A GENERALIZATION OF THE SYLVESTER-GALLAI THEOREM

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This article describes Sten Hansen’s paper A GENERALIZATION OF A THEOREM OF SYLVESTER ON THE LINES DETERMINED BY A FINITE POINT SET. In the whole paper, the action is assumed to take place in the d-dimensional real projective space if not stated differently.

The projective Sylvester-Gallai theorem: Given any finite set of noncollinear points in the real projective plane, there exists at least one line which contains exactly two of the given points.

Hansen generalized this result to higher dimensions. In order to prove Hansen’s theorem, we need some preparation:

An ordinary hyperplane (in the d-dimensional space) is a hyperplane such that all of the given points in this hyperplane but one are in a (d−2)-dimensional subspace. An elementary hyperplane is a hyperplane which contains exactly d of the given points. Every elementary hyperplane is ordinary.

Γ₀ is the set of all 0-dimensional subspaces spanned by the given points. E.g. Γ₀ is the given points itself. Γ is the union of all Γ₀, p = 0,...,d, and is called the configuration.

A d-dimensional configuration is called elementary, if it consists of exactly d+1 points. In this case all hyperplanes are elementary as well.

The hyperplanes spanned by Γ₀ divide the space into regions. A region whose interior is not met by any hyperplane spanned by Γ₀ is called a cell.

AB shall denote the subspace spanned by the subspaces A and B.

**Lemma 1.** Let σₙ be a d-dimensional simplex whose vertices belong to Γ₀ and A₀ a point outside of the simplex. Then there is a (d−2)-dimensional face σₙ₋₂ of the simplex such that the hyperplane Bₙ₋₁ spanned by A₀ and the face satisfies:

\[ Bₙ₋₁ \cap σₙ = σₙ₋₂. \]  

(1)
Figure 1: Example for Lemma 1

Figure 2: wedges

Figure 3
PROOF OF LEMMA 1. Every two hyperplanes divide the space into two wedges. According to Hanson a simplex is the intersection of closed wedges (why?)(which is not the case for a general polytope: Every closed wedge in Figure 3 contains the point $A_0$.) Since $A_0$ is not in the simplex, there must be at least one wedge which contains the simplex but not $A_0$. Define $B_{d-1}$ as the hyperplane spanned by $A_0$ and the intersection of the two hyperplanes building this wedge. $B_{d-1}$ is in the other wedge built by the same two hyperplanes and thus does not intersect the simplex but in the face contained int the intersection of these two hyperplanes.

LEMMA 2. Let $A_0$ be in $\Gamma_0$, $C_{d-1}$ in $\Gamma_{d-1}$, $\delta_{d-1}$ a cell of $C_{d-1}$, $Q_0$ an interior point of the cell $\delta_{d-1}$, $A_0$ not in $C_{d-1}$, $P_0$ such that $A_0P_0$ does not meet the cell $\delta_{d-1}$. Then, both segments of $P_0Q_0$ intersect a hyperplane spanned by $A_0$ and a $(d - 2)$-dimensional face of $\delta_{d-1}$.

PROOF OF LEMMA 2. $A_0$ together with the $(d - 2)$-dimensional faces of the cell $\delta_{d-1}$ build a cone. Both closed segments of the line $P_0Q_0$ have one point outside the cone, namely $P_0$, and one point inside the cone, namely $Q_0$. Therefore, the line somewhere has to meet the boundary of the cone. But a boundary of the cone is exactly a hyperplane spanned by $A_0$ and a $(d - 2)$-dimensional face of $\delta_{d-1}$.

HANSEN’S THEOREM. Suppose the configuration $\Gamma$ is not elementary and let $\delta_d$ be a $d$-dimensional cell. Then there is an ordinary hyperplane $A_{d-1} = B_0C_{d-2}$ where $B_0 \in \Gamma_0$ is the only point outside $C_{d-2}$

$$A_{d-1} \cap \Gamma_0 \setminus \{B_0\} \subset C_{d-2} \in \Gamma_{d-2}$$

such that $A_{d-1}$ does not intersect the cell too much in the following sense:

$$A_{d-1} \cap \delta_d \subset C_{d-2}.$$

PROOF OF HANSEN’S THEOREM.

The proof is made by induction over the dimension. After a moment of thought, for $d = 1$ the theorem is obvious. Use the convention that a $(-1)$-dimensional subspace is the empty set.
Figure 5: Counterexample for Hansen’s theorem in the case of an elementary configuration. Every hyperplane intersects too much of the cell, which is the simplex itself here.

Induction Hypothesis: assume the theorem to be true for dimension $d - 1$.

Choose a simplex whose vertices belong to $\Gamma_0$ including the given cell $\delta_d$ such that this simplex contains no other points of $\Gamma_0$ than its vertices. This is possible by just combining the points, which lie outside the interior of the cell by definition, in a suitable way (possibly in the projective sense). Notice also that no hyperplane can intersect $\delta_d$ by definition of a cell. Call this simplex $\sigma_d$.

There is no point of $\Gamma_0$ in the simplex but its vertices and $\Gamma$ is not elementary, therefore there is a point $F_0$ outside the simplex and Lemma 1 can be applied. Lemma 1 ensures the existence of a $(d - 2)$-dimensional face $\sigma_{d-2}$ and hyperplane $B_{d-1} = F_0 \sigma_{d-2}$ such that

\[ B_{d-1} \cap \sigma_d = \sigma_{d-2}. \tag{4} \]

Note that $\delta_d \subset \sigma_d$ and the $\sigma_{d-2}$ face is part of a $d - 2$-dimensional subspace $S_{d-2}$. Therefore:

\[ B_{d-1} \cap \delta_d \subset B_{d-1} \cap \sigma_d = \sigma_{d-2} \subset S_{d-2}. \tag{5} \]

This is exactly property (3) in Hansen’s theorem. If $B_{d-1}$ is elementary, property (2) is satisfied as well and the proof is complete. Hence, we assume $B_{d-1}$ to be non-elementary from now on.

Choose any point $P_0 \in \Gamma_0$ outside $B_{d-1}$.

Choose a line $L_1$ which passes $P_0$ and an interior point of the cell $\delta_d$ such that it does not intersect any of the $(d - 2)$-dimensional subspaces arising from the intersections of the hyperplanes (no $(d - 2)$-dimensional subspace but the one containing $P_0$). In other words, the line $L_1$ intersects every $(d - 1)$-dimensional cell (but the cells containing $P_0$) only in an interior point.
The point where the line $L_1$ intersects a hyperplane will be called $Q_0$. Choose a non-elementary hyperplane not containing $P_0$ (there are some of them, e.g. $B_{d-1}$) such that one of the two open segments of $P_0Q_0$ intersects neither the given cell $\delta_d$ nor any non-elementary hyperplane. Note that you might need to use the properties of the projective space for this. Call this hyperplane $Q_{d-1}$.

By construction of $L_1$, $Q_0$ is an interior point of a cell $\delta_{d-1}$ of $Q_{d-1}$. This cell together with the point $P_0$ define a cone which contains the originally given cell $\delta_d$. This cone consists of the lines starting from $P_0$ and passing a point of the cell $\delta_{d-1}$. It will be called $\gamma_d$.

As $Q_{d-1}$ is non-elementary we can use the induction hypothesis: there is a subspace $C_0S_{d-3}$ in $Q_{d-1}$ where $C_0$ is the only point in $\Gamma_0$ outside $S_{d-3}$, i.e. $C_0S_{d-3} \cap \Gamma_0 \{C_0\} \subset S_{d-3}$, such that this subset does not intersect $\delta_{d-1}$ too much in the following sense: $C_0S_{d-3} \cap \delta_{d-1} \subset S_{d-3}$.

We are now looking for a hyperplane satisfying the properties of the theorem in dimension $d$. Define $S_{d-2} = P_0S_{d-3}$ and consider the hyperplane $C_0S_{d-2}$.

$C_0S_{d-2} \cap \gamma_d \subset S_{d-2}$ (A). This is true thanks to the analogue property of $C_0S_{d-3}$ and $\delta_{d-1}$. In sloppy words: $C_0S_{d-3}$ does not intersect $\delta_{d-1}$ too much, therefore $C_0S_{d-3}P_0 = C_0S_{d-2}$ does not intersect the cone($P_0$, $\delta_{d-1}$) too much.

Since $\delta_d \subset \gamma_d$: $C_0S_{d-2} \cap \delta_d \subset S_{d-2}$. This is 3 of Hansen’s theorem. If $C_0$ is the only point of $\Gamma_0$ outside $S_{d-2}$ the proof is complete. Hence, we assume there is a $A_0 \in \Gamma_0$ outside $S_{d-2}$ but in $C_0S_{d-2}$ (B).

$A_0$ is not in $S_{d-2}$, so it is not in $S_{d-3}$ ($S_{d-3} \subset S_{d-2}$). $C_0$ is the only point of $C_0S_{d-3}$ outside $S_{d-3}$, therefore $A_0$ is not in $C_0S_{d-3}$. Also note that $C_0S_{d-2} \cap Q_{d-1} = C_0S_{d-3}$. All this implies that $A_0$ is not in $Q_{d-1}$. 
(A) together with (B) imply that \( A_0 \) is not in the cone \( \gamma_d \). Therefore \( P_0A_0 \) does not meet \( \delta_{d-1} \) and Lemma 2 can be applied: the segment of \( P_0Q_0 \) which does not meet the originally given cell \( \delta_d \) (such a segment exists by construction of \( Q_{d-1} \)) intersects a hyperplane \( A_0T_{d-2} \), where \( T_{d-2} \) contains a \((d-2)\)-dimensional cell of \( \delta_{d-1} \).

The way \( Q_{d-1} \) was chosen implies that \( A_0T_{d-2} \) is elementary and thus satisfies property 2 in Hansen’s theorem.

\( Q_{d-1} \) divides the cone \( \gamma_d \) into two parts one of which contains the cell \( \delta_d \). \( A_0T_{d-2} \) intersects the closure of this part only in \( T_{d-2} \). Therefore we have property 3 in Hansen’s theorem: \( A_0T_{d-2} \cap \delta_d \subset T_{d-2} \) and the proof is complete.

References

Sten Hansen, A Generalization of a Theorem of Sylvester on the Lines determined by a finite Point Set, Math. Scand. 16 (1965), 175-180