup f, \mathcal{F}') \). Consider the set of extended topes \( T' = \{ T \in \{ +, - \}^{E \cup f} | \text{there exists } v \in V(G) \text{ s.t. } T_E = \}

3
\(\mathcal{L}(v)\) and \(\sigma(v) \in \{T_f, 0\}\). By definition of \(\mathcal{T}'\) exists for every tope \(T' \in \mathcal{T}'\) a unique vertex \(v \in V(G)\) such that \(\mathcal{L}(v) = T'_E\). If \(\sigma(v) = 0\) then also \(\tau T' \in \mathcal{T}'\) because of the symmetry of oriented matroids. In the pseudosphere arrangement, \(\sigma(v) = 0\) corresponds to the new introduced sphere cuts the tope \(\mathcal{L}(v)\). Hence \(v\) has to be split into two vertices \(v^-, v^+ \in V(G')\), with \(\mathcal{L}(v^-)\) and \(\mathcal{L}(v^+)\) the two topes \(\mathcal{L}(v)\) is cut into. \(\sigma(v) \in \{+, -\}\) means that the tope \(\mathcal{L}(v)\) lies completely on the + or − side of the new sphere and is not cut by it. For them, the corresponding vertices are simply marked with the sign, \(v^s\) for \(s = \sigma(v) = T'_f\). An edge in \(E(G')\) indicates the adjacency of the corresponding topes, which means they differ in exactly one element. \(\{v^-, w^-\}\) and \(\{v^+, w^+\}\) are the edges between the topes which lie on the + and − side respectively. And the edges \(\{v^-, v^+\}\) come from split vertices.

\[\square\]

The goal is to find for a given tope graph all possible localization, so properties are needed to determine which signatures are localizations.

**Lemma** (4.2.3 & 4.2.5) Let \(G\) be the tope graph of an oriented matroid and \(\sigma\) a localization of \(G\). Then hold

(L1) \(\sigma(\overline{v}) = -\sigma(v)\) for all \(v \in V(G)\)

(L2) \(E(G) \cap (V^- \times V^+) = \emptyset\)

(L3) \(d_G(v, w) = d_G(v, w)\) for all \(v, w \in V^\oplus\), and \(d_G(v, w) = d_G(v, w)\)

for all \(v, w \in V^\oplus\)

(L4) \(G^-\) (and also \(G^+\)) is a connected subgraph of \(G\).

**Proof** Let \(\mathcal{M} = (E, \mathcal{F})\) be an oriented matroid with tope set \(\mathcal{T}\), tope graph \(G\), and associating bijection \(\mathcal{L} : V(G') \to \mathcal{T}\). Let \(\mathcal{M}' = (E', \mathcal{F}')\) be the single element extension of \(\mathcal{M}\) defined by a given localization \(\sigma\) of \(G\). Denote by \(\mathcal{T}'\) the tope set of \(\mathcal{M}'\) and by \(G'\) the tope graph of \(\mathcal{M}'\).

(L1): By Proposition 1.2.4 is \(d_G(x, y) = |D(\mathcal{L}(x), \mathcal{L}(y))|\) and hence \(\mathcal{L}(\overline{v}) = -\mathcal{L}(v)\), since the antipode is the furthermost point. Assume \(\sigma(v) = \sigma(\overline{v}) \neq 0\).

W.l.o.g. \(\sigma(v) = +\). The definition of a localization yields then

\[T_E = \mathcal{L}(v)\text{ implies } T_f = + \text{ for } T \in \mathcal{T}'\text{ and}
\]

\[T_E = \mathcal{L}(\overline{v}) = -\mathcal{L}(v)\text{ implies } T_f = + \text{ for } T \in \mathcal{T}'.\]

By the symmetry property of tope sets (A2), \(-T\) is a tope for all topes \(T\). So if \(T \in \mathcal{T}'\) is such that \(T_E = -\mathcal{L}(v), T_f = +\). So \(-T\) has \((-T)_E = \mathcal{L}(v)\) and \((-T)_f = -\). A contradiction to \(\sigma(v) = +\).

(L2): An edge \(\{v, w\} \in E(G)\) means that the to \(v\) and \(w\) corresponding topes \(V, W \in \mathcal{T}\) are adjacent, that is they differ in exactly one sign \(g \in E\). Is now \(\sigma(v) = -\) and \(\sigma(w) = +\), the corresponding extended topes \(V', W'\) differ in exactly two signs, \(g\) and the new \(f \in E' \setminus E\). By the reorientation property
(A1), \( \overline{\tau}V' \in \mathcal{T}' \) or \( \overline{\tau}V' = \overline{\tau}W' \in \mathcal{T}' \). But the first one contradicts \( \sigma(v) = - \) and the second \( \sigma(w) = + \).

(L3): Consider \( v, w \in V^\oplus \). Let \( v^-, w^- \in V(G') \) be the corresponding vertices with \( L(v)_f = L(w)_f = - \). Since \( d_G(x, y) = |D(L(x), L(y))| \), a shortest path \( p \) from \( v \) to \( w \) in \( G \) defines a corresponding path \( p' \) in \( G' \) between \( v^-, w^- \), which is again a shortest path. Vertices on \( p' \) correspond to topes \( T \in \mathcal{T}' \) with \( T_f = - \), again because \( d_G(x, y) = |D(L(x), L(y))| \). So the path \( p \) is contained in \( G^\oplus \), and hence \( d_{G^\oplus}(v, w) = d_G(v, w) \). The same holds by symmetry for \( G^\ominus \).

(L4): \( M' \) is the single element extension of \( M \), which means \( M = M' \setminus f \) for some \( f \in E' \). By the definition of \( \mathcal{T}' \) is clear that the extended topes with \( T_f = - \) and \( \overline{\tau}T \notin \mathcal{T}' \) are in a one-to-one correspondence with the vertices of \( G \) which are mapped to \( - \) by \( \sigma \), \( V^- = \{ v \in V(G) | \sigma(v) = - \} \). By Theorem 1.3.1, any two topes in \( \mathcal{T}' \) are connected by a sequence of topes in \( \mathcal{T}' \), each two consecutive differ only in one sign, which means consecutive ones are adjacent, and hence the corresponding vertices connected. So the subgraph \( G' \) of \( G \) induced by the vertex set \( V^- \) is connected. Analogously, \( G^+ \) is connected.

\[ \square \]

4.2.6 Definition Let \( \sigma \) be a signature of \( G \), for \( G \) the tope graph of an oriented matroid. \( \sigma \) is a weak acycloidal signature of \( G \) if (L1), (L2) and (L4) are satisfied and a strong acycloidal signature of \( G \) if (L1), (L2), (L3) and (L4) are satisfied.

4.2.7 Proposition It can be checked in polynomial time if a given signature is a localization of tope graphs of oriented matroids.

Proof By Proposition 4.2.2, the extended tope graph can be constructed easily from a tope graph and a localization. If the so constructed graph is the tope graph of an oriented matroid or not, can be decided in time bounded by \( O(n^3 f_d^2 + n^2 f_d^2) \) by Corollary 1.7.2, where \( n \) is the diameter of the new graph and \( f_d \) is the size of its vertex set.

\[ \square \]

As seen in Section 3.3, a single element extension may or may not increase the rank of the oriented matroid. Which one occurs can be recognized from the localization:
4.2.8 Lemma Let $G$ be the tope graph of an oriented matroid $M$ and $\sigma$ a localization of $G$. The rank of a single element extension $M'$ according to $G$ and $\sigma$ is the same as the rank of $M$ unless $\sigma(v) = 0$ for all $v \in V(G)$, then $\text{rank}(M') = \text{rank}(M) + 1$.

Proof Let $G$ be the tope graph of an oriented matroid $M$ with associating bijection $L : V(G) \to \mathcal{T}$, and $\sigma$ a localization of $G$, defining a single element extension $M'$ of $M$, where $f$ is the new element. By Corollary 0.4.9 (i), the rank of $M'$ is the same as the rank of $M$ unless $f$ is a coloop of $M'$, and then $\text{rank}(M') = \text{rank}(M) + 1$.

Claim: $f$ is a coloop of $M' \iff \sigma(v) = 0$ for all $v \in V(G)$

Proof: $\Rightarrow$: That $f$ is a coloop of $M'$ means that there exists a covector $X \in F$ with $X = \{f\}$. This $X$ and also $-X$ are clearly cocircuits. Taking the composition of $X, -X$ and any tope $T \in \mathcal{T}'$ shows $\tau T \in \mathcal{T}'$ by Corollary 0.6.4. But $T_f$ is determined by $\sigma(v)$ for the corresponding vertex $v$, which means $\sigma(v)$ has to be 0 for all $v \in V(G)$.

$\Leftarrow$: $\sigma(v) = 0$ for all $v \in V(G)$ means that for each $T \in \mathcal{T}'$ is $\tau T \in \mathcal{T}'$. By Proposition 0.7.3, the set of covectors is determined by the set of topes by $F' = \{X \in \{-, +, 0\}^E | X \circ T \in \mathcal{T}' \text{ for all } T \in \mathcal{T}'\}$. So there is a cocircuit $X \in D'$ with $X = \{f\}$, since $X \circ T = \tau T \in \mathcal{T}'$ for all $T \in \mathcal{T}'$. And this $X$ shows that $f$ is a coloop.

The following algorithm determines for a given tope graph of an oriented matroid all weak acycloidal signatures. It starts from the constant 0 signature. From this it constructs a new signature by changing the 0 of a vertex to $-$, if the properties (L1), (L2) and (L4) remain satisfied. This is guaranteed by:

(L1): The sign of the antipode is changed to +.

(L2): No neighbouring vertex has a + sign or is the antipode.

(L4): There exists a neighbour with a $-$ sign.

With this conditions the new signature is also a weak acycloidal signature. Repeating this gives all weak acycloidal signatures.

It is important to treat the initial state differently, since otherwise the above condition for (L4) forces the algorithm to do nothing. This is done here by the boolean variable $\text{isconstantzero}$.

To reduce the amount of repetitions the algorithm expects the vertices of the graph to be enumerated, where they can be enumerated in an arbitrary way. This allows that if the same signature is constructed more than once, only the first one is considered, the rest is ignored.
Input: The tope graph $G$ of an oriented matroid.
Output: A list $W$ of all weak acycloidal signatures of $G$.

begin WeakAcycloidalSignaturesReverseSearch($G$);
    determine all antipodes in $G$;
    let $\sigma$ be the signature with $\sigma(v) = 0$ for all $v \in V(G)$;
    $W := \{\sigma\}$; $W_{\text{new}} := \{\sigma\}$;
    isconstantzero := true;
    while $W_{\text{new}} \neq \emptyset$ do
        take any $\tau \in W_{\text{new}}$ and remove $\tau$ from $W_{\text{new}}$;
        for all $v \in V(G)$ with $\tau(v) = 0$ do
            if (there is no $\{v, w\} \in E(G)$ with $\sigma(w) = +$ or $w = \overline{v}$ and
                there is $\{v, w\} \in E(G)$ with $\sigma(w) = -$)
                or isconstantzero then
                $\sigma := \tau$; $\sigma(v) := -$; $\sigma(\overline{v}) := +$;
                determine $V^-$ from $\sigma$;
                find the smallest $u \in V^-$ such that the subgraph induced
                by $V^\prime \setminus \{u\}$ is connected;
                if $u = v$ then $W := W \cup \{\sigma\}$; $W_{\text{new}} := W_{\text{new}} \cup \{\sigma\}$ endif
            endif
        endfor
    isconstantzero = false;
endwhile;
return $W$
end WeakAcycloidalSignaturesReverseSearch.

4.3.1 Proposition The Algorithm WeakAcycloidalSignaturesReverseSearch determines the set of all weak acycloidal signatures of $G$ in time of at most $O(\ell \cdot |V(G)|^2 \cdot |E(G)|)$, where $\ell$ is the number of weak acycloidal signatures of $G$.

The algorithm generates for a given tope graph of an oriented matroid all weak acycloidal signatures. Those can be tested for being strong acycloidal signatures, by checking if (L3) holds, and then for being localizations. The last step is possible in polynomial time by Proposition 4.2.7. Proposition 4.2.2 shows a way to construct the extended tope graph for each localization. And the reasoning in the beginning gives a way to check the graphs for being isomorphic.