Set Systems of Bounded Vapnik-Chervonenkis Dimension and a Relation to Arrangements

Diplomarbeit

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Introduction

A classical problem in combinatorial geometry is to find a combinatorial characterization of arrangements of hyperplanes (or configurations of points, which are dual to arrangements, as we will see in chapter 1), i.e. establish a scheme that maps the infinite number of arrangements into finitely many classes in such a way that arrangements with the same image can be regarded as equal with respect to some property one is interested in.

J.E.Goodman and R.Pollack [GP80a] have given an example which explains the usefulness of such a classification scheme: Let $H(n)$ be the smallest natural number, such that every set of $H(n)$ points in the plane (no three of them collinear) contains the vertices of an empty convex n-gon. It is known that $H(5) = 10$ [Ha] and that $H(7)$ does not exist [Ho]. Whether $H(6)$ exists, is still an open problem. Suppose you want to attack this question algorithmically and test each configuration of a given number of points whether it contains an empty convex hexagon. Besides from the immense complexity for large configurations, how do you generate all "essentially distinct" configurations? A classification that reflects convexity properties could solve this problem in principle.

In fact, there exists a combinatorial structure that reflects many interesting properties of planar configurations, namely *circular sequences*, which were introduced by Perrin [Pe] and treated in detail by Goodman and Pollack [GP80a], [GP84]. An overview is given in [Ed].

Another representation of 2-dimensional arrangements was given by Ringel [Ri]; both approaches work only for nondegenerate arrangements and configurations.

A very powerful and general classification of d-dimensional arrangements (which need not even be in general position) in terms of so-called *oriented matroids*, was intended by J.Folkman and J.Lawrence [FL] and reproved by A.Mandel [Ma]. A new approach to the 2-dimensional case was given by R.Cordovil [Co].

It was first observed by Ringel in his paper that the structure he uses to characterize arrangements of lines actually covers a larger class of arrangements, namely arrangements of *pseudolines*, which are topologically similar to straight arrangements. He shows that there are simple arrangements of pseudolines which are not *stretchable*, i.e. whose cell complex is not equivalent to that of any arrangement of lines.

Ringel's observation has an equivalent for any combinatorial structure encoding arrangements – so it seems that straightness cannot be recognized by purely combinatorial means.

In this paper we develop a characterization of simple d-dimensional arrangements of pseudohyperplanes in terms of certain set systems of Vapnik-Chervonenkis dimension d. These set systems are called *pseudogeometric range spaces* and were introduced by E.Welzl [We], who has observed that every simple arrangement determines such a range space.

We show that the converse is also true; as a tool, we introduce an interesting new class of range spaces derived from the pseudogeometric spaces, and we characterize both classes by simple maximality conditions. Our techniques are then applied to derive some results on two more topics related to arrangements.

Given a set of hyperplanes H in d-space, where one of the open halfspaces of each $h \in H$ is called the *positive halfspace* of h (denoted by h^+), we obtain an *arrangement of halfspaces*

$\mathcal{A}(H^+)$, which consists of the same faces as the underlying arrangement of hyperplanes together with the information, which of the positive halfspaces contain a given point.

Now every cell c of the arrangement can be labelled with the set of hyperplanes, whose positive halfspaces contain c; the collection of the labels of all cells determines the *description of cells* of $\mathcal{A}(H^+)$, denoted by $\mathcal{C}(H^+)$ (figure 1.1). Formally, $\mathcal{C}(H^+)$ is a pair (H, R), where H is the set of hyperplanes and R a subset of 2^H. Such a pair is called a *range space*. We refer to H as the set of *elements* of the range space, while R is the set of *ranges* of $\mathcal{C}(H^+)$.

If the arrangement has at least one vertex, then $\mathcal{C}(H^+)$ is a range space of Vapnik-Chervonenkis dimension d. Furthermore, if the arrangement is *simple* (or in *general position*), then $\mathcal{C}(H^+)$ reaches the maximum number of ranges that a range space of VC-dimension d can have. Welzl calls a range space with this property *complete* of dimension d.

Chapter 2 studies complete range spaces and develops their basic properties. Besides from a new concept of *range space duality* and a corresponding duality theorem, all the concepts of this chapter are taken directly or in a slightly modified form from an unpublished manuscript of Welzl [We]. Some of them have already appeared in literature.

If we are given a one-dimensional arrangement of halfspaces (i.e. an arrangement of rays on the line), and we connect two ranges of the corresponding description of cells by an edge whenever they differ by a single element, we obtain the *distance-1-graph*, which in this case has the structure of a path. In a general complete space of VC-dimension 1 this graph is only a tree, which gives a necessary condition for a complete space to be the description of cells of some arrangement. This necessary condition generalizes to higher dimensions and leads to the definition of *pseudogeometric range spaces*, which are the subject of chapter 3 (figure 3.1).

The basic properties and characterizations are again taken from [We].

We prove two theorems motivated by Levi's Enlargement Lemma for arrangements of pseudolines [Le], [Gr] and newly introduce the concepts of *closure* and *boundary* of a range space. This leads to an interesting characterization of pseudogemetric spaces by a maximality condition for the number of ranges in the boundary. Furthermore, the duality theorem for complete spaces is shown to hold also for pseudogeometric spaces.

In chapter 4 we discuss a third class of range spaces, called $\overline{PG}$-spaces, which are spaces that arise as the closure of some pseudogeometric space.

$\overline{PG}$-spaces can be obtained as the description of cells of so-called *arrangements of hemispheres*, and as well as complete and pseudogemetric spaces they can be characterized by a certain maximality condition. Once more a duality theorem is established for $\overline{PG}$-spaces.

In chapter 5 we show that $\overline{PG}$-spaces correspond to simple oriented matroids, and this will lead to a major result of the paper, namely that $\overline{PG}$-spaces characterize simple *arrangements of pseudohemispheres* and pseudogeometric spaces correspond to *arrangements of pseudohalfspaces*.

The terminolgy and the basic properties of oriented matroids we develop in section 5.2 are taken from [Ma] as well as the formal definitions of arrangements of pseudohemispheres and pseudohalfspaces in section 5.5.

The last two chapters about *geometric embeddability* and *elementary transformations* of complete range spaces were inspired by conjectures and suggestions of Welzl. These chapters should be seen from the point of view of chapter 5, for they are motivated by properties of arrangements.

Chapter 6 generalizes planarity to range spaces by introducing an embedding scheme that avoids intersections of certain convex hulls. A main motivation for this embeddability concept is the k-set problem, and we show how embeddability is related to the k-set problem. Furthermore, we characterize the complete spaces, which allow a good embedding in a certain sense.

Finally, chapter 7 discusses the problem of replacing a range of a complete space by another one in such a way that the completeness-property is maintained. This is motivated by *simplex transformations* in arrangements of pseudohyperplanes. We characterize the ranges that can be replaced and show that simplex transformations have an equivalent in complete and pseudogeometric spaces.

Using a result of Ringel [Ri] we show that any two pseudogeometric spaces of VC-dimension 2 and the same boundary can be transformed into each other by using only simplex transformations, a result that does not generalize to higher dimensions.

Chapters 1 through 4 should be read in consecutive order, while chapters 5, 6 and 7 are independent from each other, but are based on the first four chapters.

Throughout the paper, some details and proofs are omitted. In this case they either are easily obtained in a straightforward manner or the reader is referred to the literature.

Furthermore, some set operations as well as the concept of *duality* occur in a double meaning. As an example, consider the symmetric difference, denoted by $\triangle$. If we have sets R, R' which are subsets of the same domain X, then $R \triangle R'$ is the usual symmetric difference defined by $R \triangle R' := (R \cup R') - (R \cap R')$.

If we have $R \subset 2^X$, $r' \subset X$, then the set operations should be applied to the elements of R, i.e. $R \triangle r' := \{r \triangle r' \mid r \in R\}$ in this case. Whenever we refer to this non-standard definition, we explicitly mention it the first time it occurs. Usually, it is clear from the context which meaning is currently valid.

As far as duality is concerned, there exists a *geometric duality* and a *range space duality*; both concepts do not occur very often in this paper, and again it will be clear from the context to which concept we refer.

Contents

Chapter 1

Arrangements of Halfspaces

1.1 The Description of Cells

Consider a finite set H of n hyperplanes in d-dimensional Euclidean space E^d. H defines an *arrangement of hyperplanes* $\mathcal{A}(H)$, i.e. a dissection of E^d into connected pieces of various dimensions, called *faces* of the arrangement, such that the disjoint union of all faces is E^d. A face of dimension k is called a k*-face*; 0-faces are *vertices* of the arrangement, d-faces are called *cells*. Observe that each face f is open in the affine subspace $\text{aff}(f)$, where $\text{aff}(f)$ denotes the affine hull of the points in f. Cells are open in E^d (for a more formal and detailed treatment of arrangements see [Ed]). In the sequel we will always assume that the number of hyperplanes is at least d, otherwise $\mathcal{A}(H)$ is equivalent to a lower-dimensional arrangement.

Every hyperplane $h \in H$ defines two open halfspaces, and by arbitrarily choosing one of these two halfspaces to be called h^+ or the *positive halfspace* of h, (and the other one h^- or the negative halfspace of h), we get an *arrangement of halfspaces* $\mathcal{A}(H^+)$, where $H^+ := \{h^+ \mid h \in H\}$.

In this way an arrangement of hyperplanes determines $2^{|H|}$ arrangements of halfspaces.

$\mathcal{A}(H^+)$ is defined to consist of the same faces as $\mathcal{A}(H)$ together with the information, which of the halfspaces from H^+ contain a given point $p \in E^d$. This information enables us to define the *description of cells* of a set of halfspaces. $\mathcal{A}(H^+)$ contains several cells, and every cell c can be labelled with the set $v(c)$ of all $h \in H$, such that c is contained in the positive halfspace of h.

1.1.1 Definition

Let H^+ be a finite set of halfspaces. The ordered pair

$$\mathcal{C}(H^+) := (H, \{v(c) \mid c \text{ cell of } \mathcal{A}(H^+)\}),$$

where $v(c) := \{h \in H \mid c \subset h^+\}$, is called the *description of cells* of H^+.

It is clear that $v(c) \neq v(c')$, if $c \neq c'$, since c and c' are separated by at least one hyperplane h and therefore lie in different halfspaces of h. (figure 1.1).

The description of cells can as well be obtained by labelling each point $p \in E^d$ with the set of hyperplanes whose positive halfspaces contain p and then considering the collection of labels of all points which do not lie on any of the hyperplanes. Clearly, these are exactly the points which lie in the cells of the arrangement.

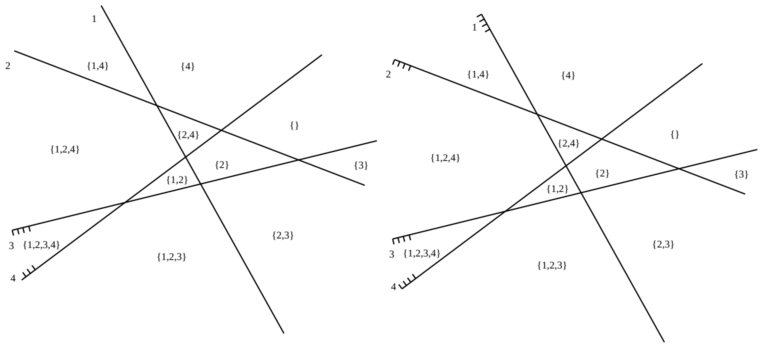

Figure 1.1: Description of cells of an arrangement of halfspaces

The concept of the description of cells is similar to that of *position vectors* given by Edelsbrunner: Let $h_1, ..., h_n$ be the hyperplanes from H. Now every face f of $\mathcal{A}(H^+)$ is labelled with an n-dimensional vector $\lambda(f)$, where $\lambda_i(f)$ is $+1$, 0 or -1, depending on whether f is contained in h_i^+, h_i or h_i^-.

At first glance it seems that this labelling contains more information than our description of cells, since every face is assigned a position vector, while we label the cells only. But it is easy to see, that the position vectors of all k-faces, where $k < d$, are determined by the position vectors of the cells.

Ringel [Ri] has studied the description of cells of 2-dimensional arrangements, and in chapter 7 we will use one of his results.

1.2 Geometric Duality

By using a geometric duality it is possible to obtain $\mathcal{C}(H^+)$ from a set of *directed points* instead of halfspaces. Assume that no hyperplane $h \in H$ is vertical.[1] We define a duality transform that maps hyperplanes to points and vice versa. It is well known that a hyperplane h can be written in the form

$$h : \alpha = a_1 x_1 + ... + a_d x_d,$$

i.e. h is the set of points satisfying this equation for appropriate scalar values $\alpha, a_1, ..., a_d$. The two halfspaces of h are obtained by changing "$=$" to "$<$" and "$>$", respectively. A hyperplane is vertical, iff $a_d = 0$.

Given a point $p = (p_1, ..., p_d) \in E^d$, its dual is defined as the hyperplane

$$\tilde{p} : p_d = 2p_1 x_1 + ... + 2p_{d-1} x_{d-1} - x_d.$$

Conversely, given a hyperplane $h : \alpha = a_1 x_1 + ... + a_d x_d$, its dual point is

$$\tilde{h} = (-\frac{a_1}{2a_d}, ..., -\frac{a_{d-1}}{2a_d}, -\frac{\alpha}{a_d}).$$

[1] This does not mean that the vertical direction is distinguished in some way; the assumption is made simply in order to apply a convenient duality transform. At the cost of making things a little harder to visualize, we could use another duality that works also for vertical hyperplanes.

This shows that the duality is self-inverse, but is not defined for vertical hyperplanes.

Using this duality we can map an arrangement of non-vertical hyperplanes to a *configuration* of points (and vice versa).

The duality is incidence-preserving in the following sense:

1.2.1 Observation

Point p lies on (above, below) hyperplane h, iff point $\tilde{h}$ lies on (above, below) hyperplane $\tilde{p}$.

Details concerning this duality transform can be found in [Ed].

We extend the correspondence to halfspaces and directed points in the following obvious way: a *directed point* is a pair (p, dir), where $p \in E^d$ and $dir \in \{up, down\}$. We can visualize a directed point as a point with a ray attached to it that points into the direction given by dir. For a non-vertical hyperplane h let h_{below} denote the halfspace below h, h_{above} the one above h. We dualize h_{below} to the directed point $(\tilde{h}, up)$ and h_{above} to $(\tilde{h}, down)$. How a directed point is dualized to a halfspace is immediate from this.

Now the following holds and is an easy consequence of the above observation:

1.2.2 Observation

Point p is contained in halfspace h^+, iff $\tilde{h} \notin \tilde{p}$ and the ray emanating from $\tilde{h^+}$ stabs hyperplane $\tilde{p}$.

This observation yields the dual approach to the description of cells: given a configuration of directed points in E^d, label each non-vertical hyperplane h of E^d with the set of all points whose rays stab h. Then the collection of labels of all hyperplanes containing none of the points is the description of cells of the dual arrangement of halfspaces (figure 1.2).

1.3 Simplicity

So far we have considered arbitrary arrangements of halfspaces, but in the following we will restrict ourselves to the simple case: an arrangement of halfspaces in E^d is called *simple*, iff the underlying arrangement of hyperplanes is simple, which means that any d hyperplanes have a unique point in common and any $d+1$ have empty intersection. This translates to the dual space in the following way: a configuration of directed points in E^d is simple, iff the underlying configuration of points is simple, and this is the case if any d points lie on a unique non-vertical hyperplane and there is no hyperplane containing $d+1$ of the points. Note that this implies that no line through two of the points is vertical.

The reason to deal with simple arrangements only is the following

1.3.1 Observation

Let $\mathcal{A}(H^+)$ be an arrangement of n halfspaces with description of cells $\mathcal{C}(H^+)$. Then there exists a simple arrangement $\mathcal{A}(G^+)$ of n halfspaces, such that $\mathcal{C}(H^+) \subset \mathcal{C}(G^+)$.

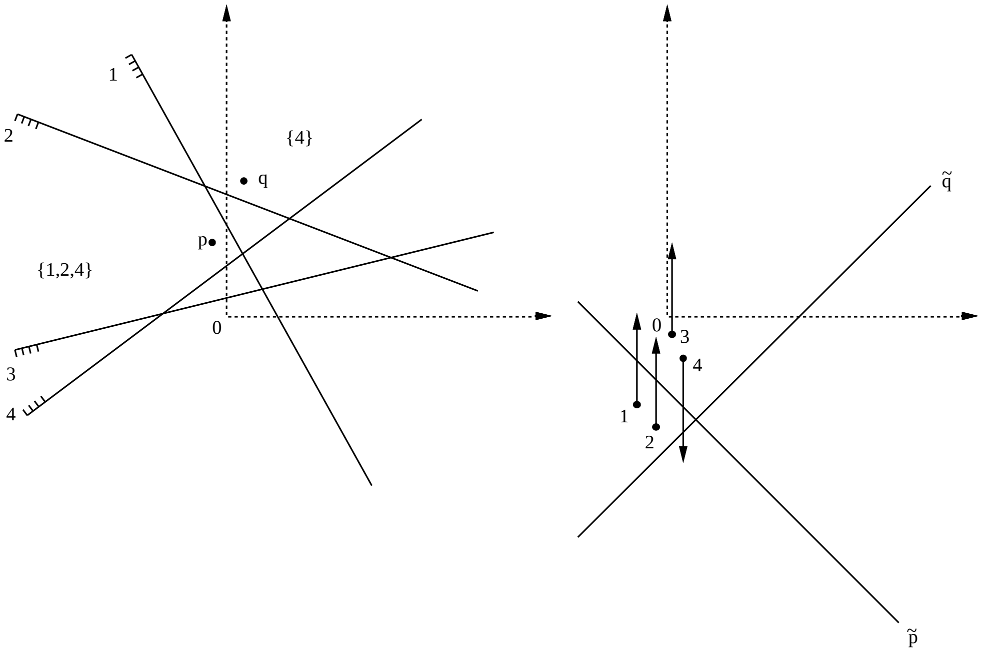

Figure 1.2: arrangement of halfspaces and dual configuration of directed points

To see that this is true, observe that there is only a finite number of cells in an arrangement and each of them has non-vanishing volume. It follows that the hyperplanes can be slightly perturbed without destroying any of the cells. This perturbation can be performed in such a way that all degeneracies disappear, i.e. the arrangement becomes simple. So this simplification transform (figure 1.3) generates some new cells and hence some new labels in the description of cells without changing the old ones.

This shows that simple arrangements maximize the description of cells and this fact is reason enough for us to rule out non-simplicity in the sequel, for it turns out that only a maximal description of cells gives rise to a well-behaved structure in our approach.

We will use the dual view in terms of a configuration of directed points in chapter 6. In the other chapters all the concepts are motivated and explained using the primal arrangement of halfspaces; this turns out to be more handy in what follows. However, we encourage the reader to visualize a newly introduced notion also in the dual space.

For a finite set of halfspaces we now want to work out the basic properties of $\mathcal{C}(H^+)$, introducing as the main tool the concept of *range spaces*.

1.4 Range Spaces

If X is a set and R a collection of subsets of X (possibly empty), the pair $S = (X, R)$ is called a *range space*. X is the *underlying set* of S, consisting of the *elements* of S, the elements of R are called *ranges* of S. If X is a finite set, S is called *finite*. With the exception of examples, all the range spaces in this paper are assumed to be finite, so this is not always mentioned explicitly.

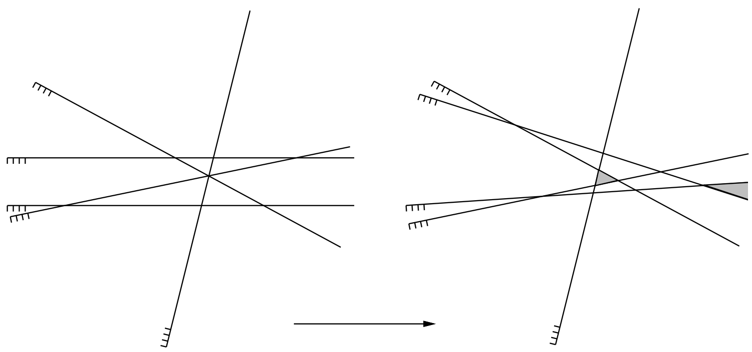

Figure 1.3: simplifying an arrangement of halfspaces – new cells are generated

When geometric concepts are absent, the term *hypergraph* is more commonly used to denote (X, R). By talking of range spaces we usually have in mind that X is some set of points and R a collection of geometric ranges (like in the following examples).

Range spaces (especially of finite VC-dimension, a combinatorial parameter that we introduce in the next section) play an important role in connection with geometric range queries, ϵ-nets [HW], [CW], concept learning [Fd], [BEHW], [Va], and discrepancy [MWW].

1.4.1 Examples

(i) $S = (E^d, B)$, where B is the set of all closed balls in E^d
(ii) $S = (E^d, P)$, where P is the set of all polytopes in E^d

We will come back to these examples when we introduce the VC-dimension of a range space.

Another example of a range space is the description of cells of H^+. The underlying set is H and the ranges are the labels of the cells of $\mathcal{A}(H^+)$. A range space arising in this way will be called *geometric*, if $\mathcal{A}(H^+)$ is simple. Geometric spaces will serve as our main example to explain the notions we will introduce next.

1.4.2 Definition

Let $S = (X, R)$ be a range space, $Y \subset X$. We define

$$\begin{aligned} S - Y &:= (X - Y, R - Y), \text{ where } R - Y := \{r - Y \mid r \in R\} \\ S^Y &:= (X - Y, R^Y), \text{ where } R^Y := \{r \in R \mid r \cap Y = \emptyset, r \cup Y' \in R \, \forall Y' \subset Y\} \\ S|_Y &:= (Y, R|_Y), \text{ where } R|_Y := \{r \cap Y \mid r \in R\} \\ -S &:= (X, -R), \text{ where } -R := 2^X - R \end{aligned}$$

We refer to $S - Y$, S^Y and $S|_Y$ as *subspaces* of S. $-S$ is called the *dual* of S (note that this range space duality has nothing to do with the geometric duality introduced in the second section of this chapter and is also different from the standard range space duality, as defined, for example, in [CW]).

The case where Y is a singleton will be of special interest in the sequel, and this case is basic in the sense that any subspace can equivalently be defined using only subspaces determined by a single element:

if in general $y_1, ..., y_k$ is any ordering of the elements of Y, clearly

$$S - Y = (...(S - \{y_1\}) - ...) - \{y_k\},$$
$$S|_Y = (...(S|_{\{y_1\}})|...)|_{\{y_k\}}.$$

Via an easy induction, part (i) of the following lemma also implies

$$S^Y = (...(S^{\{y_1\}})...)^{\{y_k\}}.$$

1.4.3 Lemma

Let $S = (X, R)$ be a range space, $x, y \in X, Y \subset X$. Then the following holds:

(i) $(R^Y)^{\{x\}} = R^{Y \cup \{x\}}, x \notin Y$
(ii) $|R| = |R - \{x\}| + |R^{\{x\}}|$
(iii) $R - Y = R|_{X-Y}$
(iv) $R^{\{x\}} - \{y\} \subset (R - \{y\})^{\{x\}}$
(v) $-(R - Y) = (-R)^Y$
(vi) $-(R^Y) = (-R) - Y$

Proof:

(i) $r \in (R^Y)^{\{x\}} \Leftrightarrow r, r \cup \{x\} \in R^Y \Leftrightarrow r \cup Y', r \cup \{x\} \cup Y' \in R$ for all $Y' \subset Y \Leftrightarrow r \cup Y' \in R$ for all $Y' \subset Y \cup \{x\} \Leftrightarrow r \in R^{Y \cup \{x\}}$.

(ii) By deleting x from the ranges of R to obtain $R - \{x\}$, exactly the pairs of ranges (r, r') with $x \notin r$ and $r' = r \cup \{x\}$ collapse to one range. Since there are $|R^{\{x\}}|$ such pairs, we conclude that $|R - \{x\}| = |R| - |R^{\{x\}}|$.

(iii) It suffices to observe that $r - Y = r \cap (X - Y)$, for all $r \in R$.

(iv) If $r \in R^{\{x\}} - \{y\}$, then $r \in R^{\{x\}}$ or $r \cup \{y\} \in R^{\{x\}}$, which means $r, r \cup \{x\} \in R$ or $r \cup \{y\}, r \cup \{x, y\} \in R$. In both cases we have $r, r \cup \{x\} \in R - \{y\}$, so $r \in (R - \{y\})^{\{x\}}$.

(v) $r \in -(R - Y) \Leftrightarrow r \notin R - Y \Leftrightarrow \forall Y' \subset Y : r \cup Y' \notin R \Leftrightarrow \forall Y' \subset Y : r \cup Y' \in -R \Leftrightarrow r \in (-R)^Y$.

(vi) similar to (v). ■

When we consider $S := \mathcal{C}(H^+)$ and $h \in H$, then $S - \{h\}$ arises from S by removing h from the label of every cell. So $S - \{h\}$ is simply the description of cells we get after removing the halfspace h^+ from H^+. Consequently, $S - Y$ is the description of cells of $H^+ - Y^+$.

$S^{\{h\}}$ describes exactly the cells who are not in h^+ but are separated only by h from a cell in h^+. Clearly, these are the cells of h^- incident to h; the remaining halfspaces induce a $(d-1)$-dimensional arrangement of halfspaces in h, and there is a one-to-one correspondence between the cells of this subarrangement and the cells described by $S^{\{h\}}$; it follows that $S^{\{h\}}$ can be regarded as the description of cells of the subarrangement.

In general, S^Y corresponds to the $(d - |Y|)$ - dimensional subarrangement induced in $\bigcap_{h \in Y} h$ by $H^+ - Y^+$ (figure 1.4)

Finally, if $G \subset H$, $\mathcal{C}(H^+)|_G$ describes the cells of the arrangement $\mathcal{A}(G^+)$, so it is equal to $\mathcal{C}(G^+)$.

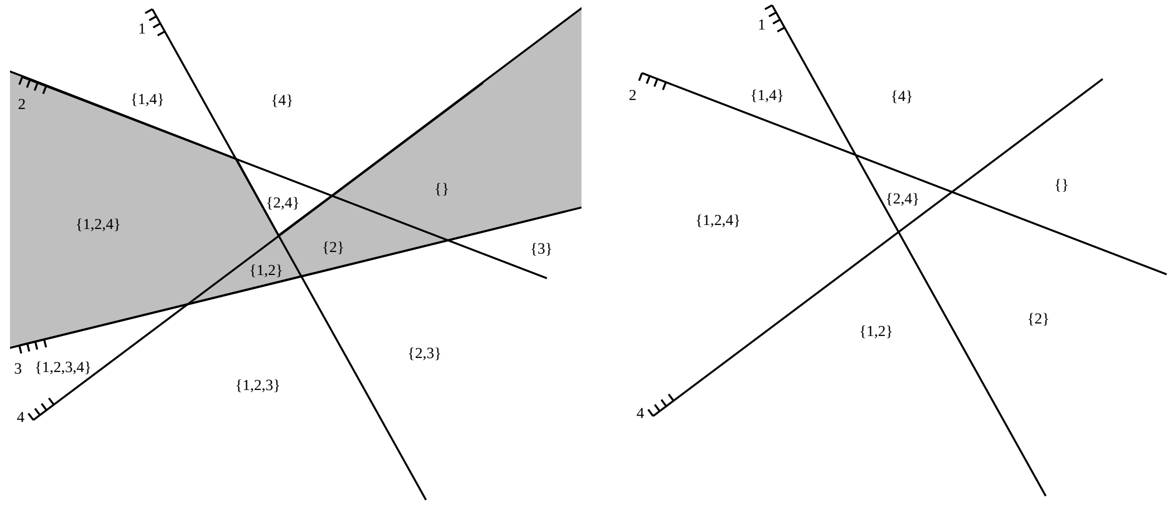

Figure 1.4: $S^{\{h\}}$ and $S - \{h\}$ for description of cells S, $h = 3$

1.5 VC-dimension

As already mentioned, a range space $S = (X, R)$ can be assigned a dimension, the so-called *Vapnik-Chervonenkis Dimension* [VC71]; we say $Y \subset X$ is *shattered* in R, iff $S|_Y = (Y, 2^Y)$, i.e. every subset of Y can be obtained by intersecting Y with a range from R. This leads to the following

1.5.1 Definition

Let $S = (X, R)$ be a range space. The number

$$\text{VC-dim}(S) := \begin{cases} -1 & \text{if } R = \emptyset; \\ k & \text{if } k \text{ is the cardinality of the largest} \\ & \text{subset of } X \text{ that is shattered in } R \\ \infty & \text{if arbitrarily large subsets are shattered} \end{cases}$$

is called the *Vapnik-Chervonenkis dimension* or *VC-dimension* of S.

Clearly, in a finite range space the VC-dimension is bounded by $|X|$.

Now we will give the VC-dimensions of the examples from 1.4.1:

(i) The VC-dimension of this space is $d + 1$. To see this, note first that the set of balls shatters any $d + 1$ points which form a simplex in E^d. If on the other hand a set A of at least $d + 2$ points is given, then Radon's theorem [Ed] ensures that there is a partition of A into subsets A_1 and A_2, such that conv$(A_1)\cap$ conv$(A_2) \neq \emptyset$. Assume A is shattered; then there are balls B_1, B_2 with $B_1 \cap A = A_1$ and $B_2 \cap A = A_2$. Let h be a hyperplane separating $B_1 - B_2$ from $B_2 - B_1$. If the balls are disjoint or have just one point in common, such a hyperplane trivially exists – otherwise we choose h to be the hyperplane spanned by the sphere that is the intersection of the boundaries of B_1 and B_2. Clearly, $B_1 \cap B_2$ contains no element of A, so h also separates A_1 from A_2, a contradiction to the fact that the convex hulls intersect. Hence it follows that A is not shattered.

(ii) This space has VC-dim$(S) = \infty$: Let A be a set of points on a sphere and A' an arbitrary subset of A. Clearly conv(A') is a polytope containing exactly the elements from A', so A is shattered, where $|A|$ can be arbitrarily large.

We will simplify notation and refer to the VC-dimension simply as the dimension of a range space S and write dim(S) instead of VC-dim(S).

1.5.2 Remark

If $S = (X, R)$ is of dimension d, then $S^{\{x\}}$ is of dimension at most $d - 1$, for all $x \in X$.

Proof:
Assume $Y \subset X - \{x\}$ is shattered in $R^{\{x\}}$. We show that $Y \cup \{x\}$ is shattered in R; let Y' be a subset of $Y \cup \{x\}$.
If $Y' \subset Y$, there is $r \in R^{\{x\}} \subset R$ with $r \cap (Y \cup \{x\}) = r \cap Y = Y'$.
Otherwise there is $r \in R^{\{x\}}$ with $r \cap Y = Y' - \{x\}$. But then $(r \cup \{x\}) \cap (Y \cup \{x\}) = Y'$, where $r \cup \{x\} \in R$. Since S is of dimension d, $|Y \cup \{x\}| \leq d$, so $|Y| \leq d - 1$, and this proves the remark. ■

Of course, now we are interested in the dimension of our description of cells $\mathcal{C}(H^+)$:

1.5.3 Theorem

Let H^+ be a set of n halfspaces, defining a simple d-dimensional arrangement. Then the following holds:

(i) $\mathcal{C}(H^+)$ has $\Phi_d(n)$ ranges, where $\Phi_d(n) := \sum_{i=0}^{d} \binom{n}{i}$
(ii) $\mathcal{C}(H^+)$ has dimension d

Proof:
(i) The number of k-faces of a simple arrangement of n hyperplanes in E^d is

$$f_k(n) := \sum_{i=0}^{k} \binom{d-i}{k-i} \binom{n}{d-i},$$

as shown in [Ed]. This implies $f_d(n) = \Phi_d(n)$, so the arrangements $\mathcal{A}(H)$ and $\mathcal{A}(H^+)$ contain $\Phi_d(n)$ cells. Since we have already observed that different cells are mapped to different ranges of $\mathcal{C}(H^+)$, the first part of the theorem follows.

(ii) Let G be a subset of H. Recall that $\mathcal{C}(H^+)|_G$ is the description of cells of G^+. Hence, using part (i) of the lemma, $\mathcal{C}(H^+)|_G$ has $\Phi_d(|G|)$ ranges. If $|G| = d$, this number equals 2^d, i.e. G is shattered. If G has $k > d$ elements, $\Phi_d(|G|)$ is smaller than 2^k, so G is not shattered. VC-dimension d is immediate from this. ■

Chapter 2

Complete Range Spaces

2.1 The Defining Property

Surprisingly, it turns out that $\Phi_d(n)$ is an upper bound on the number of ranges that a range space of dimension d with n elements can have, and the existence of $\mathcal{C}(H^+)$ shows that this bound is tight. The class of range spaces who reach this number of ranges, has very interesting properties and will be the subject of this chapter.

First we give the $\Phi_d(n)$-bound that was independently proved in [Sa] and [VC74]. For the following we define $\Phi_{-1}(n) := 0$.

2.1.1 Theorem

Let $S = (X, R)$ be a range space of dimension d with $|X| = n$ elements. Then $|R| \leq \Phi_d(n)$.

Proof:
The assertion is true for $d = -1$, $d = 0$ and for $n = d \geq 0$, since in this case $|R| = 2^d = \Phi_d(n)$; now let $S = (X, R)$ be a range space of dimension $d \geq 1$, $|X| = n > d$, and assume the theorem holds for any range space of dimension at most $d - 1$ and for any range space of dimension d with at most $n - 1$ elements.

Choose $x \in X$. Clearly, the subspace $S - \{x\}$ is of dimension at most d, so

$$|R - \{x\}| \leq \Phi_d(n - 1)$$

by hypothesis. From 1.5.2 we know that $S^{\{x\}}$ is of dimension at most $d - 1$, which means

$$|R^{\{x\}}| \leq \Phi_{d-1}(n - 1).$$

Since $|R| = |R - \{x\}| + |R^{\{x\}}|$ (1.4.3 (ii)), we conclude $|R| \leq \Phi_d(n - 1) + \Phi_{d-1}(n - 1)$.

Using the equality $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$ we easily obtain $\Phi_d(n - 1) + \Phi_{d-1}(n - 1) = \Phi_d(n)$, and the theorem is established. ■

2.1.2 Definition

A range space $S = (X, R)$ of dimension d with $|X| = n$ elements is called *complete* (of dimension d), iff $|R| = \Phi_d(n)$.

2.2 Characterizations and Duality

It will be useful to have some equivalent characterizations of a complete range space. This is given in the following

2.2.1 Theorem

Let $S = (X, R)$ be a range space of dimension $d \geq 1$ with $n := |X| > d$. Then the following statements are equivalent:

(i) S is complete of dimension d
(ii) $\forall x \in X : S^{\{x\}}$ is complete of dimension $d-1, S-\{x\}$ is complete of dimension d
(iii) $\forall x \in X : S^{\{x\}}$ is complete of dimension $d-1$
(iv) $\exists x \in X : S^{\{x\}}$ is complete of dimension $d-1, S-\{x\}$ is complete of dimension d
(v) $|R^A| = 1$ for all $A \subset X, |A| = d$

Proof:
(i) $\Rightarrow$ (ii) Since dim $(S^{\{x\}}) \leq d-1$ (1.5.2) and dim $(S-\{x\}) \leq d$, we have

$$\Phi_d(n) = |R| = |R^{\{x\}}| + |R-\{x\}| \leq \Phi_{d-1}(n-1) + \Phi_d(n-1) = \Phi_d(n),$$

so we conclude that $|R^{\{x\}}| = \Phi_{d-1}(n-1)$ and $|R-\{x\}| = \Phi_d(n-1)$.

We are done, if we can show that $S^{\{x\}}$ is of dimension $d-1$ and $S-\{x\}$ is of dimension d; since $n > d > 0$, we know that $\Phi_{d-1}(n-1) > \Phi_{d'-1}(n-1), \Phi_d(n-1) > \Phi_{d'}(n-1)$ for any $d' < d$, which means that $S^{\{x\}}$ and $S-\{x\}$ contain too many ranges to be of dimensions less than $d-1$ and d, resp., and this proves the implication.

(ii) $\Rightarrow$ (iii), (iv) trivial

(iii) $\Rightarrow$ (i) We proceed by induction on n. If $n = d+1$, let A be a set of cardinality d shattered in R. There is exactly one $x \in X - A$, and A is also shattered in $R-\{x\}$. Clearly, $R-\{x\}$ contains exactly all the subsets of A, so $|R-\{x\}| = 2^d = \Phi_d(n-1)$, which implies $|R| = |R^{\{x\}}| + |R-\{x\}| = \Phi_{d-1}(n-1) + \Phi_d(n-1) = \Phi_d(n)$.

Now assume $n > d+1$. Again choose $x \in X$, such that $S-\{x\}$ is of dimension d. For all $z \in X-\{x\}$, $(S-\{x\})^{\{z\}}$ is of dimensions at most $d-1$, so

$$\Phi_{d-1}(n-2) \geq |(R-\{x\})^{\{z\}}| \geq |R^{\{z\}} - \{x\}| = \Phi_{d-1}(n-2),$$

since $S^{\{z\}}$ is complete of dimension $d-1$, which – using (i)$\Rightarrow$ (ii) – implies completeness for $S^{\{z\}} - \{x\}$. This shows $|(R-\{x\})^{\{z\}}| = \Phi_{d-1}(n-2)$ for all $z \in X-\{x\}$.

Using the same argument as in (i) $\Rightarrow$ (ii) we see that $(S-\{x\})^{\{z\}}$ is of dimension $d-1$ and hence complete of dimension $d-1$ for all $z \in X-\{x\}$, so by hypothesis $S-\{x\}$ is complete of dimension d, and we conclude

$$|R| = |R-\{x\}| + |R^{\{x\}}| = \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n),$$

so S is complete.

(iv) ⇒ (i) $|R| = |R^{\{x\}}| + |R - \{x\}| = \Phi_{d-1}(n-1) + \Phi_d(n-1)$, since $S^{\{x\}}$ and $S - \{x\}$ are complete, so $|R| = \Phi_d(n)$.

(i) ⇔ (v) To see that "⇒" holds, iterate (i)⇒(iii) and observe that $\Phi_0(n) = 1$. For the inverse implication we proceed by induction on d, noting that for $d = 1$ the assertion is equivalent to implication (iii)⇒(i).

If $d > 1$, consider $S^{\{x\}}$, $x \in X$. Remark 1.5.2 implies that

$$0 = \dim(S^A) \leq \dim(S^{\{x\}}) - (d-1)$$

for $|A| = d, x \in A$, so $S^{\{x\}}$ is of dimension $d-1$. Furthermore, $|(R^{\{x\}})^B| = |R^{B \cup \{x\}}| = 1$ for $B \subset X - \{x\}, |B| = d-1$, so $S^{\{x\}}$ is complete by hypothesis. This holds for all $x \in X$, so again implication (iii)⇒(i) shows that S is complete.

■

One could conjecture that we might leave out the global assumption that S is of dimension d – but then the theorem gets false: if we have a range space satisfying statement (iv) of the theorem for a certain d, S itself does not have to be of dimension d. For an example, consider $X = \{1,2,3\}, R_1 := \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}, R_2 := \{\emptyset, \{2\}, \{2,3\}, \{1,2,3\}\}$. $S_1 := (X, R_1)$ is of dimension 2 and non-complete, while $S_2 := (X, R_2)$ is complete of dimension 1. Since we have $R_1^{\{1\}} = R_2^{\{1\}} = \{\{2,3\}\}$ and $R_1 - \{1\} = R_2 - \{1\} = \{\emptyset, \{2\}, \{2,3\}\}$, both S_1 and S_2 satisfy statement (iv).

2.2.2 Remark

Characterization (v) shows that a complete space of dimension d shatters any subset of cardinality d: Given $|A| = d$, consider the unique range r in R^A. From the definition of R^A we know that $r \cup A' \in R$ for all $A' \subset A$, and $A' = A \cap (r \cup A')$ shows that A is shattered.

r is called a *vertex* of S, and later on we will be able to show that a complete range space is completely determined by its vertices.

2.2.3 Corollary

Let $S = (X, R)$ be a complete range space, $x, y \in X$. Then

$$S^{\{x\}} - \{y\} = (S - \{y\})^{\{x\}}.$$

Proof:

The assertion is easily seen to hold if $\dim(S) \leq 0$ or $|X| = \dim(S)$. Otherwise we know from 1.4.3 (iv) that $R^{\{x\}} - \{y\} \subset (R - \{y\})^{\{x\}}$; using the preceeding theorem it follows that $|R^{\{x\}} - \{y\}| = |(R - \{y\})^{\{x\}}|$, and this implies $S^{\{x\}} - \{y\} = (S - \{y\})^{\{x\}}$. ■

The property that subspaces of complete spaces are complete turns out to be very important; it enables us to perform inductive proofs, whenever we have a statement involving complete range spaces.

As far as $\mathcal{C}(H^+)$ is concerned, this is not very surprising, since we have already seen that the subspaces of $\mathcal{C}(H^+)$ correspond to arrangements that are obtained by deleting some of the halfspaces from H^+ or to lower-dimensional subarrangements on the hyperplanes $h \in H$, whose descriptions of cells, of course, are complete, too.

A nice and useful statement involving the dual range space is the following

2.2.4 Theorem

$S = (X, R)$ is complete of dimension d, iff $-S$ is complete of dimension $|X| - d - 1$.

Proof:

It suffices to prove one implication; for $d = -1$ and $d = n := |X|$ the theorem is easily seen to be valid. Now assume $0 \leq d < n$. Because of $2^n - \Phi_d(n) = \Phi_{n-d-1}(n)$ it remains to show that $-S$ is of dimension at most $n - d - 1$.

Assume on the contrary, there is $A \subset X$, $|A| = n-d$ shattered in $-R$. Then $|X - A| = d$, and theorem 2.2.1(v) states that there is a unique range $r \in R^{X-A}$. Since $r \subset A$, there is $r' \in -R$, such that $A \cap r' = r$. This implies that r' is a superset of r and $r' - r$ contains no element of A. But then r' is of the form $r' = r \cup B, B \subset X - A$, which is a contradiction, since $r \in R^{X-A}$ implies that all the ranges of this form are contained in R. ■

2.3 The Distance-1-graph and Swapping

In order to find out more about the structure of a complete range space, we introduce a graph on S, called the distance-1-graph of S. To this end we define a distance function on R by

$$\text{dist}(r_1, r_2) := |r_1 \triangle r_2|,$$

where $r_1 \triangle r_2$ is the symmetric difference of r_1 and r_2 (Note that (R,dist) is a metric space).

2.3.1 Definition

Let $S = (X, R)$ be a range space; the undirected edge-labelled graph

$$D^1(S) := (R, E), \text{ where } E := \{\{r_1, r_2\} \subset R \mid \text{dist}(r_1, r_2) = 1\}$$

is called the *distance-1-graph* of S.

The label $\lambda(e)$ of an edge $e = \{r_1, r_2\} \in E$ is defined as the unique element in $r_1 \triangle r_2$.

So the D^1-Graph of S joins two ranges of R with an edge, when they differ by exactly one element $x \in X$, and x is the label of this edge.

For an example consider the geometric range space $S = \mathcal{C}(H^+)$. An edge of $D^1(S)$ joins two ranges, when they are the labels of adjacent cells of the arrangement $\mathcal{A}(H^+)$. The label of the edge is the hyperplane separating the two cells (figure 2.1). We remark that the D^1-graph is invariant under changing H^+, i.e. each of the $2^{|H|}$ arrangements of halfspaces that come from an arrangement of hyperplanes $\mathcal{A}(H)$ define the same D^1-graph. To facilitate our further considerations, we will generalize this statement to arbitrary range spaces, introducing the notion of *swapping* a range space.

Let us consider the effect of interchanging the positive and the negative halfspace of a hyperplane $h \in H$. A cell, that previously was contained in h^+ is now contained in h^- and vice versa, i.e. its new label is the symmetric difference of its old label and $\{h\}$. Motivated by this we give the following

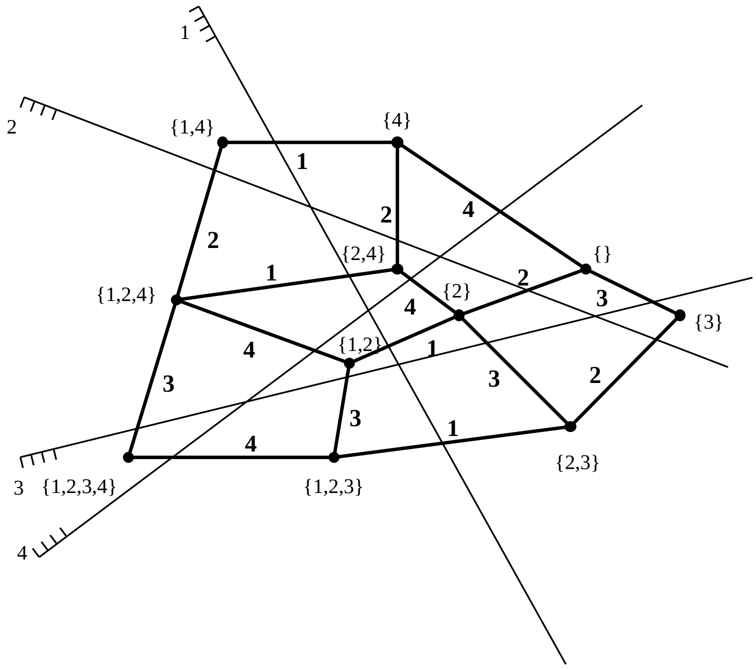

Figure 2.1: D^1-graph of a geometric range space

2.3.2 Definition

Let $S = (X, R)$ be a range space, $D \subset X$. The range space

$$S \triangle D := (X, R \triangle D), \text{ where } R \triangle D := \{r \triangle D \mid r \in R\},$$

is called S *swapped* D.

Clearly $|R| = |R \triangle D|$; Note that the distance-1-graph is invariant under swapping, since $r \triangle r' = (r \triangle D) \triangle (r' \triangle D)$, for all $r, r' \in R$.

Furthermore, swapping does not change the dimension of S: Consider $A \subset X$. Observing that

$$A \cap r = A \cap r' \Leftrightarrow A \cap (r \triangle D) = A \cap (r' \triangle D),$$

we see that $|\{A \cap r \mid r \in R\}| = |\{A \cap (r \triangle D) \mid r \in R\}|$, i.e. A is shattered in $R \triangle D$ if and only if A is shattered in R. This immediately leads to the following

2.3.3 Observation

Let $S = (X, R)$ be a range space, $D \subset X$. S is complete of dimension d, if and only if $S \triangle D$ is complete of dimension d.

The invariance of the completeness-property under swapping will be an important tool in the sequel. Whenever it is convenient, we may assume that a fixed range is equal to a certain subset of X, which simplifies many proofs. A phrase like "by swapping assume $r = \emptyset$" means that we swap

S in such a way that r is mapped to $\emptyset$, and work with the swapped space instead of the original one. As far as the D^1-graph and similar concepts are concerned, this is no loss of generality.

Now we are able to prove a fundamental result about the distance-1-graph of a complete range space of dimension $d \geq 0$, namely that it is connected. We actually derive a stronger result: Given two ranges $r, r' \in R$, there exists a path from r to r' in $D^1(S)$ with length dist(r, r'), and this path is shortest possible, since at least every element from $r \triangle r'$ must occur as the label of an edge on the path.

First we need a lemma:

2.3.4 Lemma

Let $S = (X, R)$ be complete of dimension $d \geq 1$, $X \in R$, $r \neq X$ a range of S. Then there exists $z \in X$, such that $r \in R^{\{z\}}$.

Proof:

We proceed by induction on $n := |X|$. For $n = d$, any subset of X is a range, so the lemma holds in this case.

Now assume $n > d$, and let the assertion be true for all complete spaces of less then n elements.

Consider $r \in R, r \neq X$. Then there is $x \in X$, such that $x \notin r$. If $r = X - \{x\}$, we are done. Otherwise we know that $r \in R - \{x\}$, and by hypothesis there exists $z \in X - \{x\}$ with $r \in (R - \{x\})^{\{z\}} = R^{\{z\}} - \{x\}$ (2.2.3). So either $r \in R^{\{z\}}$ or $r \cup \{x\} \in R^{\{z\}}$, which implies $r \in R^{\{x\}}$. ■

2.3.5 Theorem

Let $S = (X, R)$ be a complete range space, $r, r' \in R, r \neq r'$. Then there is a path of length D :=dist(r, r') between r and r' in $D^1(S)$.

Proof:

By swapping we can assume that $r' = X$.

Now apply the preceeding lemma to $r_0 := r$, and suppose $r_0 \in R^{\{z_0\}}$. Then $r_1 := r_0 \cup \{z_0\}$ is a range in R. Analogously define $r_i := r_{i-1} \cup \{z_{i-1}\}$, $i > 1$; since dist(r_i, r') decreases by one in every step, we conclude that $r_D = r'$, and the $r_i, 0 \leq i \leq D$ define the desired path. ■

If S is complete of dimension 1, the D^1-graph is of a special structure:

2.3.6 Theorem

Let $S = (X, R)$ be a complete range space of dimension 1. Then $D^1(S)$ is a tree, and every $x \in X$ occurs exactly once as an edge label of $D^1(S)$.

Proof:

$S^{\{x\}}$ is complete of dimension 0, so $|R^{\{x\}}| = \Phi_0(|X| - 1) = 1$, for all $x \in X$. Clearly the number of edges labelled with x equals $|R^{\{x\}}|$, so there is exactly one edge labelled with x, for all $x \in X$.

It follows that $D^1(S)$ has $|X|$ edges.

Assume, D^1 contains a cycle of edges $e_1, ..., e_k$. It is clear that the label of e_i has to occur an even number of times on this cycle, for all i, i.e. at least twice, which is a contradiction, so there is no cycle.

$D^1(S)$ has $\Phi_1(|X|) = |X| + 1$ nodes and $|X|$ edges and contains no cycle, so it is a tree. ■

The fact that any two ranges of a complete range space S are joined by a (shortest possible) path in $D^1(S)$, will turn out to be a key theorem in the sequel. A trivial consequence is that any range has at least one neighbour in $D^1(S)$. This can be used to prove the following easy

2.3.7 Lemma

Let $S = (X, R)$ be complete of dimension $d \geq 1$. Then any range has at least d neighbours in $D^1(S)$.

Proof:

Clearly, the lemma holds for $d = 1$. If $d > 1$, $r \in R$, let r' be a neighbour of r in $D^1(S)$. By swapping assume $r' = r \cup \{x\}$, $x \in X$. Then $r \in R^{\{x\}}$, and by hypothesis r has $d - 1$ neighbours in $D^1(S^{\{x\}})$. These are neighbours also in $D^1(S)$, so together with r' this sums up to d neighbours. ■

A more interesting consequence is the next theorem that we have already mentioned:

2.3.8 Theorem

Let $S = (X, R)$ be complete of dimension $d \geq 0$. Then S is completely determined by its vertices, i.e.

$$R = \bigcup_{|A|=d} \{r_A \cup A' \mid A' \subset A\},$$

where r_A is the vertex in R^A.

Proof:

Recall that the vertices are the unique ranges $v \in R^A$, $|A| = d$. A is said to determine the vertex v.

We show that for any $r \in R$ there is $|A| = d$ such that $r = v \cup A'$ for some $A' \subset A$, v the vertex determined by A. Since on the other hand all the sets of the form $v \cup A'$ are ranges of R, the theorem follows.

We use induction on d, noting that for $d = 0$ the theorem holds; now assume $d > 0$ and let $r \in R$ be given. Let r' be a neighbour of r in $D^1(S)$, $x \in X$ the label of the edge (r, r'). Let u be the one of r, r' that is contained in $R^{\{x\}}$; by hypothesis there is $B \subset X - \{x\}$, such that $u = v \cup B'$, $B' \subset B$, v the vertex of $R^{\{x\}}$ determined by B. v is also a vertex of R, determined by $B \cup \{x\}$; clearly, the theorem holds if $r = u$. If $r' = u$, then $r = v \cup (B' \cup \{x\})$, and since $B' \cup \{x\} \subset B \cup \{x\}$, v is an appropriate vertex for r also in this case. ■

Chapter 3

Pseudogeometric Range Spaces

3.1 The Defining Property

Up to now we have shown that the description of cells of a set of halfspaces in E^d is a complete range space of VC-dimension d, and we have given some basic properties of complete spaces. Now it is natural to ask, whether to a given complete space S there exists a set of halfspaces, such that S is its description of cells – or equivalently: is every complete space geometric?

To attack this question, consider a set H^+ of halfspaces in E^1, i.e. on the line. A hyperplane h is a point on the line, the halfspace h^+ is one of the two rays starting at h.

The cells of the corresponding 1-dimensional arrangement are ordered along the line and their labels are joined by the D^1-graph of $\mathcal{C}(H^+)$ in this order. This means, the D^1-graph is not only a tree (as shown in theorem 2.3.6), but a path (figure 3.1).

So we have established a necessary condition that a geometric space of dimension 1 has to fulfill, and there are 1-dimensional complete spaces who violate this condition (the smallest example is $S = (X, R)$ with $X = \{1, 2, 3\}$ and $R = \{\emptyset, \{1\}, \{2\}, \{3\}\}$, where $\emptyset$ is of degree three in $D^1(S)$).

There is an analogous condition for d-dimensional geometric spaces, $d > 1$. Recall that $\mathcal{C}(H^+)^Y$ corresponds to the $d - |Y|$- dimensional subarrangement induced in $\bigcap_{h \in Y} h$ by the remaining halfspaces $H^+ - Y^+$. In the case where $|Y| = d - 1$, this subarrangement is of

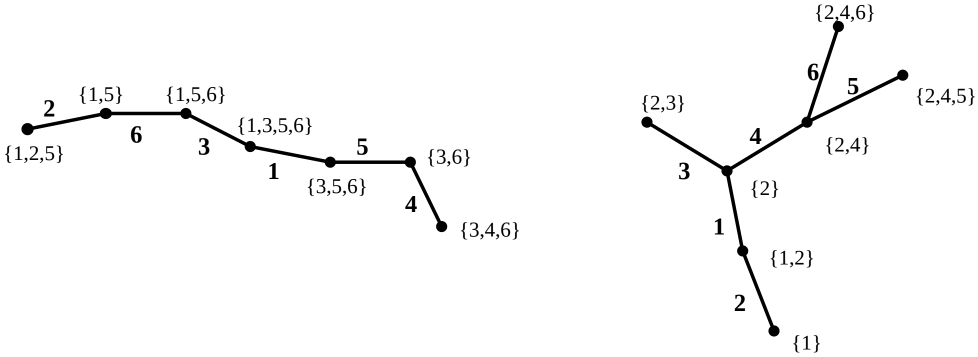

Figure 3.1: geometric and non-geometric complete space of dimension 1

dimension 1, so the D^1-graph of $\mathcal{C}(H^+)^Y$ again is a path.

This necessary condition can be used as a defining property of a subclass of the complete spaces, called pseudogeometric range spaces, which are the subject of this chapter.

3.1.1 Definition

Let $S = (X, R)$ be complete of dimension d. S is called *pseudogeometric* or PG-space, iff $d \leq 0$ or – in case $d \geq 1$, if $D^1(S^Y)$ is a path, for all $Y \subset X$, $|Y| = d - 1$.

Note that since pseudogeometric spaces are determined by properties of the distance$-1-$graph, there is an equivalent to observation 2.3.3 also for PG-spaces, so swapping is a useful tool here either.

3.2 Characterizations and Duality

In theorem 2.2.1 we have given five characterizations of complete spaces; equivalents of statements (i) - (iii) are easily seen to hold also in the context of pseudogeometric spaces; this is shown in the following theorem, where two additional characterizations are given.

Unfortunately statement 2.2.1 (iv) cannot be added to the list, i.e. if for a space S of dimension d there exists $x \in X$, such that $S^{\{x\}}$ and $S - \{x\}$ are pseudogeometric of dimensions $d - 1$ and d resp., S itself does not have to be pseudogeometric. We give a 2-dimensional counterexample: $S = (X, R)$ with

$$X = \{1,2,3,4\}, R = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{2,3,4\}\}.$$

S is of dimension 2 and complete, since all $R^{\{x\}}$ are complete of dimension 1. Now

$$R - \{4\} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}, R^{\{4\}} = \{\emptyset, \{1\}, \{2\}, \{2,3\}\};$$

it is easy to check, that $S - \{4\}$ and $S^{\{4\}}$ are both pseudogeometric, but S is not pseudogeometric, since $R^{\{1\}} = \{\emptyset, \{2\}, \{3\}, \{4\}\}$ shows that $D^1(S^{\{1\}})$ is not a path.

Furthermore, it is clear that an equivalent to 2.2.1(v) cannot hold for pseudogeometric spaces, since the 0-dimensional subspaces do not carry any structural information.

Note that the following theorem is stated for $d \geq 2$ – if $d = 1$, the implications (iii)⇒(i),(ii) as well as (ii)⇒(i) may not hold! However, the proof shows that (i)⇔(iv)⇔(v) as well as (i)⇒(ii) holds also for $d = 1$.

3.2.1 Theorem

Let $S = (X, R)$ be a complete of dimension $d \geq 2$ and $|X| > d$. Then the following statements are equivalent:

(i) S is pseudogeometric of dimension d

(ii) $\forall x \in X : S^{\{x\}}$ is pseudogeometric of dimension $d - 1$, $S - \{x\}$ is pseudogeometric of dimension d

(iii) $\forall x \in X : S^{\{x\}}$ is pseudogeometric of dimension $d - 1$

(iv) $\forall Y \subset X, |Y| = d + 2$, $S|_Y$ is pseudogeometric of dimension d

(v) $\forall Y \subset X, |Y| = d + 2$, $S|_Y$ is geometric of dimension d

Proof:

(i) $\Rightarrow$ (ii) Consider a fixed $x \in X$, $Y \subset X - \{x\}, |Y| = d - 2$. Clearly, $D^1((S^{\{x\}})^Y)$ is a path, since S is pseudogeometric and $(S^{\{x\}})^Y = S^{Y \cup \{x\}}$ (1.4.3 (i)), so $S^{\{x\}}$ is PG.

If $D^1(S^Y)$ is a path, this is also the case for $D^1(S^Y - \{x\}) = D^1((S - \{x\})^Y)$ (2.2.3), so $S - \{x\}$ is pseudogeometric of dimension d.

(ii) $\Rightarrow$ (iii) trivial

(iii) $\Rightarrow$ (i) Let Y be a subset of X, $|Y| = d - 1, x \in Y$. Then $S^Y = (S^{\{x\}})^{Y-\{x\}}$, and since $S^{\{x\}}$ is pseudogeometric of dimension $d - 1$, $D^1((S^{\{x\}})^{Y-\{x\}})$ is a path, so S itself is pseudogeometric.

(i) $\Leftrightarrow$ (iv) The implication "$\Rightarrow$" follows by iterating the second part of (i) $\Rightarrow$ (ii). To prove the other one, assume S is non-PG. Then there is $Z \subset X, |Z| = d - 1$, such that there is a range r of degree at least three in S^Z. Let a, b, c denote the labels of three edges incident to r in $D^1(S^Z)$.

Then a node of degree three is still present in $S^Z|_{\{a,b,c\}} = (S|_{Z \cup \{a,b,c\}})^Z$. This shows that $S|_Y$ is non-PG, where $Y = Z \cup \{a, b, c\}$, $|Y| = d + 2$.

(iv) $\Leftrightarrow$ (v) We will show that there is only "one" pseudogeometric space with $d + 2$ elements, i.e. up to swapping and relabelling the elements all PG-spaces with $d + 2$ elements are equal. From this it follows that an arrangement of $d + 2$ halfspaces in E^d generates any desired PG-space with $d + 2$ elements after renaming the hyperplanes and interchanging some positive with negative halfspaces in a suitable way. Hence the PG-space must be geometric.

The proof is based on the duality theorem for pseudogeometric spaces and is given as a corollary to the theorem. ■

3.2.2 Theorem

> $S = (X, R)$ is pseudogeometric of dimension d, iff $-S$ is pseudogeometric of dimension $n - d - 1$.

Proof:

If $d = -1$ or $d = 0$, then $-S$ is complete of dimension $n := |X|$ or $n - 1$. Since the 1-dimensional subspaces $(-S)^Y$ contain 2 or 3 ranges in this case, the corresponding D^1-graph must be a path.

For $n = d$ or $n = d + 1$, $-R$ is empty or contains one range, so $-S$ is PG of dimension -1 or 0 by definition.

Now assume $d \geq 1$, $n \geq d + 2$. If S is PG, then following 3.2.1(i)$\Rightarrow$(iv), $S|_Y$ is PG for all $Y \subset X, |Y| = d + 2$, and $T := (S|_Y)^Z = (S - (X - Y))^Z$ consists of exactly three elements and four ranges which form a path in the corresponding D^1-graph, for all $|Z| = d - 1$.

Then also $-T$ has this property. Since $-T = -((S - (X - Y))^Z) = ((-S) - Z)^{X-Y}$, where $|X - Y| = n - d - 2$ =dim$(-S) - 1$, we know that $(-S) - Z = (-S)|_{X-Z}$ is PG, $|X - Z| = n - d + 1$ =dim$(-S) + 2$. Now 3.2.1 (iv) $\Rightarrow$ (i) shows that $-S$ itself is pseudogeometric. ■

3.2.3 Corollary

> Up to swapping and relabelling the elements, all PG-spaces $S = (X, R)$ of di-

mension $d \geq 1$ with $|X| = d + 2$ elements are equal.

Proof: We equivalently show that there is only "one" dual space. Let $x_1, ..., x_n$ be an ordering of X. The theorem shows that $-S$ is PG of dimension 1. The D^1-graph is a path, and after swapping with a range on one end of the path, the ranges are ordered by inclusion, starting with the empty set. By appropriate relabelling we obtain $R = \{\emptyset\} \cup \{R_i \mid 1 \leq i \leq n\}$, $R_i = \{x_1, ..., x_i\}$. ■

We remark that in any dimension $d \geq 1$ there exist complete spaces that are not pseudogeometric: For $n \geq d$ let X be a set with n elements and define R to be the set of all subsets of X with at most d elements. Clearly, $S_d(X) := (X, R)$ is of dimension d. Furthermore, since there are $\binom{n}{i}$ subsets with i elements, $S_d(X)$ has $\sum_{i=0}^{d} \binom{n}{i} = \Phi_d(n)$ ranges, which means that it is complete. $S_d(X)$ is called the *canonical* complete space of dimension d over X.

Consider $Y \subset X$, $|Y| = d - 1$.

It is an easy observation that $R^Y = \{\emptyset\} \cup \{\{z\} \mid z \in X - Y\}$. This shows that $\emptyset$ has degree $|X - Y|$ in $D^1(S_d(X)^Y)$, so $S_d(X)$ is not pseudogeometric if $|X| \geq d + 2$.

Since the structure of $S_d(X)$ depends only on the cardinality of the set X, we define $S_d(n) := S_d(\{1, ..., n\})$.

3.3 Levi-type Theorems

The two theorems in this section are inspired by Levi's lemma for arrangements of pseudolines [Le], [Gr], and with the interpretation of geometric spaces in terms of arrangements of halfspaces in mind, the definition of a *segment* and a *line* that we give next, will be quite intuitive. Note that for non-PG-spaces the following won't work anymore; the canonical complete space $S_1(3)$ is an easy counterexample.

3.3.1 Definition

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$. The set of ranges on a path in $D^1(S)$ is called a *segment*, iff no $x \in X$ occurs more than once as an edge label on the path. If r, r' are the ranges on both ends of the path, the segment is said to *join* r and r'.

A *line* is a segment joining ranges $r, X - r \in R$ (figure 3.2).

From 2.3.5 we know that for any pair of ranges r, r' there is a segment joining them; the segment has dist$(r, r') + 1$ elements and the edge labels on the segment are exactly the elements from $r \triangle r'$. Note that a subsegment between ranges u and u' can be replaced by any other segment joining u, u' – the result again is a valid segment for r, r'.

We say that a set of ranges $R' \subset R$ *admits* a segment, iff there is a segment containing R'. We will show that in the case of pseudogeometric spaces any two ranges admit a line and give necessary and sufficient conditions under which an arbitrary set of ranges admits a line. Before that we prove a more technical lemma to facilitate the following considerations.

3.3.2 Lemma

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$.

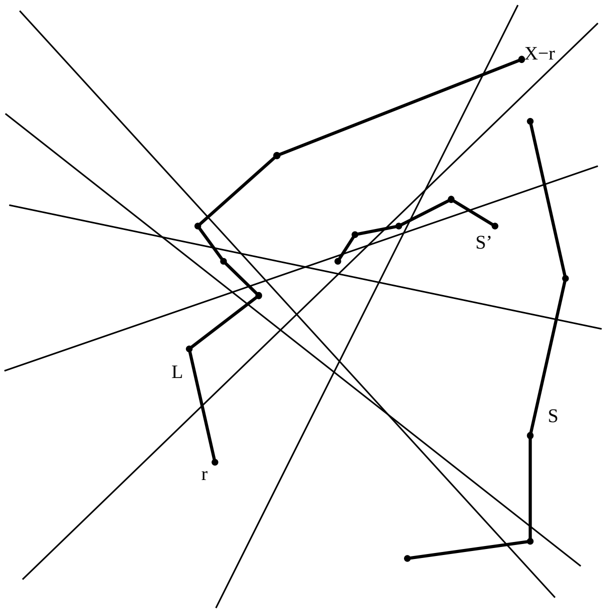

Figure 3.2: segment S, line L in geometric space. S' is not a segment

(i) If $R' \subset R$ is a segment, $|R'| \geq 2$, then (X, R') is of dimension 1.

(ii) $R' \subset R$ admits a segment iff there is $Y \subset X$, such that the elements from $R' \triangle Y$ can be linearly ordered by inclusion (and occur on the segment in this order).

(iii) $r, r' \in R$ admit a line, iff there are $t, X - t \in R$, such that $r - r' \subset t$ and $r' - r \subset X - t$ ($t, r, r', X - t$ occur in this order on the line).

Proof:

(i) Assume that $A := \{x, y\} \subset X$ is shattered in R'. Then there is $U \subset R', |U| = 4$, which shatters A. Let $u_1, ..., u_4$ be the elements of U in the order in which they occur on the segment. The segment property requires the sets $u_i \triangle u_{i+1}, i = 1, 2, 3$ to be pairwise disjoint. On the other hand, each of these sets contains a non-empty subset of A, so either x or y must be contained in two of the sets, which is a contradiction, so R' is of dimension 1.

(ii) If R' admits a segment, let $r_1, ..., r_k$ be an ordering of the elements of R' along the segment. Similar to (i), this means $(r_1 \triangle r_i) \cap (r_i \triangle r_{i+1}) = \emptyset, i < k$. Now swap with $Y := r_1$ and let r'_i denote $r_i \triangle Y$. Then $r'_i \cap (r'_i \triangle r'_{i+1}) = \emptyset$, which implies $r'_i \subset r'_{i+1}$.

Now suppose $R' = \{r_i \mid i \leq k\}$ and swapping with $Y \subset X$ yields $r'_1 \subset ... \subset r'_k, r'_i = r_i \triangle Y$. Clearly then $D_i := r'_i \triangle r'_{i+1} = r'_{i+1} - r'_i$, so $D_i \cap D_{i+1} = \emptyset$. This means, by piecing together segments joining r'_i and r'_{i+1} in $D^1(S \triangle Y)$ we obtain a segment containing $R' \triangle Y$ in $S \triangle Y$. Since the distance-1-graph is invariant under swapping this shows that R' admits a segment in S.

(iii) If r, r' admit a line then there are ranges $t, X - t$, such that X is the disjoint union of $t \triangle r$, $r \triangle r'$ and $r' \triangle (X - t)$. We show that $r - r' \subset t$ (symmetric argumentation works for $r' - r \subset X - t$): assume $x \in r - r' \subset r \triangle r'$; this implies $x \notin t \triangle r$. Since $x \in r$, we conclude that $x \in t$.

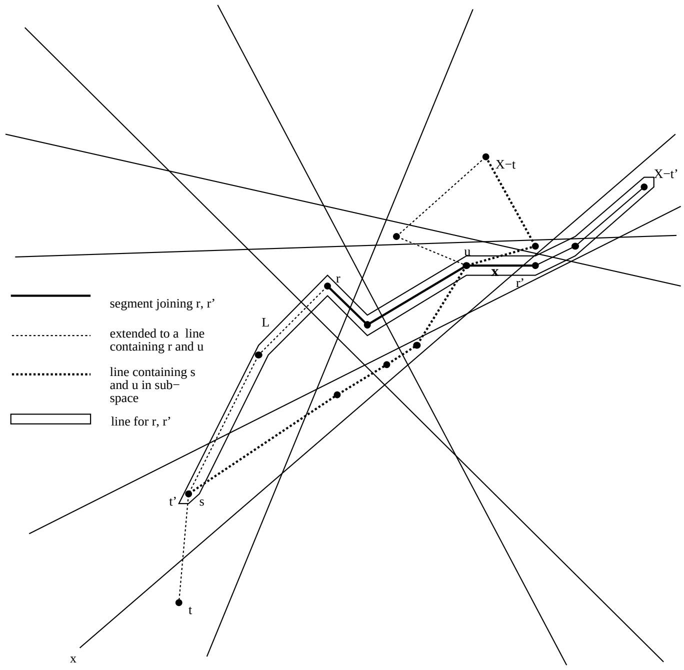

Figure 3.3: Constructing a line containing r, r' by induction

If on the other hand $r - r' \subset t$, $r' - r \subset X - t$, then easy calculation shows $r \triangle t \subset r' \triangle t$, so we have $\emptyset = t \triangle t \subset r \triangle t \subset r' \triangle t \subset (X - t) \triangle t = X$; following part (ii) of the lemma this means that $t, r, r', X - t$ admit a segment, and this segment is a line containing r, r'. ■

Now we are ready to prove

3.3.3 Theorem

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$. Then any two ranges r, r' admit a line.

Proof:

The assertion is true for $d = 1$, since in this case R itself is a line. Furthermore, if dist$(r, r') = 0$, i.e. $r = r'$, then part (iii) of the lemma shows that it is sufficient to find one pair of complementary ranges $t, X - t$. Such a pair always exists, as follows by easy induction on d: For $d = 1$, take the ranges on both ends of $D^1(S)$. If $d > 1, x \in X$, there are ranges $t, X - (t \cup \{x\}) \in R^{\{x\}}$ by hypothesis, which implies $t, X - t \in R$.

Now let $S = (X, R)$ be pseudogeometric of dimension $d > 1$, $r, r' \in R$ with distance $D :=$ dist$(r, r') > 0$ and assume the theorem holds for any PG-space of dimension less than d and any pair of ranges with distance less than D in R.

Consider a segment joining r and r' and let u be the range followed by r' on this segment (figure 3.3). By swapping with $\{x\}$, if necessary, we can assume $r' = u \cup \{x\}$ for some $x \in X$. Since dist$(r, u) = D - 1$, r and u admit a line L by hypothesis, so there are ranges $t, X - t$ with $r - u \subset t, u - r \subset X - t$. If $x \in X - t$, then we obtain $r - r' \subset t, r' - r \subset X - t$, so we are done.

Otherwise $x \in t$, and since $x \notin r$, by traversing L from r to t, we encounter a range $s \in R^{\{x\}}$. By hypothesis s and u admit a line in $S^{\{x\}}$, so we have $t', X - (t' \cup \{x\}) \in R^{\{x\}}$ with $s - u \subset t'$, $u - s \subset X - (t' \cup \{x\})$, which yields $s - r' \subset t'$, $r' - s \subset X - t'$. Now s can be replaced with r in this formula, since we have $r - r' \subset s - r'$, $r' - r \subset r' - s$, which follows from the fact that s, r and $u = r' - \{x\}$ appear on the original line L in this order.

Together with the fact that $X - t'$ is a range in R, this shows that r and r' admit a line in S. ■

3.3.4 Corollary

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$, $x \in X$.
If $\emptyset, X - \{x\} \in R$, then $\{x\} \in R$ or $X \in R$.

We will use the theorem to prove the following generalization of it:

3.3.5 Theorem

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$, $R' \subset R$, $|R'| \geq 4$. The following statements are equivalent:

(i) R' admits a line
(ii) Any four ranges from R' admit a line
(iii) Any three ranges from R' admit a line and (X, R') is of dimension
1

Proof:
(i)⇒ (ii) trivial
(ii)⇒ (iii) Suppose, (X, R') is of dimension $d \geq 2$; then (X, U) is of dimension 2 for some $U \subset R'$, $|U| = 4$. Now part (i) of the lemma shows that U cannot admit a line.
(iii)⇒ (i): theorem 3.3.3 can equivalently be expressed as follows: any segment can be extended to a line. This means, it suffices to show that R' admits a segment. To this end consider ranges $r, r' \in R'$ with maximal distance. After swapping assume $r = \emptyset$. We show that now the ranges from R' are linearly ordered by inclusion, so part (ii) of the lemma yields the desired conclusion.

Consider $u, u' \in R'$. We have to show $u \subset u'$ or $u' \subset u$. $\{r, u, r'\}$ as well as $\{r, u', r'\}$ admit a segment, and since r and r' have maximal distance, these segments join r and r'. $r = \emptyset$ then implies $r \subset u, u' \subset r'$. If u and u' are not comparable with respect to $\subset$, there is $x \in u - u'$, $y \in u' - u$. $r = \emptyset$ contains neither x nor y, while r' contains both of them. This means, $\{x, y\}$ is shattered in $\{r, u, u', r'\}$, a contradiction to the assumption that R' is of dimension 1. ■

3.4 Closure and Boundary

We have already introduced certain subspaces associated with a range space; now we will define two more spaces, namely the *closure* and the *boundary* of a range space. Although these two spaces can be derived from any range space, they have interesting properties and a geometric interpretation especially in the context of pseudogeometric spaces.

3.4.1 Definition

Let $S = (X, R)$ be a range space. The spaces

$$\overline{S} = (X, \overline{R}),\ \text{where } \overline{R} := R \cup \{X - r \mid r \in R\}$$

and

$$\delta S = (X, \delta R),\ \text{where } \delta R := \{r \in R \mid X - r \in R\}$$

are called the *(complementary) closure* and the *(complementary) boundary* of S.

When we consider the PG-space $\mathcal{C}(H^+)$, the ranges of $\delta\mathcal{C}(H^+)$ are exactly the labels of the unbounded cells of the arrangement $\mathcal{A}(H^+)$. The geometric interpretation of $\overline{\mathcal{C}(H^+)}$ is not so obvious; it will be given in the next chapter, where we introduce arrangements of hemispheres.

3.4.2 Lemma

Let $S = (X, R)$ be a range space. Then for all $x \in X$

(i) $(\delta S)^{\{x\}} = \delta(S^{\{x\}})$
(ii) $\overline{S} - \{x\} = \overline{S - \{x\}}$

and

(iii) $-\delta S = \overline{-S}$
(iv) $-\overline{S} = \delta(-S)$

Proof:
(i) $r \in (\delta R)^{\{x\}} \Leftrightarrow r, r \cup \{x\} \in \delta R \Leftrightarrow r, X - r, r \cup \{x\}, X - (r \cup \{x\}) \in R$
$\Leftrightarrow r, X - (r \cup \{x\}) \in R^{\{x\}} \Leftrightarrow r \in \delta(R^{\{x\}})$.
(ii) similar to (i)
(iii) $r \in -\delta R \Leftrightarrow r \notin \delta R \Leftrightarrow r \notin R \vee X - r \notin R \Leftrightarrow r \in -R \vee X - r \in -R \Leftrightarrow r \in \overline{-R}$
(iv) similar to (iii) ■

Surprisingly, corresponding statements $(\delta S) - \{x\} = \delta(S - \{x\})$ and $\overline{S}^{\{x\}} = \overline{S^{\{x\}}}$ do not hold for general range spaces – not even for complete spaces (once more an easy counterexample is $S_1(3)$), but using corollary 3.3.4 we are able to prove that these equalities hold for pseudogeometric range spaces. This is the following

3.4.3 Lemma

Let $S = (X, R)$ be a pseudogeometric range space of dimension $d \geq 1$. Then for all $x \in X$

(i) $(\delta S) - \{x\} = \delta(S - \{x\})$
(ii) $\overline{S}^{\{x\}} = \overline{S^{\{x\}}}$

Proof:
(i) $r \in (\delta R) - \{x\} \Leftrightarrow r \in \delta R \vee r \cup \{x\} \in \delta R \Leftrightarrow r, X - r \in R \vee r \cup \{x\}, X - (r \cup \{x\}) \in R$
$\Rightarrow r, X - (r \cup \{x\}) \in R - \{x\} \Leftrightarrow r \in \delta(R - \{x\})$.

So far this is true for any range space; for PG-spaces, however, the " $\Rightarrow$ " becomes a " $\Leftrightarrow$ ": Assume $r, X-(r\cup\{x\}) \in R-\{x\}$. If $r, X-r \in R$ or $r\cup\{x\}, X-(r\cup\{x\}) \in R$, we are done, so the critical cases are (a) $r, X-(r\cup\{x\}) \in R$ or (b) $r\cup\{x\}, X-r \in R$.

Consider case (a): after swapping with r, 3.3.4 shows that $r\cup\{x\} \in R$ or $X-r \in R$, so $r, X-r \in R$ or $r\cup\{x\}, X-(r\cup\{x\}) \in R$ must hold, and this proves the " $\Leftarrow$ ". Case (b) is treated analogously.

(ii) We use duality: $-(\overline{S}^{\{x\}}) = (-\overline{S}) - \{x\} = \delta(-S) - \{x\} = \delta((-S)-\{x\}) = \delta(-(S^{\{x\}})) = -\overline{S^{\{x\}}}$. ■

3.4.4 Remark

The proof shows that $(\delta R)-\{x\} \subset \delta(R-\{x\})$ and $\overline{R^{\{x\}}} \subset \overline{R}^{\{x\}}$ for any range space. This fact will be useful later.

3.5 A Characterizing Maximality Condition

Recall that $\delta\mathcal{C}(H^+)$ contains the labels of the unbounded cells of the arrangement $\mathcal{A}(H^+)$. Clearly, an arrangement of halfspaces always has unbounded cells, so the set of ranges of $\mathcal{C}(H^+)$ is non-empty. We will show that this is the case for all pseudogeometric range spaces of dimension $d \geq 1$, moreover: all PG-spaces of a fixed dimension d and a fixed number of elements n have the same number of ranges in their boundary (from this it follows that they also have the same number of ranges in their closures)

3.5.1 Theorem

Let $S = (X, R)$ be a pseudogeometric range space of dimension $d \geq 1$ with $|X| = n$ elements. Then

(i) $|\delta R| = 2\Phi_{d-1}(n-1)$
(ii) $|\overline{R}| = 2\Phi_d(n-1)$.

Proof:

(i) We proceed by induction; for $d = 1$, the edges from $D^1(S)$ form a path of length n, and each $x \in X$ occurs as a label on the path (2.3.6, 3.1.1). Hence there are complementary ranges r and $X-r$ only on both ends of this path, so there are $2 = 2\Phi_0(n-1)$ ranges in the boundary. Furthermore, if $n = d$, then $|\delta R| = |R| = 2^d = 2\cdot 2^{d-1} = 2\Phi_{d-1}(n-1)$. So the theorem holds for $d = 1$ and $n = d$.

Now assume (i) holds for any PG-space of dimension $d-1$ or dimension d and $n-1$ elements, and consider $S = (X, R)$, pseudogeometric of dimension $d \geq 2$, $|X| = n > d$.

Then $|\delta R| = |(\delta R)-\{x\}| + |(\delta R)^{\{x\}}|$ (1.4.3), and using 3.4.3 we obtain

$$\begin{aligned} |\delta R| &= |\delta(R-\{x\})| + |\delta(R^{\{x\}})| \\ &= 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) \text{ by hypothesis} \\ &= 2\Phi_{d-1}(n-1). \end{aligned}$$

(ii) Evidently $|\overline{R}| = 2|R| - |\delta R|$, which yields $|\overline{R}| = 2\Phi_d(n) - 2\Phi_{d-1}(n-1) = 2\Phi_d(n-1)$.
■

While complete spaces reach the maximal number of ranges that a range space of fixed dimension and a fixed number of elements can have, the pseudogeometric range spaces maximize the number of ranges in the boundary. Moreover, among the complete spaces they are characterized by this property.

First of all we show that any range space of dimension d with n elements has at most $2\Phi_{d-1}(n-1)$ ranges in its boundary. This implies the claimed maximality of the PG-spaces.

3.5.2 Theorem

Let $S = (X, R)$ be a range space of dimension $d \geq 1$ with $n := |X|$ elements. Then $|\delta R| \leq 2\Phi_{d-1}(n-1)$.

Proof:

We proceed by induction; Let S be of dimension 1 and suppose, δS has more than $2\Phi_0(n-1) = 2$ ranges. Then there are at least four ranges in the boundary, and by swapping we can assume that for some $r \subset X$ we have $\{\emptyset, X, r, X - r\} \subset R$. Choose $x \in r$, $y \in X - r$. $\{x, y\}$ is shattered in $\{0, X, r, X - r\}$, so S is not of dimension 1, which is a contradiction. So the theorem holds for $d = 1$. Furthermore it is true for $n = d$, since in this case $|\delta R| = |R| = 2^d = 2\Phi_{d-1}(n-1)$.

Now consider a range space S of dimension $d > 1$ and $n > d$ elements and assume the theorem holds for all spaces of dimension less than d or dimension d and less than n elements.

Remark 3.4.4 shows that $|(\delta R) - \{x\}| \leq |\delta(R - \{x\})|$; furthermore, from 3.4.2 it follows $|(\delta R)^{\{x\}}| = |\delta(R^{\{x\}})|$, so

$$\begin{aligned} |\delta R| &= |(\delta R) - \{x\}| + |(\delta R)^{\{x\}}| \\ &\leq |\delta(R - \{x\})| + |\delta(R^{\{x\}})| \\ &\leq 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) \text{ by hypothesis} \\ &= 2\Phi_{d-1}(n-1). \end{aligned}$$

■

Now we are able to extend Theorem 3.2.1 and give a few more characterizations of pseudogeometric spaces:

3.5.3 Theorem

Let $S = (X, R)$ be a range space of dimension $d \geq 1$ with $|X| = n$ elements. Then the following statements are equivalent:

(i) S is pseudogeometric of dimension d
(ii) S is complete of dimension d and $|\delta R| = 2\Phi_{d-1}(n-1)$
(iii) S is complete of dimension d and $|\overline{R}| = 2\Phi_d(n-1)$
(iv) $|\delta R| = 2\Phi_{d-1}(n-1)$, $|\overline{R}| = 2\Phi_d(n-1)$

Proof:

(i) ⇒ (ii) has already been shown; For the inverse implication we use induction on d. If $d = 1$ and $|\delta R| = 2\Phi_0(n-1) = 2$, then there is a path of length n in $D^1(S)$ connecting the two ranges in the boundary (2.3.5). Since $D^1(S)$ itself has only n edges (2.3.6), it is identical with this path, so S is pseudogeometric.

Now assume $d > 1$; similar as in 3.5.2 we obtain

$$\begin{aligned} 2\Phi_{d-1}(n-1) &= |\delta R| = |(\delta R) - \{x\}| + |(\delta R)^{\{x\}}| \\ &\leq 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) \\ &= 2\Phi_{d-1}(n-1), \end{aligned}$$

so in particular $2\Phi_{d-2}(n-2) = |(\delta R)^{\{x\}}| = |\delta(R^{\{x\}})|$ (3.4.2), which by hypothesis shows that $S^{\{x\}}$ is pseudogeometric of dimension $d - 1 > 0$, for all $x \in X$. This means, S itself is pseudogeometric (3.2.1).

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) We know that $|\overline{R}| = 2|R| - |\delta R|$, and easy calculation shows that two of the quantities determine the third one in the way it was claimed. ■

Finally, we give the dimensions of closure and boundary of a PG-space:

3.5.4 Lemma

Let $S = (X, R)$ be pseudogeometric of dimension d. Then

(i) dim $(\delta S) = d$, if $d \geq 1$

(ii) dim $(\overline{S}) = \begin{cases} d & \text{if } S = \overline{S} (\Leftrightarrow |X| = d) \\ d+1 & \text{otherwise} \end{cases}$, if $d \geq 0$

Proof:

(i) Clearly, δS has dimension at most d. S is pseudogeometric of dimension $d \geq 1$, so $|\delta R| = 2\Phi_{d-1}(|X| - 1)$ (3.5.1). If on the other hand S is of dimension $d' \leq d$, then it follows from theorem 3.5.2 that δS has at most $2\Phi_{d'-1}(|X| - 1)$ ranges, which implies $d = d'$, if $d \geq 1$.

(ii) We show that $\overline{S}$ has dimension at most $d + 1$: If $d = 0$, then $S = (X, \{r\}), r \subset X$, so $\overline{S} = (X, \{r, X - r\})$ is of dimension 1. If $|X| = d$, then $S = \overline{S}$, so in this case $\overline{S}$ is of dimension d. Now let $S = (X, R)$ be pseudogeometric of dimension $d > 0$ with $n := |X| > d$ and assume the assertion is true for all PG-spaces of dimension less than d or dimension d and less than n elements. Choose $x \in X$. Then $S^{\{x\}}, S - \{x\}$ are pseudogeometric of dimensions $d - 1$ and d, repectively, for all $x \in X$, so $\overline{S^{\{x\}}}$ is of dimension at most d and $\overline{S - \{x\}}$ is of dimension at most $d + 1$ by hypothesis.

Let d' denote the dimension of $\overline{S}$. If $n > d'$, let $A \subset X$ be a set of cardinality d' shattered in $\overline{R}$, and consider $y \in X$, such that $y \notin A$. Then A is shattered already in $\overline{R} - \{y\} = \overline{R - \{y\}}$ (3.4.2), so $d' = |A| \leq d + 1$.

If $n = d'$, then each subset of X is a range of $\overline{S}$. This implies that for $x \in X$ each subset of $X - \{x\}$ is a range of $\overline{S}^{\{x\}} = \overline{S^{\{x\}}}$ (3.4.3). $\overline{S^{\{x\}}}$ is of dimension at most d, so $|X - \{x\}|$ can have at most d elements. Thus $d' = n \leq d + 1$.

The desired conclusion is immediate from this: if $S = \overline{S}$, dimension d is obvious. Otherwise $|\overline{R}| > |R|$, and since S is complete of dimension d, $\overline{S}$ must be of higher dimension. ■

Chapter 4

$\overline{PG}$-spaces

4.1 Definition and a Characterizing Maximality Condition

We have seen that the boundary and the closure are important concepts to characterize PG-spaces, and we have already given a geometric interpretation of the boundary in the case where the PG-space is the description of cells of an arrangement of halfspaces. Now we will take a closer look at the closure and give an interpretation in terms of arrangements of hemispheres. To allow a more formal treatment, we give the following definition:

4.1.1 Definition

Let $S = (X, R)$ be a range space of dimension $d \geq 1$. S is called a $\overline{PG}$-space, if there exists a pseudogeometric range space T, $T \neq S$, such that $S = \overline{T}$,
T is called an *underlying* space of S.

The underlying space of a $\overline{PG}$-space of dimension d is not unique, but from lemma 3.5.4 it follows that any underlying space has dimension $d - 1$.

We have seen that complete and pseudogeometric range spaces can be characterized by certain maximality conditions: complete spaces by defintion maximize the total number of ranges, while among the complete spaces exactly the PG-spaces maximize the number of ranges in the boundary. This characterization is very useful in deciding whether a given complete space S is pseudogeometric, because it is a "top-level-criterion", i.e. unlike the defining characterization of PG-spaces (3.1.1) it does not require any knowledge about subspaces of S.

Our first theorem in this section shows that a characterizing maximality condition can also be found for $\overline{PG}$-spaces. As a corollary we obtain the result that the boundary of a PG-space is the closure of some lowerdimensional PG-space. The proof of the theorem will be much clearer with the following geometric interpretation in mind:

Let H be a set of hyperplanes in E^d with corresponding set of halfspaces H^+. The description of cells $\mathcal{C}(H^+)$ is a (geometric) PG-space. We have shown that the boundary of this space contains the labels of the unbounded cells of the arrangement. In order to obtain a geometric interpretation of its closure, we make use of a different representation of the d-dimensional euclidean space: think of E^d as the tangential hyperplane touching the unit sphere $S^d \subset E^{d+1}$ in the north pole. Now E^d can be mapped bijectively to the open northern hemisphere of S^d using central projection.

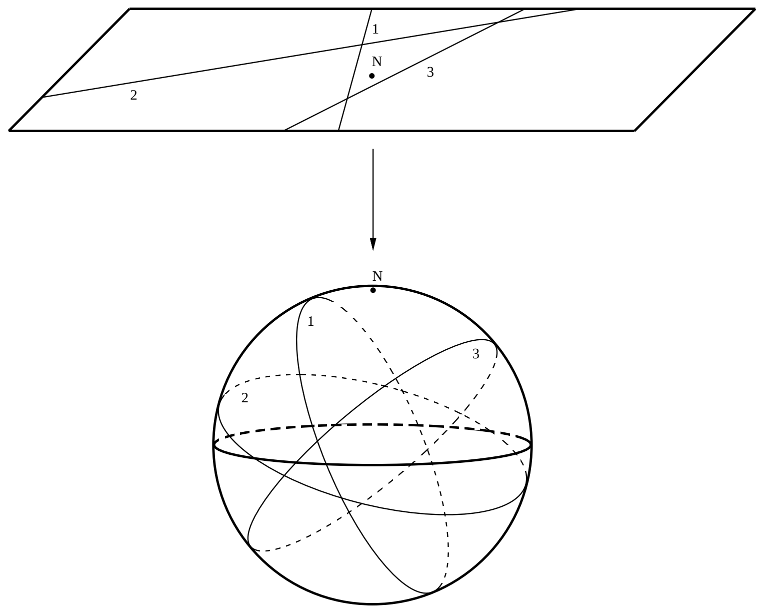

Figure 4.1: mapping an arrangement of hyperplanes to an arrangement of great spheres

This transformation takes a hyperplane h of E^d to a relatively open great halfsphere of dimension $d-1$.

In a unique way this halfsphere can be continued to a full great $(d-1)$-sphere in S^d, so an arrangement of hyperplanes in E^d induces an arrangement of great spheres in S^d; moreover, if we have positive and negative halfspaces associated with the hyperplanes, this information in an obvious way determines positive and negative hemispheres associated with the great spheres, so that we obtain an *arrangement of hemispheres* in S^d (figure 4.1).

We can define the description of cells of this arrangement analogously to the one of a set of halfspaces: each cell c is labelled with the set of great spheres, whose positive hemispheres contain c; assuming the great sphere generated by the hyperplane h is also called h, this description of cells is the closure of the PG-space $\mathcal{C}(H^+)$. This is easy to see, since by extending the arrangement of halfspaces in the open northern hemisphere to an arrangement of hemispheres we generate an antipodal cell with complementary label for each cell in the northern hemisphere.

So if S is a pseudogeometric space arising from a set of halfspaces in E^d, the $\overline{PG}$-space $\overline{S}$ arises from the set of hemispheres in S^d, that can be obtained by mapping the halfspaces to the northern hemisphere of S^d and extending them to full hemispheres.

A $\overline{PG}$-space arising in this way from a geometric space will be called a *geometric $\overline{PG}$-space*.

We come back to arrangements of hemispheres when we introduce oriented matroids.

Now we can establish the anounced theorem: A few notions are necessary: A range space $S = (X, R)$ is called *closed*, if $S = \overline{S} = \delta S$; theorem 3.5.2 says that S of dimension $d \geq 1$ can have at most $2\Phi_{d-1}(|X|-1)$ ranges in this case, so we call S *maximal closed* of dimension d, if S reaches this maximal number of ranges.

4.1.2 Theorem

Let $S = (X, R)$ be a range space, $d \geq 1$. S is maximal closed of dimension d, if and only if S is a $\overline{PG}$-space of dimension d.

Proof:
If S is $\overline{PG}$ of dimension d, then S is the closure of a $(d-1)$-dimensional PG-space T with $S \neq T$. $S = \overline{T}$ has $2\Phi_{d-1}(|X|-1)$ ranges (3.5.1 (ii)), so S is maximal closed of dimension d.

To obtain the inverse implication, we proceed by induction on d. If S is maximal closed of dimension 1, this means $S = (X, \{r, X-r\}), r \subset X$. Now $T = (X, \{r\})$ is of dimension 0 and hence pseudogeometric with $S = \overline{T}$.

Now suppose $d > 1$ and let $S = (X, R)$ be maximal closed of dimension d, $n := |X|$. Consider $x \in X$. It is easy to see that $S^{\{x\}}$ and $S - \{x\}$ are closed of dimensions at most $d-1$ and d, respectively, so we have

$$\begin{aligned} 2\Phi_{d-1}(n-1) = |R| &= |R^{\{x\}}| + |R - \{x\}| \\ &\leq 2\Phi_{d-2}(n-2) + 2\Phi_{d-1}(n-2) \\ &= 2\Phi_{d-1}(n-1). \end{aligned}$$

This especially shows $|R^{\{x\}}| = 2\Phi_{d-2}(n-2)$, so $S^{\{x\}}$ is maximal closed of dimension $d-1$. By hypothesis there exists a PG-space $S_x = (X - \{x\}, R_x)$ of dimension $d-2$ with $S^{\{x\}} = \overline{S_x}$.

Let x^+ denote the set of all ranges of S that contain x, i.e. $x^+ := \{r \in R \mid x \in r\}$. Now we construct the range space

$$T = (X, R_x \cup x^+)$$

and we claim that T is pseudogeometric of dimension $d-1$ with closure S, which proves the theorem.

Obviously, $S = \overline{T}$. The number of ranges of T is

$$|R_x| + |x^+| = \Phi_{d-2}(n-1) + \frac{1}{2} 2\Phi_{d-1}(n-1) = \Phi_{d-1}(n).$$

Furthermore, T has $2|R_x| = 2\Phi_{d-2}(n-1)$ ranges in the boundary. If we can show that T is of dimension $d-1$, then theorem 3.5.3 ensures that T is pseudogeometric.

From the number of ranges of T it is clear that T is of dimension at least $d-1$; to prove that it is of dimension at most $d-1$, we consider $A \subset X$, such that A is shattered in $R_x \cup x^+$. We distinguish two cases:

(a) $x \notin A$:
Since $R_x \subset R^{\{x\}}$, we have $r \cup \{x\} \in x^+$ for all $r \in R_x$, and because of $A \cap r = A \cap (r \cup \{x\})$ we know that A is already shattered in x^+. In the remarks to definition 2.3.2 we have shown that the property of being shattered is invariant under swapping, so A is also shattered in $x^- := R - x^+ = x^+ \triangle X$. This implies that $A \cup \{x\}$ is shattered in R: x^+ generates all subsets that contain x, while the subsets not containing x are obtained by intersecting $A \cup \{x\}$ with the ranges from x^-. S is of dimension d, so $|A| \leq d-1$.

(b) $x \in A$:
By intersecting A with the ranges from x^+ we only get subsets of A that contain x. This means, $A - \{x\}$ is shattered in R_x. S_x is of dimension $d-2$, so again $|A| \leq d-1$.

We have shown that if A is shattered in $R_x \cup x^+$, then $|A| \leq d-1$, so T is of dimension $d-1$, which completes the proof. ■

4.1.3 Corollary

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$. Then δS is $\overline{PG}$ of dimension d.

Proof:

From lemma 3.5.4 we know that δS is of dimension d. Moreover, δS has $2\Phi_{d-1}(|X| - 1)$ ranges (3.5.1), so it is maximal closed of dimension d and hence $\overline{PG}$ by the theorem. ∎

4.1.4 Remark

The proof of the theorem shows how to obtain an underlying space of a $\overline{PG}$-space S from the underlying space of some $S^{\{x\}}$: If $S_x = (X - \{x\}, R_x)$ is an underlying PG-space of the $\overline{PG}$-space $S^{\{x\}}$, then $T = (X, x^+ \cup R_x)$ is an underlying space of S. Because of symmetry this especially shows: Given a $\overline{PG}$-space (X, R) and some $x \in X$, there is an underlying PG-space containing x^+ and another one containing $x^- := R - x^+$. x^+ and x^- are called the *halfspaces* of x in R.

4.2 More Characterizations and Duality

The following theorem that establishes a few equivalent chacterizations of $\overline{PG}$-spaces is of a kind that should be familiar by now:

4.2.1 Theorem

Let $S = (X, R)$ be closed of dimension $d \geq 2, |X| > d$. Then the following statements are equivalent:

(i) S is a $\overline{PG}$-space of dimension d
(ii) S is maximal closed of dimension d
(iii) $\forall x \in X : S^{\{x\}}$ is $\overline{PG}$ of dimension $d - 1$, $S - \{x\}$ is $\overline{PG}$ of dimension d.
(iv) $\forall x \in X : S^{\{x\}}$ is $\overline{PG}$ of dimension $d - 1$
(v) $\exists x \in X : S^{\{x\}}$ is $\overline{PG}$ of dimension $d - 1$, $S - \{x\}$ is $\overline{PG}$ of dimension d
(vi) $|R^A| = 2$ for all $A \subset X, |A| = d - 1$.

Proof:

(i) $\Leftrightarrow$ (ii) is theorem 4.1.2. Once we have this first characterization, we can use the proof of theorem 2.2.1 (which is the equivalent of this theorem for complete spaces) to obtain characterizations (iii) to (vi) by simply changing ”complete” to ”maximal closed” and ”$\Phi_d(n)$” to ”$2\Phi_{d-1}(n-1)$”. We need only to observe that if S is closed, then for all $x \in X, S^{\{x\}}$ and $S - \{x\}$ are also closed.

Note that the theorem is stated for $d \geq 2$, because a $\overline{PG}$-space must be of dimension at least 1 by definition. ∎

We conclude this chapter with a duality theorem for $\overline{PG}$-spaces that slightly differs from the ones given for complete and PG-spaces with respect to dimension:

4.2.2 Theorem

Let $S = (X, R)$ be $\overline{PG}$, $n := |X| >$dim$(S)d$. Then $-S$ is $\overline{PG}$ of dimension $n - d$.

Proof:

Let T be an underlying PG-space of S. T is of dimension $d - 1$, so $-T$ is PG of dimension $n - d \geq 1$ by the duality theorem for pseudogeometric spaces 3.2.2.

Furthermore, $-S = -\overline{T} = \delta(-T)$, so $-S$ is the boundary of $-T$ and hence $\overline{PG}$ of dimension $n - d$ by corollary 4.1.3. ■

Chapter 5

Oriented Matroids and $\overline{PG}$-spaces

5.1 Introduction

We have started this paper with the consideration of arrangements of halfspaces and the description of cells of such an arrangement. In order to investigate the description of cells we have introduced the notion of range spaces.

A simple arrangement of halfspaces determines a complete range space, but the converse was easily seen to be false. This led to the concept of pseudogeometric spaces, a subclass of the complete spaces defined by an additional property of arrangements of halfspaces that is not satisfied by complete spaces in general.

While arrangements of halfspaces determine PG-spaces, arrangements of hemispheres define $\overline{PG}$-spaces, which is the third class of range spaces we have introduced so far (like in the other chapters, we speak of simple arrangements here, but do not always mention it explicitly).

Now the same question we have asked in the context of complete spaces comes up again here: is it true that any PG-space is the description of cells of an arrangement of halfspaces, and that any $\overline{PG}$-space determines an arrangement of hemispheres?

Once more we give a negative answer – nevertheless, PG- and $\overline{PG}$-spaces seem to be the most appropriate approach to arrangements: in this chapter we show that PG- and $\overline{PG}$-spaces correspond to *arrangements of pseudohalfspaces* and *arrangements of pseudohemispheres*, respectively.

An exact definition of these objects is postponed to the end of this chapter – intuitively, such arrangements consist of ”distorted” halfspaces or hemispheres, which intersect in the same way as straight halfspaces or hemispheres do.

From a topological point of view there is no difference – the notion of ”straightness” that distinguishes halfspaces and hemispheres from pseudohalfspaces and pseudohemispheres is purely geometric.

A well-studied case is the 2-dimensional one, and there is a lot of literature concerning arrangements of pseudolines (for a survey of the subject up to 1972 see [Gr]; new results can be found in [GP80b], [GP82], [GP85]).

A remarkable result is the existence of arrangements of pseudolines which are not *stretchable*, i.e. which are not equivalent to any straight arrangement in a certain sense [Ed], [Ri] – translated to the terminology of range spaces this means: there exist pseudogeometric spaces which are not the description of cells of any arrangement of halfspaces. An example of a simple, non-stretchable arrangement of pseudolines can be found in [Ri].

It is common to all known combinatorial structures encoding arrangements, that they cannot "recognize" straight arrangements, and this especially holds for a very powerful and general structure, namely *oriented matroids*.

Folkman and Lawrence [FL] have shown that an arrangement of pseudohemispheres determines an oriented matroid; moreover, every oriented matroid comes from such an arrangement. Later A.Mandel [Ma] gave an alternative proof of this result.

We show that $\overline{PG}$-spaces are exactly simple oriented matroids, and this implies the correspondences we have anounced above.

We proceed as follows: First we introduce oriented matroids using the terminology of A.Mandel and prove some of their basic properties. Then we establish the relation to $\overline{PG}$-spaces. Finally, we give definitions of arrangements of pseudohalfspaces and -hemispheres and show how PG- and $\overline{PG}$-spaces are related to these structures.

We remark that although the axioms for an oriented matroid used by Mandel are different from the ones used in [FL] and [BL], both axiomatizations can be shown to be equivalent. For our purposes Mandel's terminology is more convenient, so we have decided to use his approach instead of the one from the standard papers on the subject.

5.2 Definition and Basics on Oriented Matroids

Let X be a finite set. A *signed vector* on X is a mapping $F : X \to \{0, +1, -1\}$. $F(x)$ will be denoted by F_x. The *support* of F is defined as the set $\underline{F} := \{x \in X \mid F_x \neq 0\}$. $F^0 := X - \underline{F}$ is called the *zeroset* of F.

In the context of signed vectors 0 is the vector satisfying $0_x = 0$ for all $x \in X$. $-F$ is defined by $(-F)_x := -(F_x)$. If F and G are signed vectors, then we declare the *product* $F \cdot G$ by $(F \cdot G)_x := F_x$, if $F_x \neq 0, G_x$ otherwise. For $Y \subset X$, $F|_Y$ denotes the restriction of F to Y.

Finally we say $x \in X$ *separates* F and G, iff $F_x = -G_x \neq 0$.

Note that $(F \cdot G)_x = (G \cdot F)_x$ iff x does not separate F and G.

5.2.1 Definition

Let X be a finite set, $\mathcal{V}$ a set of signed vectors on X satisfying

(OM1) $0 \in \mathcal{V}$

(OM2) $F \in \mathcal{V}$ implies $-F \in \mathcal{V}$

(OM3) If $F, G \in \mathcal{V}$, then $F \cdot G \in \mathcal{V}$

(OM4) If $F, G \in \mathcal{V}$, such that x separates F and G, then there is $H \in \mathcal{V}$, such that $H_x = 0$ and $H_y = (F \cdot G)_y = (G \cdot F)_y$ for each y not separating F and G (we say H results from the *elimination* of x between F and G).

Then the pair $(X, \mathcal{V})$ is called an *oriented matroid*.

The axioms are inspired by properties of vector spaces:

5.2.2 Example

For $x \in E^d$, define a signed vector $\sigma(x)$ on $D := \{1, ..., d\}$ by $\sigma(x)_i := +1, 0$ or -1 depending on whether $x_i > 0, = 0$ or < 0. Let V be a linear subspace of E^d. Then the pair $(D, \{\sigma(x) \mid x \in V\})$ is an oriented matroid.

Proof:

Properties (OM1) and (OM2) are obviously satisfied. To prove the other two, consider $X = \sigma(x), Y = \sigma(y)$ and let g be the relatively open line segment connecting x and y. If $\underline{X}$ is the support of X, there is an open neighbourhood U of x, such that $\forall u \in U, m \in \underline{X} : \sigma(u)_m = \sigma(x)_m$. Let z be any element in $U \cap g$. Then $Z := \sigma(z)$ is the product of X and Y.

If $i \in D$ separates X and Y, then g intersects the hyperplane $x_i = 0$ in a unique point z, and $Z := \sigma(z)$ results from the elimination of i between X and Y. ■

Similar argumentation works for an arrangement of hemispheres:

5.2.3 Example

Let S be a set of great $(d-1)$-spheres s with positive and negative hemispheres s^+ and s^- in the unit sphere S^d. For $x \in S^d$ the signed vector $\sigma(x)$ is defined by $\sigma(x)_s := +1, 0$ or -1, depending on whether x is contained in s^+, s or s^-. Then $(S, \{0\} \cup \{\sigma(x) \mid x \in S^d\}$ is an oriented matroid.

Proof:

Again the first two properties are obviously satisfied; for $X = \sigma(x), Y = \sigma(y)$ we now let g be the relatively open shortest circular arc joining x and y. The product of X and Y is obtained by defining a neighbourhood U as above and choosing a point in $U \cap g$; if the great sphere s separates X and Y, the signed vector Z required by (OM4) is $\sigma(z)$, where z is the intersection point of g and s. ■

An oriented matroid arising in this way from an arrangement of hemispheres is called *linear*. It is clear that the signed vectors of this linear oriented matroid correspond to the faces of the arrangement.

If f and g are faces of the arrangement with corresponding signed vectors F and G, then f is a subface of g, iff F arises from G by changing some components of G to zero.

This observation motivates us to introduce a partial order on signed vectors on a set X: We define $F \leq G :\Leftrightarrow \forall x \in X : F_x = 0$ or $F_x = G_x$, so in the case where we have a linear oriented matroid, this order is isomorphic to the usual incidence order among the faces of the corresponding arrangement of hemispheres.

This leads to general terminology: If $M = (X, \mathcal{V})$ is an oriented matroid, then the elements of $\mathcal{V}$ are called *faces* of M. If $F \leq G$, F is a *subface* of G. The subface relation is *proper*, iff $F < G$, i.e. $F \leq G$ and $F \neq G$.

The maximal vectors in the order $\leq$ are called *topes*, while we refer to the minimal nonzero vectors as *vertices* of M.

Note that $F \leq F \cdot G$, for all $F, G \in \mathcal{V}$, so F is always a subface of its product with any other face.

The vertices are not only the basic vectors in the order $\leq$, they also completely determine the oriented matroid. This follows from the fact that the product of faces is a face together with the following

5.2.4 Theorem

Let $M = (X, \mathcal{V})$ be an oriented matroid. Every nonzero face F of M is the product of its vertices (i.e. the vertices dominated by F in the order $\leq$).

Proof:

Note first, that the order in which the vertices are multiplied is irrevelant in this case, since subfaces of F are never separated by an element, in which case the product is commutative.

Now assume there is a face F contradicting the theorem; choose F minimal with respect to this property. Let B be a vertex of F. Since B is nonzero, F and $-B$ are separated by at least one element x. Eliminating x between F and $-B$ yields a face $G \neq F$ with $G_y = (F \cdot (-B))_y = F_y$ for all $y \in B^0$. Choose $G \neq F$ with this property, such that G is separated from F by as few elements as possible. Then no element separates F and G, otherwise eliminating this element between F and G yields a face $G' \neq F$ that is separated from F by fewer elements than G, while maintaining $G'_y = F_y$ for all elements in the zeroset of B, contradicting the choice of G.

But now we know that $G < F$: $F_y = 0$ implies $y \in B^0$, so $G_y = 0$, while $F_y \neq 0$ implies $G_y = F_y$ or 0, since y does not separate F and G.

F cannot be a vertex itself, so there is $z \in B^0$ with $G_z = F_z \neq 0$, which shows that G is nonzero. Due to the minimality of F, G is the product of its vertices. Furthermore, $F = B \cdot G$, and since the vertices of G are also vertices of F, F is the product of some of its vertices, a contradiction. ■

We now come back to the consideration of the order $\leq$; The pair $(\mathcal{V}, \leq)$ is called the *complex* of M; it is a poset which is in a certain sense well-behaved:

5.2.5 Proposition

Let $(\mathcal{V}, \leq)$ be the complex of an oriented matroid. Then the following holds:

(i) For any $F \in \mathcal{V}$, all saturated $\leq$-chains from 0 to F have the same length (and this length is called the *rank* of F, denoted by $r(F)$. Note that $r(0) = 0$)

(ii) All topes have the same rank

A poset with property (i) is said to satisfy the *Jordan - Dedekind chain condition*; if in addition to (i), (ii) also holds, the poset is called *pure-dimensional*.

5.2.6 Definition

Let M be an oriented matroid, F a face of M. The *dimension* of F is defined as $d(F) := r(F) - 1$. The dimension of M is $d(M) := d(T)$, where T is a tope of M (so $d(M)$ is the length of a chain from a vertex to a tope).

Following this definition, the linear oriented matroid from example 5.2.3 has the same dimension as its generating arrangement.

The proof that the complex of an oriented matroid is pure-dimensional can be found in A.Mandel's thesis [Ma]; it relies on properties of the underlying matroid, which exists for any oriented matroid:

5.2.7 Definition

Let X be a finite set, $\mathcal{S}$ a collection of subsets of X. The pair $(X, \mathcal{S})$ is called a *matroid* on X, iff

(M1) $X \in \mathcal{S}$
(M2) $A, B \in \mathcal{S}$ implies $A \cap B \in \mathcal{S}$
(M3) If $A \in \mathcal{S}$ with $x, y \notin A$ and there exists $B \in \mathcal{S}$ containing $A \cup \{x\}$ but not $\{y\}$, then there exists $C \in \mathcal{S}$ containg $A \cup \{y\}$ but not $\{x\}$

5.2.8 Lemma

For an oriented matroid $M = (X, \mathcal{V})$, let $\mathcal{V}^0$ denote the collection of all zerosets of faces of M.

$(X, \mathcal{V}^0)$ is a matroid on X. It is called the *underlying matroid* of M.

Proof:

Since $0 \in \mathcal{V}$, (M1) is clear. To see that (M2) holds, it suffices to observe that if F and G are faces of M with zerosets F^0 and G^0, then their product $F \cdot G$ has zeroset $F^0 \cap G^0$.

It remains to prove (M3): Let F, G be faces, $x, y \in X$, such that $x, y \notin F^0$ and $F^0 \cup \{x\} \subset G^0 \not\ni y$. Clearly, F_x, F_y and G_y are nonzero, while $G_x = 0$. Assume $F_y = -G_y$ (which can be achieved by substituting G with $-G$, if necessary) and let H be a face resulting from elimination of y between F and G.

Since $F_z = G_z = 0 \; \forall z \in F^0$, this is also true for H. Furthermore, H satisfies $H_y = 0$, and (since x does not separate F and G), $H_x = (F \cdot G)_x \neq 0$. This means H^0 contains $F^0 \cup \{y\}$, but not $\{x\}$, as required. ■

5.3 Simplicity

At the beginning of this chapter we have said that there is a correspondence between $\overline{PG}$-spaces and *simple* oriented matroids, so a definition of simplicity is what we give next.

This definition will be quite clear, when we figure out what it means in the case of linear oriented matroids (example 5.2.3).

Recall that an arrangement of (at least d) halfspaces in E^d (hemispheres in S^d resp.) is simple, iff any d of the underlying hyperplanes (great spheres resp.) meet in a common vertex (0-sphere resp.), and any $d + 1$ have empty intersection (section 1.3). There also exists an equivalent, dimension-independent characterization for arrangements of great spheres, which uses the fact that great spheres cannot be parallel: a *flat* of the arrangement is the intersection of great spheres; now the arrangement is simple, iff no nonempty flat contained in a great sphere is the intersection of some of the remaining great spheres.

A similar notion can be introduced for oriented matroids:

5.3.1 Definition

Let $M = (X, \mathcal{V})$ be an oriented matroid. For $Y \subset X$, define

$$\mathcal{V}_Y := \{F \in \mathcal{V} \mid F_y = 0 \; \forall y \in Y\}.$$

If Y is the zero set of some face of M, $\mathcal{V}_Y$ is called a *flat* of M. M is *simple*, iff for any $x \in X$, no nonzero flat contained in $\mathcal{V}_{\{x\}}$ is the intersection of the members of a (possibly empty) subset of $\{\mathcal{V}_{\{y\}} \mid y \in X - \{x\}\}$.

So a linear oriented matroid is simple, iff its generating arrangement of hemispheres is simple. (Mandel uses the term *general position* instead of *simple* and reserves the word *simple* to describe another property of oriented matroids, which we don't need here).

As it turns out, $\mathcal{V}_Y$ is a flat for any $Y \subset X$, which simplifies the following considerations. To prove this we have to find for a given Y a face F, such that $\mathcal{V}_Y = \mathcal{V}_{F^0}$. This is done by defining F to be the product (in some order) of all members of $\mathcal{V}_Y$. Clearly, $F \in \mathcal{V}_Y$, so $\mathcal{V}_{F^0} \subset \mathcal{V}_Y$. On the other hand, if $G \in \mathcal{V}_Y$, then $F^0 \subset G^0$, since $F^0 = \bigcap_{H \in \mathcal{V}_Y} H^0$, so $G \in \mathcal{V}_{F^0}$, which means $\mathcal{V}_Y \subset \mathcal{V}_{F^0}$; hence $\mathcal{V}_Y = \mathcal{V}_{F^0}$.

Simple oriented matroids have a very regular structure, and in the following we give two equivalent characterizations of simplicity, which are less abstract then the definition using flats.

5.3.2 Theorem

Let $M = (X, \mathcal{V})$ be an oriented matroid of dimension $d \geq 0$. M is simple, iff

(i) The topes of M have support set X.

(ii) If F, G are in $\mathcal{V}$, such that $0 \neq F < G$ and there is no $H \in \mathcal{V}$ with $F < H < G$ (we say, G *covers* F), then F results from G by changing exactly one component to zero.

Proof:

Let M be simple; we show that (i) and (ii) hold:

(i) Assume $T_x = 0$ for some tope T. Then we know that $F_x = 0$ for all faces F (otherwise $T < T \cdot F$ for some $F \in \mathcal{V}$, which is a contradiction to T being a tope). Since $\mathcal{V}_{\{x\}}$ contains $T \neq 0$, we have that $\mathcal{V}_{\{x\}}$ is a nonzero flat with $\mathcal{V}_{\{x\}} = \mathcal{V} = \bigcap_{y \in \emptyset} \mathcal{V}_{\{y\}}$, contradicting the simplicity of M.

(ii) Assume on the contrary that G covers F, and $|F^0 - G^0| \geq 2$ for the corresponding zerosets. Choose $x \in F^0 - G^0$. Then $H_x = 0$ for every face H in the flat $\mathcal{V}_{F^0 - \{x\}}$ – otherwise for some H either $F \cdot H$ or $F \cdot (-H)$ would lie properly between F and G in the order $\leq$, which cannot happen.

But then $\mathcal{V}_{F^0}$ is a nonzero flat contained in $\mathcal{V}_{\{x\}}$ with $\mathcal{V}_{F^0} = \mathcal{V}_{F^0 - \{x\}} = \bigcap_{y \in F^0 - \{x\}} \mathcal{V}_{\{y\}}$, so M is non-simple.

Now let M be non-simple; it has to be shown that M cannot satisfy both (i) and (ii).

There exists $x \in X$ and a subset $Y \not\ni x$ of X, such that a nonzero flat contained in $\mathcal{V}_{\{x\}}$ is equal to $\bigcap_{y \in Y} \mathcal{V}_{\{y\}} = \mathcal{V}_Y$. Fix x and choose a minimal Y having this property.

If $\mathcal{V}_Y = \mathcal{V}$, then $\mathcal{V} \subset \mathcal{V}_{\{x\}}$, so $F_x = 0$ for all faces of M, which means that (i) does not hold.

Otherwise we know that $Y \neq \emptyset$; $\mathcal{V}_Y$ is nonzero, so let $0 \neq F$ be a maximal face of $\mathcal{V}_Y$. Clearly, $F_x = 0$. If F is a tope, (i) is violated, so assume there exists a face G covering F. From the maximality of F it follows that $G_z \neq 0$ for some $z \in Y$. If $G_{z'} \neq 0$ for some other $z \neq z' \in Y$, (ii) does not hold, so we may assume $G \in \mathcal{V}_{Y - \{z\}}$.

We claim that this finally implies $G_x \neq 0$. To prove this, we assume on the contrary that $G_x = 0$. From the minimality of Y we know that $\mathcal{V}_{Y - \{z\}} \not\subset \mathcal{V}_{\{x\}}$, so there exists a face $H \in \mathcal{V}_{Y - \{z\}}$ with $H_x \neq 0$. By replacing H with $G \cdot H$ we can assume that H dominates G. Now we know that $x, z \notin H^0$ and G^0 contains $H^0 \cup \{x\}$ but not $\{z\}$. Hence there exists a face E such that E^0 contains $H^0 \cup \{z\}$ but not $\{x\}$. Since $Y \subset H^0 \cup \{z\} \subset E^0$, we have $E \in \mathcal{V}_Y$, which is a contradiction to $x \notin E^0$.

It follows that $G_x \neq 0$, which shows that F and G violate (ii), and this completes the proof. ■

The next characterization is strongly related to the original definiton of simplicity for arrangements:

5.3.3 Corollary

Let $M = (X, \mathcal{V})$ be an oriented matroid of dimension $d \geq 0$. M is simple, iff all vertices have zerosets of cardinality d.

Proof:
The vertices are the minimal nonzero vectors of $\mathcal{V}$, so they are exactly the faces of dimension 0. If M is simple of dimension d, the theorem implies that a face of dimension $k \geq 0$ has zeroset of cardinality $d - k$, which proves one implication. If M is non-simple, then there is some length-d-chain from a vertex to a tope, where the tope has not full support set or two consecutive faces in the chain differ by more than one element in their zerosets. In both cases, the vertex in question has more than d elements in its zeroset. ■

5.3.4 Theorem

A simple oriented matroid is completely determined by the set of topes.

Proof:
Let $M = (X, \mathcal{V})$ be a simple oriented matroid of dimension d. If $d = -1$, i.e. $\mathcal{V} = \{0\}$, there is nothing to prove, so assume $d \geq 0$.

We will show that a signed vector F with zeroset of cardinality k is in $\mathcal{V}$ if and only if all the 2^k signed vectors $G \geq F$ with support set X are in $\mathcal{V}$. Clearly, this implies the theorem.

We proceed by induction on k, noting that for $k = 0$ the theorem holds. For $k > 0$ let F be a face of M with $|F^0| = k$. F is not a tope, so there is $G \in \mathcal{V}$ covering F with $|G^0| = k - 1$. Furthermore, $H := F \cdot (-G) \in \mathcal{V}$ with $H^0 = G^0$; If x denotes the unique element in $F^0 - G^0$, we have $G_x = -H_x \neq 0$. By hypothesis, all 2^{k-1} full-support-vectors dominating G are in $\mathcal{V}$, and the same holds for H. Since x separates G and H, no signed vector can dominate both G and H, which means that there are $2 \cdot 2^{k-1} = 2^k$ full-support-vectors in $\mathcal{V}$ dominating F.

Conversely, assume that for given F with zeroset of cardinality k all 2^k full-support-vectors are in $\mathcal{V}$. Choose $x \in F^0$ and let G and H be signed vectors obtained from F by switching F_x to $+1$ and -1 respectively. Clearly there are in each case 2^{k-1} full-support-vectors in $\mathcal{V}$ dominating G and H, so G and H are in $\mathcal{V}$ by hypothesis. Since $G_y = H_y$ for all $y \neq x$, eliminating x between G and H yields $F \in \mathcal{V}$. ■

We have seen that a non-simple oriented matroid can be obtained from a non-simple arrangement of hemispheres, but there are other instances of non-simplicity that do not occur in a linear oriented matroid as defined up to now. However, by slightly extending the definition of arrangements we can produce two more kinds of degeneracies, namely *loops* and *coincident elements*. We don't need this in the sequel, but we want to point out that simplicity is a stronger constraint for oriented matroids than it is for arrangements as we have defined them here.

An arrangement of hemispheres has an underlying arrangement of great spheres. Assume that we allow the great spheres to occur more than once in the arrangement, i.e. the set of great spheres becomes a multiset. An arrangement of hemispheres based on this multiset produces an oriented matroid $M = (X, \mathcal{V})$, which can contain *coincident elements*: $x, y \in X$ are called coincident, iff either $\forall F \in \mathcal{V} : F_x = F_y$ or $\forall F \in \mathcal{V} : F_x = -F_y$. Clearly, $\mathcal{V}_{\{x\}} = \mathcal{V}_{\{y\}}$ in this case, and if these flats are nonzero, M is non-simple.

If we allow the arrangement to contain degenerate great spheres equal to the whole sphere which carries the arrangement (the hemispheres of such great spheres are empty), then a corresponding oriented matroid M may contain *loops*, which are elements $x \in X$, such that $F_x = 0$ for all $F \in \mathcal{V}$. From 5.3.2 (i) it follows that M is non-simple also in this case.

5.4 Representing $\overline{PG}$-spaces as Simple Oriented Matroids

To prepare the proof of the correspondence between $\overline{PG}$-spaces and simple oriented matroids we now introduce *minors* of an oriented matroid M, which play the same role for M as the subspaces S^Y and $S - Y$ do for a range space S.

5.4.1 Lemma

Let $M = (X, \mathcal{V})$ be an oriented matroid, $Y \subset X$. The pairs

$$M \text{ ctr } Y := (X - Y, \mathcal{V} \text{ ctr } Y),$$

where $\mathcal{V} \text{ ctr } Y := \{F|_{X-Y} \mid F \in \mathcal{V}, F_y = 0 \; \forall y \in Y\}$, and

$$M \text{ del } Y := (X - Y, \mathcal{V} \text{ del } Y),$$

where $\mathcal{V} \text{ del } Y := \{F|_{X-Y} \mid F \in \mathcal{V}\}$, are again oriented matroids. They are said to arise from M by *contracting* resp. *deleting* Y.

Contractions and deletions are common operations in the theory of oriented matroids, and it is a very simple straightforward exercise to check that contracting or deleting an arbitrary subset yields an oriented matroid – we leave this to the reader.

It is not surprising that for a linear oriented matroid these minors have an interpretation in terms of the generating arrangement. This interpretation equals the one for subspaces of a geometric range space: Given the arrangement of hemispheres corresponding to M, M del Y occurs after deleting the hemispheres spanned by the great spheres in Y, while M ctr Y corresponds to the lower-dimensional subarrangement in the sphere that is the intersection of the great spheres in Y. Note that while M itself does not have coincident elements, in case of a non-simple arrangement a certain M ctr Y might; so in some settings it may be useful to extend the definition of arrangements in the way we have shown above. However, since we are only interested in the simple case anyway, we do not consider this any further.

5.4.2 Theorem

Let $M = (X, \mathcal{V})$ be simple of dimension $d \geq 0$, $|X| > d + 1$, $x \in X$. Then

(i) M ctr $\{x\}$ is simple of dimension $d - 1$
(ii) M del $\{x\}$ is simple of dimension d

Proof:

(i) Dimension $d-1$ follows, if we can show that a maximal face F in $\mathcal{V}_{\{x\}}$ is covered by a tope of M. Let F be such a maximal face. Since F does not have support set X, it is not a tope, so there exists $G > F$. Assume that G is not a tope, i.e. there is $H > G$. F is maximal in $\mathcal{V}_{\{x\}}$, so $x \notin G^0$. On the other hand there must be some element $y \in G^0 - H^0$. This means, G^0 contains $H^0 \cup \{y\}$ but not $\{x\}$, so from 5.2.8 we know that there is a zeroset E^0 containing $H^0 \cup \{x\}$ but not $\{y\}$. Since $y \in F^0$, this implies $F < F \cdot E \in \mathcal{V}_{\{x\}}$, contradicting the maximality of F.

Simplicity immediately follows from 5.3.3: the vertices of M ctr $\{x\}$ are exactly the signed vectors $F|_{X-\{x\}}$, where F is a vertex in $\mathcal{V}_{\{x\}}$. M is simple, so $|F^0| = d$ for all these, which implies $|F|^0_{X-\{x\}}| = d-1$.

(ii) Let T be a tope of M. Since $T_x \neq 0$ and T is the product of its vertices, there is a vertex $F \leq T$ of M with $F_x \neq 0$. F has zeroset of cardinality d, and since $|X| > d+1$, there is another element $z \neq x$ with $F_z \neq 0$. Consider a length-d-chain from F to T. Deleting x maps this chain to a length-d-chain from $F|_{X-\{x\}} \neq 0$ to $T|_{X-\{x\}}$ in M del $\{x\}$, so M del $\{x\}$ is of dimension d.

Simplicity follows because M del $\{x\}$ again satisfies properties 5.3.2 (i) and (ii). ■

Recall how a simple d-dimensional arrangement of hemispheres determines a $d+1$- dimensional $\overline{PG}$-space: take a cell of the arrangement and label it with the set of great spheres whose hemispheres contain the cell. The set of all such labels defines the range space. In a very similar fashion the oriented matroid of dimension d determined by the arrangement is obtained: take a face of the arrangement and label it with a signed vector on the set of great spheres, where the component corresponding to a certain great sphere is $0, +1$ or -1 depending on whether the face is contained in the great sphere, in its positive or its negative hemisphere.

The difference is that while we consider only the cells, i.e. the full-dimensional faces to define the range space, all faces are needed to get the oriented matroid. However, a simple arrangement yields a simple oriented matroid, which is determined by its set of topes, as we have shown. Since the topes are exactly the labels of the cells of the arrangement, we conclude that a simple linear oriented matroid is uniquely defined by the cells of its generating arrangement.

This observation immediately yields a correspondence between simple linear oriented matroids and geometric $\overline{PG}$-spaces on a set X: If $\Psi : \{+1,-1\}^X \to 2^X$ is the canonical bijection defined by $\Psi(F) := \{x \in X \mid F_x = +1\}$, then $\mathcal{T}$ is the set of topes of a unique simple linear oriented matroid of dimension $d \geq 0$ on X if and only if $(X, \Psi(\mathcal{T}))$ is a geometric $\overline{PG}$-space of dimension $d+1$.

What we want to show is that this statement holds even if we leave out the words ”geometric” and ”linear”, i.e. if we do not have a generating, well-behaved arrangement of hemispheres indirectly relating the range space to the oriented matroid.

The following two theorems establish a direct correspondence; they are based on the equivalence theorem 4.2.1 for $\overline{PG}$- spaces:

5.4.3 Theorem

Let $M = (X, \mathcal{V})$ be a simple oriented matroid of dimension $d \geq 0$ with set of topes $\mathcal{T}$. Then $(X, \Psi(\mathcal{T}))$ is a $\overline{PG}$-space of dimension $d+1$.

Proof:

We proceed by induction on d; if $d = 0$, then M has exactly two topes T and $-T$. To see this, assume there exists a third one U. Because of simplicity all of them have full support set. Since $U \neq T, -T$, there is x separating U and T and y not separating U and T. Hence elimination of x between U and T yields a nonzero face which has not full support set, so it is not a tope. Since $d = 0$ implies that every nonzero vector is a tope, this is a contradiction. So if $\mathcal{T} = \{T, -T\}$, then $\Psi(\mathcal{T}) = \{r, X - r\}$ for some $r \subset X$, so $(X, \Psi(\mathcal{T}))$ is $\overline{PG}$ of dimension 1.

Now assume, M is of dimension $d > 0$. Then M ctr $\{x\}$ is simple of dimension $d - 1$ for all $x \in X$ (5.4.2); let $\mathcal{T}^{\{x\}}$ denote the set of topes of M ctr $\{x\}$. By hypothesis $(X - \{x\}, \Psi(\mathcal{T}^{\{x\}}))$ is a $\overline{PG}$-space of dimension d for all $x \in X$ (note that we are a bit sloppy about Ψ by assuming that it automatically adapts to the domain of its argument).

Furthermore it is an easy observation that if $\Psi(\mathcal{T}) = R$, then $\Psi(\mathcal{T}^{\{x\}}) = R^{\{x\}}$. This follows from the fact that the topes of M ctr $\{x\}$ correspond to the maximal faces of $\mathcal{V}_{\{x\}}$ in M, and because of simplicity F is such a maximal face, iff there are topes $G, G' := F \cdot (-G)$ covering F in M, such that G and G' are separated by $\{x\}$ but coincide in every other component.

Together this show that $\Psi(\mathcal{T})$ determines a range space S, where $S^{\{x\}}$ is $\overline{PG}$ of dimension d, for all $x \in X$. If we can show that S is of dimension $d + 1$, then it follows from 4.2.1 that S itself is a $\overline{PG}$-space.

Clearly, S is of dimension at least $d + 1$. Now assume there is $A \subset X$, $|A| \geq d + 2$ shattered in $\Psi(\mathcal{T})$. This immediately implies that any signed vector in $\{+1, -1\}^A$ is a tope of M del $(X - A)$. Repetetive application of the elimination axiom (OM4) easily shows that in this case M del $(X - A)$ consists of all signed vectors on A (and is called the *free* oriented matroid on A), so the dimension, i.e. the length of a maximal chain from a vertex to a tope is $|A| - 1 \geq d + 1$, which is a contradiction, since deleting a subset clearly cannot increase the dimension of an oriented matroid. ■

5.4.4 Theorem

Let (X, R) be a $\overline{PG}$-space of dimension $d \geq 1$. Then $\Psi^{-1}(R)$ is the set of topes of a simple oriented matroid of dimension $d - 1$ on X.

Proof:

We construct the oriented matroid itself. To this end we introduce auxiliary mappings $\Gamma_A : R^A \to \{+1, 0, -1\}^X$ defined by

$$\Gamma_A(r)_x := \begin{cases} 0, \text{ if } x \in A \\ +1, \text{ if } x \in r \\ -1, \text{ otherwise} \end{cases}$$

Then $M := (X, \{0\} \cup \bigcup_{|A| \leq d-1} \Gamma_A(R^A))$ is a simple oriented matroid of dimension $d - 1$. Its set of topes clearly is $\Gamma_\emptyset(R) = \Psi^{-1}(R)$.

We have to show that M satisfies the axioms of an oriented matroid.

(OM1) is true by definition of M; (OM2) follows from the fact that the subspaces R^A are again $\overline{PG}$-spaces and therefore closed.

To see that (OM3) is satisfied, choose faces F and G of M. We can assume $F \neq 0$. Since $F \leq F \cdot G$, it suffices to show that for a given face F with zeroset A all the $3^{|A|}$ signed vector dominating F are faces of M.

Assume, $F = \Gamma_A(r), r \in R^A$. From the definition of R^A, $r \cap A = \emptyset$ and $r \cup B \in R^C$ for all $C \subset A$, $B \subset A - C$. Clearly, all the $\Gamma_C(r \cup B)$ are distinct and dominate F. Furthermore, $\#(\{\Gamma_C(r \cup B) \mid C \subset A, B \subset A - C\}) = \sum_{i=0}^{|A|} \binom{|A|}{i} \cdot 2^{|A|-i} = (1+2)^{|A|} = 3^{|A|}$, which means that any signed vector dominating F is of the form $\Gamma_C(r'), r' \in R^C$ and therefore is a face of M.

Finally, it remains to prove (OM4); let F and G be faces of M which are separated by x. W.l.o.g. we can assume that $F_y = G_y$, for all y not separating F and G (this can be achieved by replacing F and G with $F \cdot G$ and $G \cdot F$, respectively, which are faces of M again; furthermore, a face resulting from the elimination of x between $F \cdot G$ and $G \cdot F$ will be a proper choice also for F and G).

F and G have the same zeroset, so $F = \Gamma_A(r)$, $G = \Gamma_A(r')$ for some $A \subset X$, $r, r' \in R^A$. If r and r' are complementary ranges in R^A, then 0 results from the elimination of x between F and G (this covers the case where $|A| = d-1$, in which case r, r' with $r = (X - A) - r'$ are the only two ranges in the 1-dimensional $\overline{PG}$-space R^A).

Now assume $r \neq (X - A) - r'$. Then there is $y \in X - A$, such that r and r' are in a common halfspace of y in R^A and therefore in a common PG-space S underlying R^A (see 4.1.4). x separates F and G, so assume $x \notin r$, $x \in r'$. Corollary 2.3.5 shows that there is a path between r and r' in the D^1-graph of S, where the edge labels on this path are exactly the elements in $r \triangle r'$. Let u and u' be the two ranges on this path with $u' = u \cup \{x\}$. Then $u \in R^{A \cup \{x\}}$. Consider $H := \Gamma_{A \cup \{x\}}(u)$; clearly $H_x = 0$, and if y does not separate F and G, then $F_y = G_y$, which means $y \in r$ iff $y \in r'$, so $y \notin r \triangle r'$. Hence $y \in u$ iff $y \in r, r'$, and we conclude that $H_y = F_y = G_y = (F \cdot G)_y$, so H results from the elimination of x between F and G.

Dimension $d-1$ is immediate from the definition of M – the vertices of M are the vectors $\Gamma_A(r)$, where $|A| = d-1, r \in R^A$. Since $r \in R^B$ for all $B \subset A$, a descending length-$d-1$-chain from A to $\emptyset$ in the inclusion order $\subset$ yields an ascending length-$d-1$-chain from the vertex $\Gamma_A(r)$ to the tope $\Gamma_\emptyset(r)$.

Simplicity also follows, since $\Gamma_A(r)$ has zeroset A, which has cardinality $d-1$ for any vertex.
■

5.5 Arrangements of Pseudohemispheres and -halfspaces

To conclude this chapter, we give a formal definition of arrangements of pseudohemispheres and arrangements of pseudohalfspaces and show how these structures correspond to $\overline{PG}$- and PG-spaces. We do not go into the details concerning the arrangements – they can be found in [Ma], [FL].

A *(topological) sphere* is a topological space homeomorphic to the unit sphere S^d for some d. A *hyperpsphere* of a sphere is the image of a great sphere of S^d under some homeomorphism. The *hemispheres* of a hypersphere s, denoted by s^+ and s^- are the images of the two open hemispheres of the great sphere.

5.5.1 Definition

A simple *arrangement of pseudohemispheres* is a triple (S, E, H), where S is a topological sphere, E a finite index set and $H = \bigcup_{e \in E}\{s_e, s_e^+, s_e^-\}$ a collection of subsets of S, such that

(i) for $e \in E$, s_e is a hypersphere with hemispheres s_e^+, s_e^-

(ii) for every $\emptyset \neq A \subset E$, $\bigcap_{e \in A} s_e$ is a sphere (possibly empty). This is called a *flat* of the arrangement

(iii) For every flat F and $e \in E$, either $F \subset s_e$ or $F \cap s_e$ is a hypersphere of F with hemispheres $F \cap s_e^+$ and $F \cap s_e^-$

(iv) No nonempty flat contained in some $s_e, e \in E$ is the intersection of some of the remaining $s_f, f \in E - \{e\}$

The original definition by [FL] contains another axiom, the so-called *ball axiom*, which is redundant, as shown by Mandel. This is a non-trivial result.

We restrict the definition to simple arrangements, because only these are needed here; furthermore, property (i) becomes more complicated in the general case.

Property (iii) intuitively says that a flat that is not contained in one of the hyperspheres, must cross it.

Mandel has shown that there is a one-to-one correspondence between the simple arrangements of pseudohemispheres and the simple oriented matroids of the same dimension; the oriented matroid is obtained from the arrangement in the same manner as in example 5.2.3: every point p in the underlying sphere S is labelled with a signed vector $\sigma(p)$ on E, where $\sigma(p)_e$ is $+1, 0$ or -1 depending on whether $p \in s_e^+, s_e$ or s_e^-.

Together with our result this shows that $\overline{PG}$-spaces exactly represent simple arrangements of pseudohemispheres. (Mandels's result is much more general - he doesn't need simplicity; so what we use here is only a special case of his correspondence).

An arrangement of pseudohalfspaces can now be defined as the cell complex in one of the hemispheres of an arrangement of pseudohemispheres. Formally Mandel gives the following

5.5.2 Definition

A simple *arrangement of pseudohalfspaces* is a triple (K, E, L), where K is a toplogical space homeomorphic to E^d for some d, E a finite index set and $L = \bigcup_{e \in E} \{h_e, h_e^+, h_e^-\}$ a collection of subsets of K, such that there exists an arrangement of pseudohemispheres $(S, E \cup \{x\}, H)$, $x \notin E$ satisfying

(i) $K = s_x^-$
(ii) $h_e = s_e \cap s_x^-$, $h_e^* = s_e^* \cap s_x^-$, $* \in \{+, -\}$.

Arrangements of pseudohalfspaces correspond to so-called *affine matroids*, which are related to oriented matroids in the following way:

5.5.3 Definition

A simple *affine matroid* on E is a pair $(E \cup \{x\}, B)$, $x \notin E$, where B is a collection of signed vectors on $E \cup \{x\}$, such that there exists a simple oriented matroid $(E \cup \{x\}, \mathcal{V})$ with $B = \{F \in \mathcal{V} \mid F_x = -1\}$.

The dimension of an affine matroid is defined as the dimension of the related oriented matroid. Affine matroids are studied in detail by Mandel. What we need here to show that PG-spaces correspond to simple arrangements of pseudohalfspaces, is the following theorem that finishes this section:

5.5.4 Theorem

There is a one-to-one correspondence between the PG-spaces of dimension d and the affine matroids of dimension d.

Proof:

Clearly, an affine matroid on E and its corresponding oriented matroid on $E \cup \{x\}$ are uniquely deducible from each other, so for fixed E and $x \notin E$ there is a one-to-one correspondence between the d-dimensional affine matroids on E and the d-dimensional oriented matroids on $E \cup \{x\}$.

It remains to prove that such a correspondence exists between the d-dimensional PG- spaces on E and the $d+1$-dimensional $\overline{PG}$-spaces on $E \cup \{x\}$.

To see this, map a PG-space $S = (E, R)$ to the space $\hat{S} = (E \cup \{x\}, \hat{R})$, where $\hat{R} = R \cup \{(E \cup \{x\}) - r \mid r \in R\}$. Then $|\hat{R}| = 2|R| = 2\Phi_d(|E|)$. Furthermore, $\hat{S}$ is of dimension $d+1$: if $A \subset E \cup \{x\}$ is shattered in $\hat{R}$, there are two cases: if $x \in A$, then it is an easy observation that $A - \{x\}$ is shattered in R, so $|A| \leq d+1$. If $x \notin A$, then A is already shattered in $\hat{R} - \{x\} = \overline{R}$, and since $\overline{S}$ is of dimension at most $d+1$ (3.5.4), we know that $|A| \leq d+1$ also in this case.

Together this shows that $\hat{S}$ is maximal closed of dimension $d+1$, which means that $\hat{S}$ is $\overline{PG}$ by theoerem 4.1.2.

To obtain a PG-space from a $(d+1)$-dimensional $\overline{PG}$-space, we proceed as follows: Given $\hat{S} = (E \cup \{x\}, \hat{R})$, we know from remark 4.1.4 that there exists an underlying d-dimensional PG-space $S' = (E \cup \{x\}, R')$ with $x^- = \{r \in \hat{R} \mid x \notin r\} \subset R'$. $|E \cup \{x\}|$ is at least $d+1$, so $S := S' - \{x\}$ is the desired PG-space of dimension d on E.

It remains to show that both mappings are inverse to each other: if we start with $S = (E, R)$, the first mapping gives us $\hat{S} = (E \cup \{x\}, R \cup \{(E \cup \{x\}) - r \mid r \in R\})$. If we apply the second mapping to this space, we first obtain $S' = (E \cup \{x\}, R')$ and by construction R' contains R. Clearly, then also $R' - \{x\}$ contains R, and $|R' - \{x\}| = |R|$ shows that $R' - \{x\} = R$, which means that we have obtained the same space we have started with. ■.

As a summary of this chapter we have the following

5.5.5 Result

There exist one-to-one correspondences between

(i) $\overline{PG}$-spaces of dimension $d+1$ and simple d-dimensional arrangements of pseudohemispheres

(ii) PG-spaces of dimension d and simple d-dimensional arrangements of pseudohalfspaces

Chapter 6

Geometric Embeddability

6.1 Basics

In this chapter we introduce the geometric notion of m-*embeddability* of a range space, which can be regarded as a generalization of planarity with two more degrees of freedom.

m-embeddability is an interesting subject of its own, but beyond it sheds some light on geometric features of range spaces, which contrasts to the previous chapter, where topological aspects were placed into the foreground.

The main theorem in this chapter, a characterization of the complete range spaces which are m-embeddable for a certain m, reveals an interesting relation between topologic and geometric properties of complete spaces.

Moreover, embeddability is related to the k-set problem, which in fact is the main motivation for this concept.

6.1.1 Definition

Let $S = (X, R)$ be a range space, $m \geq 0$. S is called m-*embeddable*, if X can be mapped to m-dimensional euclidean space by a function $f : X \to E^m$, such that

$$\forall r, r' \in R : \text{conv}(f(r - r')) \cap \text{conv}(f(r' - r)) = \emptyset.$$

Note that f is not an embedding in the usual sense, since it need not be injective. Nevertheless, we do not lose generality by assuming f to be injective, if $m > 0$. We can even require $f(X)$ to be a simple configuration of points as defined in section 1.3, i.e. any m points define a unique non-vertical hyperplane and no hyperplane contains $m + 1$ of the points.

To see this, assume some $Y \subset X$, $|Y| \geq 2$ is mapped by f to a single point p. Let $D(r, r')$ denote the minimal distance between conv$(f(r - r'))$ and conv$(f(r' - r))$ ($D(r, r') := \infty$, if one of $r - r'$, $r' - r$ is empty), and define $D := min\{D(r, r') \mid r, r' \in R\}$. Clearly, D is some positive constant. Let $B(p)$ be the ball with center p and radius D, and modify f in such a way that the elements of Y are mapped to distinct points in $B(p)$, which are not yet in $f(X)$. Assume that now there are ranges r, r' violating the embeddability-condition. Clearly then, one of $r - r'$, $r' - r$ must contain an element of Y – in any other case nothing would have changed. W.l.o.g. $Y \cap (r - r') \neq \emptyset$. This implies $Y \cap (r' - r) = \emptyset$, otherwise $p \in \text{conv}(f(r - r')) \cap \text{conv}(f(r' - r))$ for the original f, a contradiction. By construction, the replacement of p with $|Y|$ distinct points blows up conv$(f(r - r')$ only by such a small amount that it cannot hit conv$(f(r' - r))$, which remains unchanged; this shows that the pair r, r' does not exist, so the modified f again induces an m-embedding. Step by step every point that is the image of more than one $x \in X$ can be replaced in this way, finally ending up with f being injective. By again slightly perturbing the (now distinct) points without changing the embeddability condition we can make $f(X)$ a simple configuration of points in E^m.

The definition of m-embeddability is somewhat non-obvious at first sight, and to make it a little clearer, we show how this can be seen to generalize planarity: note that a graph is simply a range space, where every range has cardinality exactly two. Now the graph is planar, iff it is 2-embeddable according to our definition. Observe that this is true, because a planar graph can be embedded in the plane in the ordinary sense in such a way, that all edges are mapped to straight line segments [Fá].

We proceed as follows: First we give an obvious lower bound on m, if S is of fixed dimension, and then show that geometric range spaces (under a certain constraint) are m-embeddable, where m matches the lower bound. This will lead to a short excurse concerning the k-set problem.

After having handled this special case, we consider general complete spaces and characterize the subclass of spaces that as well as geometric spaces are m- embeddable with optimal m. This will be a proper subclass of the complete spaces and a proper superclass of the geometric spaces.

6.1.2 Lemma

Let $S = (X, R)$ of dimension $d \geq 1$ be m-embeddable. Then $m \geq d - 1$.

Proof:

If $m = 0$, it is easy to see that S can be embedded in E^0 if and only if the ranges are pairwise comparable with respect to inclusion; this is not the case, if $d \geq 2$.

Now assume $0 < m \leq d-2$; let A be a set of cardinality d that is shattered in R. After applying an appropriate injective embedding function we can assume that $A \subset E^m$. Radon's theorem [Ed] says that A can be partioned into sets A_1, A_2 in such a way that conv$(A_1)\cap$ conv$(A_2) \neq \emptyset$. Let r_1, r_2 be ranges with $A \cap r_1 = A_1, A \cap r_2 = A_2$. Then we have $r_1 - r_2 \supset A_1$ and $r_2 - r_1 \supset A_2$, which shows that conv$(r_1 - r_2)\cap$conv$(r_2 - r_1) \neq \emptyset$, a contradiction to embeddability. It follows that $m \geq d - 1$. ■

The lemma shows that $(d - 1)$-embeddability is the best we can hope for if S is of dimension d, and the following theorem shows that this bound is tight:

6.2 Embedding Geometric Range Spaces

6.2.1 Theorem

Let $S = (X, R)$ be a geometric range space of dimension $d \geq 1$, $\emptyset, X \in R$ (we say, S is in *standard position*). Then S is $(d - 1)$-embeddable.

Proof:

Recall that a geometric range space of dimension d is the description of cells $\mathcal{C}(H^+)$ of some simple arrangement of halfspaces $\mathcal{A}(H^+)$ in E^d. $\emptyset, X \in R$ shows that there is an unbounded cell $c_\emptyset$ contained in none of the halfspaces and another one c_X contained in all of them. It follows that

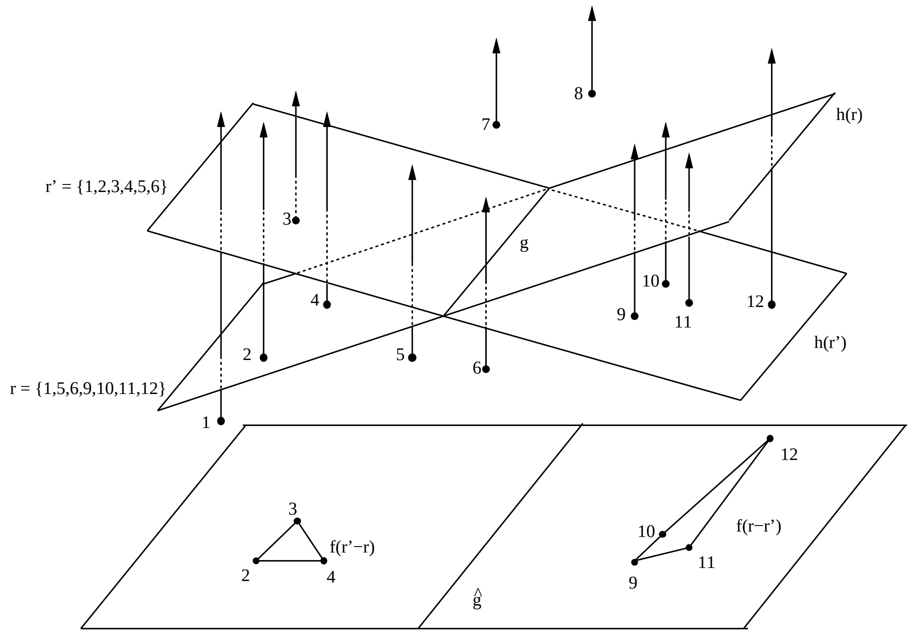

Figure 6.1: configuration of directed points in E^3 and its 2-embedding

the arrangement can be rotated in such a way that the positive halfspace h^+ is the halfspace below h for all hyperplanes $h \in H$.

In the first chapter we have shown how, by invoking duality, one can obtain this geometric space also from a simple configuration of directed points, and it is this point of view that we are taking here. A directed point can be visualized as a point in E^d with a ray attached to it that points either up or down.

Dualize every halfspace h^+ to a directed point in the way described in section 1.2. Since we know that $h^+ = h_{below}$ for all $h \in H$, all the rays of these directed points go upwards.

We have seen that the description of cells can now equivalently be obtained as follows: Label every non-vertical hyperplane of E^d with the set of points whose rays stab the hyperplane; then the collection of labels of all hyperplanes containing none of the points determines $\mathcal{C}(H^+)$.

So assume that X is the configuration of directed points dual to $\mathcal{A}(H^+)$. Let P be a horizontal hyperplane that is not stabbed by any of the rays, and project the points of X vertically onto P. Let $f(x)$ denote the image of $x \in X$ under this projection. We claim that the mapping f defines a legal $(d-1)$-embedding. To see this, let r, r' be ranges from R. There are non-vertical hyperplanes $h(r), h(r')$ defining r and r', i.e. $h(r)$ is stabbed by exactly the rays of the points in r, and the same holds for $h(r')$ and r'. The two hyperplanes can be chosen to be non-parallel and hence intersect in a common $(d-2)$-flat g. The projection of g onto P is a hyperplane $\hat{g}$ in P. Consider the two open halfspaces defined by $\hat{g}$. With respect to one of them $h(r)$ lies always above $h(r')$, while for the other one $h(r')$ lies above $h(r)$. Since the rays of all the directed points go upwards, $r - r'$ is the set of points that lie below $h(r)$ but above $h(r')$, while the points from $r' - r$ lie below $h(r')$ but above $h(r)$. From what we have just said it follows that $r - r'$ and $r' - r$ are projected to different open halfspaces of $\hat{g}$, so $\hat{g}$ separates the convex hulls of $f(r - r')$ and $f(r' - r)$, which means that f is a $(d-1)$-embedding (figure 6.1).

The reader might have noted that this argumentation fails for $d = 1$ – therefore this case needs a special treatment. Recall that the D^1-graph (2.3.1) of a geometric space of dimension 1 is a path (figure 3.1); this fact has motivated us to introduce pseudogeometric spaces (3.1.1). It is an easy observation that if $\emptyset, X \in R$, then the ranges must be ordered by inclusion along the path, and in the proof of lemma 6.1.2 we have already remarked that in this case S is 0-embeddable.

■

Note that the proof does not use the fact that the configuration of points is simple. This is not surprising, since we have already seeen in observation 1.3.1 that the set of ranges defined by a non-simple configuration of directed points is always a subset of the set of ranges determined by an appropriate simple one. This means that – as far as embeddability is concerned – simple configurations are the most difficult ones.

One might conjecture that by some additional effort the condition "$\emptyset, X \in R$" could be eliminated: a geometric space can always be swapped in such a way that standard position is achieved, so why should it not work for any geometric space?

The answer is that standard position is necessary for $(d-1)$-embeddability of complete spaces in general. This is somewhat surprising, because up to now all the properties of range spaces we have considered were invariant under swapping – moreover, swapping was a useful technical tool to facilitate most of the proofs.

By taking a look at the k-set problem the reader might get an idea why embeddability is different from the other concepts with respect to swapping.

6.3 The k-set Problem

Let X be a configuration of points in E^d, h a hyperplane (disjoint from X) with open halfspaces h^+ and h^-. The sets $h^+ \cap X$ and $h^- \cap X$ are called *semispaces* of X. A semispace of cardinality k is called a k-set.

The k-set problem is simply posed as follows:

> Given a configuration X of n points and a natural number $0 \leq k \leq n$, how many k-sets are defined by X?

Despite of its simple formulation, the k-set problem turns out to be very difficult. For every k-set there is a unique $(n-k)$-set, so it suffices to consider the range $0 \leq k \leq \frac{n}{2}$, and in some sense the most interesting case is $k = \frac{n}{2}$; there is an easy upper bound of $O(n^d)$ on the number of all semispaces [Ed], but a first non-trivial upper bound (i.e. a bound better than $O(n^3)$) on the number of $\frac{n}{2}$-sets for $d = 3$ has only recently been developed [BFL]. The currently best bound is given by [ACEGSW].

Even for $d = 2$ there is a wide gap between the best known lower and upper bound [Ed], [EW] and nothing is known about good bounds in dimension $d \geq 4$.

In order to establish the correspondence between the k-set problem and embeddability we classify the semispaces of a configuration in the following way: A semispace is called a *lower* semispace, if it is the set of points *below* one of its defining hyperplanes. An *upper* semispace is defined analogously. Note that a semispace can be lower and upper at the same time.

Now we have the following easy

6.3.1 Observation

Let X be a configuration of points and let R_l, R_u denote the set of lower and upper semispaces, $S_l = (X, R_l)$, $S_u = (X, R_u)$.

Then

(i) $\delta S_l = \delta S_u = (X, R_l \cap R_u)$
(ii) $\overline{S_l} = \overline{S_u} = (X, R_l \cup R_u)$

Proof:

Observe that the complement of a lower (upper) semispace is an upper (lower) semispace, so $R_l = \{X - r \mid r \in R_u\}$. The assertions are immediate from this. ■

The next lemma relates semispaces of a configuration to geometric spaces, and by using duality this correspondence is more or less obvious.

6.3.2 Lemma

Let $S = (X, R)$ be a range space. S is geometric of dimension $d \geq 1$ in standard position, if and only if X can be identified with a simple configuration of points in E^d in such a way that R is the set of lower semispaces of X.

Proof:

If S is geometric in standard position this equivalently means that S is the description of cells of a simple arrangement of halfspaces $\mathcal{A}(H^+)$ with $h^+ = h_{below}$ for all $h \in H$ (see the proof of theorem 6.2.1). By duality S corresponds to a simple configuration $\tilde{X}$ of directed points with all rays going upwards, and in this dual setting a range r of S is the set of directed points stabbing a certain non-vertical hyperplane h_r disjoint from the points.

If we consider only the underlying points, this means that r corresponds to the lower semispace of $\tilde{X}$ defined by h_r (see figure 6.1).

Furthermore, every lower semispace of $\tilde{X}$ is obtained in this way: Given a lower semispace, dualize its defining hyperplane (which can be choosen to be non-vertical) to a point, which lies in some cell of $\mathcal{A}(H^+)$. It follows that the semispace is equal to the label of this cell.

If we are given a simple configuration of points, then it can be seen to determine the required geometric space by simply applying the inverse duality: Add an upwards ray to each point and dualize the corresponding configuration of directed points to an arrangement of halfspaces, which by construction is simple and in standard position. Clearly, again the lower semispaces will correspond to the labels of the cells of the arrangement. ■

Now we are able to show how the k-set problem is related to embeddability:

Let X be a fixed configuration of points in E^d, k a natural number. $R_l(k)$ denotes the set of lower semispaces of cardinality k (the lower k-sets), $R(k)$ is the set of all k-sets. We are interested in an upper bound on $R(k)$, so we can assume that X is simple, which can only increase $|R(k)|$.

$R_l(k)$ is a subset of the set of all lower semispaces R_l, and since by the lemma (X, R_l) is geometric in standard position, we know that (X, R_l) and therefore also $(X, R_l(k))$ is $(d-1)$-embeddable (theorem 6.2.1).

Now assume that there is an upper bound B_k on the number of sets of cardinality k that can be embedded into E^{d-1} in a legal way, i.e.

$$B_k \geq max\{|R| \mid (X,R)\ \ (d-1)-\text{embeddable}, R \subset 2^X, |r| = k \text{ for all } r \in R\}.$$

Clearly then, the bound B_k holds for the number $|R_l(k)|$ of lower k-sets. Furthermore, we can bound the overall number $|R(k)|$ of k-sets (lower *and* upper ones) by $2B_k$.

To see this, observe that the upper k-sets correspond to the lower $(n-k)$-sets. It is an easy observation that B_k is also an upper bound for the number of sets of cardinality $n-k$ which are embeddable in a legal way (this follows from the general fact that (X,R) is m-embeddable if and only if $(X, \{X-r \mid r \in R\})$ is m-embeddable, which can be proved using the equality $r - r' = (X - r') - (X - r)$). Since every k-set of X determines a unique lower k- or $(n-k)$-set, $(d-1)$-embeddability of $(X, R_l(k))$ implies that there are at most $2B_k$ k-sets of X.

As an example we can give an upper bound of $6n-12$ on the number of 2-sets of n points in E^3, using the fact that at most $3n-6$ sets of cardinality 2 can be embedded in E^2 without intersections – this is the relation to planarity given at the beginning of this chapter (using this technique, the bound can be adjusted to the real upper bound of $3n-6$ by some additional considerations; we don't do this here).

Unfortunately, nothing is known about bounds of this kind for cardinality and dimension more than 2; from what we have just said, non-trivial bounds B_k immediately imply non-trivial bounds for the k-set problem.

6.4 Embeddability of Complete Range Spaces

Before we continue with embeddability of complete spaces we give an interesting lemma that holds for arbitrary range spaces:

6.4.1 Lemma

Let $S = (X,R)$ be m-embeddable with embedding function f, $m \geq 1$, $x \in X$. Then

(i) $S - \{x\}$ is m-embeddable.
(ii) $S^{\{x\}}$ is $(m-1)$-embeddable, if $f(x)$ is extreme in $f(X)$.

Proof:

Part (i) of the lemma is easy. Simply take $f|_{X-\{x\}}$, which is a legal embedding function for $S - \{x\}$. To see that (ii) holds, consider a hyperplane h separating $f(x)$ from $f(X - \{x\})$, and project $f(y)$, $y \neq x$ onto h using central projection with center $f(x)$. Let $g(y)$ denote the image of $f(y)$ under this projection.

Now consider $r, r' \in R^{\{x\}}$ and assume that conv$(g(r - r'))\cap$ conv $(g(r' - r))$ contains a point $p \in h$. Consider the ray l through p starting from $f(x)$. l stabs conv$(f(r - r'))$ and conv$(f(r' - r))$ in a well-defined order, since these two sets are disjoint. Let q be a point of E^m contained in conv$(f(r - r'))$, which we assume to be hit by l first. But then we have $q \in$ conv$(f(r - (r' \cup \{x\})))\cap$ conv$(f((r' \cup \{x\}) - r))$, a contradiction to m-embeddability of S. ■

Now we come to the main result of this chapter, which is a characterization of the complete spaces of dimension d that are $(d-1)$-embeddable:

6.4.2 Theorem

Let $S = (X, R)$ be complete of dimension $d \geq 1$. S is $(d-1)$-embeddable, if and only if

(i) S is pseudogeometric in standard position
(ii) δS is the closure of a geometric space of dimension $d-1$

Proof:
First assume, S is $(d-1)$-embeddable. If $d = 1$, then 0-embeddability implies that the ranges of S are linearly ordered by inclusion. It follows that the D^1-graph of S (definition 2.3.1) is not only a tree, but a path with ranges $\emptyset$ and X on both ends of it. By definition 3.1.1 S is pseudogeometric in standard position, and $\delta S = (X, \{\emptyset, X\})$ clearly satisfies condition (ii). If on the other hand S satisfies condition (i), then its ranges are linearly ordered by inclusion, so S will be 0-embeddable.

If $d > 1$, then after applying an appropriate injective embedding function we can assume that X is a simple configuration of points in E^{d-1}.

We claim that R contains all semispaces of X. To see this assume on the contrary that there is a semispace r of X not contained in R. Let h_r be its defining hyperplane. W.l.o.g r is a lower semispace with respect to h_r.

Let r' be any subset of X. Clearly then $r - r' \subset r$ lies below h_r, while $r' - r \subset X - r$ lies above h_r. It follows that r can be added to R without violating the embeddability condition. This means, $(X, R \cup \{r\})$ is again $(d-1)$-embeddable. But since $S = (X, R)$ is complete, $(X, R \cup \{r\})$ must be of dimension more than d, which is a contradiction to lemma 6.1.2.

We conclude that r is already a range of S, so R contains all the semispaces of X. The set of lower semispaces of $X \subset E^{d-1}$ determines a geometric space $S' = (X, R')$ of dimension $d-1$, as shown in lemma 6.3.2. The set of all semispaces is the closure of the set of lower semispaces and is contained in the boundary of S, so it follows that $\overline{R'} \subset \delta R$; since $\overline{S'}$ is maximal closed of dimension d (theorem 4.1.2), we know that $\overline{R'} = \delta R$, so S has the maximum number of ranges in its boundary. By theorem 3.5.3 S is pseudogeometric; standard position follows from the fact that $\emptyset$ as well as X are semispaces of X.

An underlying geometric space of δS as required by (ii) has already been found: it is $S' = (X, R')$, with R' equal to the set of lower semispaces of X.

Now assume, S satisfies conditions (i) and (ii). Consider δS. If δS is the closure of a geometric space of dimension $d-1$, δS can be visualized as the description of cells of some simple arrangement of hemispheres in the unit sphere S^{d-1} (see section 4.1). Clearly, $\emptyset, X \in \delta R$, so there are antipodal cells of the arrangement labelled with $\emptyset$ and X. Choose the equator in such a way that it cuts through these cells. Then the labels of the cells in the northern hemisphere determine a geometric space $S' = (X, R')$ of dimension $d-1$ in standard position with $\overline{S'} = \delta S$. Again we use lemma 6.3.2, which shows that X can be identified with a point set in E^{d-1} in such a way that R' is the set of lower semispaces of X. Consequently, $\overline{R} = \delta R$ is the set of all semispaces of X.

To see that S is $(d-1)$-embeddable, consider $r, r' \in R$. S is pseudogeometric, so r and r' admit a line (theorem 3.3.3), which due to lemma 3.3.2 is equivalent to the existence of ranges $t, X - t \in R$ with $r - r' \subset t, r' - r \subset X - t$. t and $X - t$ are boundary ranges, so they correspond to complementary semispaces of X, which clearly implies that conv(t)$\cap$ conv($X - t$) $= \emptyset$. Clearly, conv($r - r'$) $\subset$ conv(t) and conv($r' - r$) $\subset$ conv($X - t$), and this yields conv($r - r'$)$\cap$ conv($r' - r$) $= \emptyset$. ■

Chapter 7

Elementary Transformations and Simplices

7.1 Basics

In this chapter we discuss *elementary transformations*, which take a complete range space S to another complete space S' with the same number of ranges by performing a local modification of S.

We have already seen another transformation that maintains completeness, namely the swap-operation defined in 2.3.2. Swapping is a global operation in the sense that it affects all the ranges; on the other hand, swapping does not ”really” change the structure of a range space – the distance-1-graph which reflects many structural properties remains unchanged.

Elementary transformations are more interesting with respect to this point.

7.1.1 Definition

Let $S = (X, R)$ be a range space, $r \in R$, $r' \notin R$. We define

$S \cup \{r'\} = (X, R \cup \{r'\})$,
$S - \{r'\} = (X, R - \{r'\})$,
$S \triangle \{r, r'\} = (X, R \triangle \{r, r'\})$.

The operation $S \to S \triangle \{r, r'\}$ is called a *local swap*. If S is complete of dimension $d \geq 0$, the local swap is called an *elementary transformation*, iff $S \triangle \{r, r'\}$ is again complete of dimension d.

The notion of elementary transformations is motivated by properties of simple arrangements of pseudohalfspaces, which – as we know from chapter 5 – correspond to pseudogeometric spaces. Figure 7.1 shows an example of an elementary transformation in the 2-dimensional case: a pseudoline is moved across a vertex of the arrangement, destroying one cell and generating a new one. Note that both cells are simplicial cells.

The reader can check that this operation does not affect the PG-property of the description of cells, which might be obvious for this example, but is not that intuitive for higher dimensions.

In the sequel we will formally define what we mean by a simplex in the setting of range spaces and show that simplices in complete (and pseudogeometric) spaces give rise to an elementary transformation of the kind shown by the example.

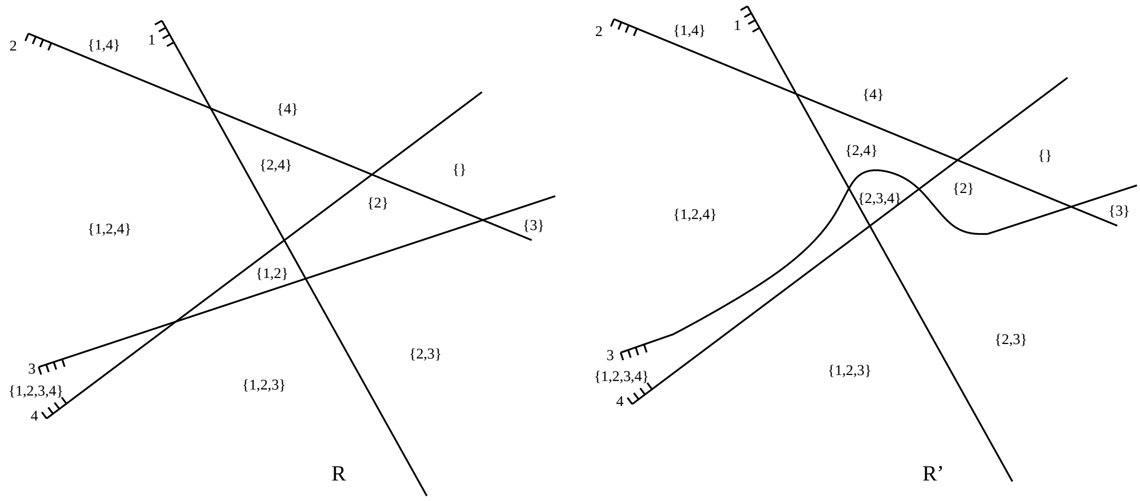

Figure 7.1: elementary transformation; $R' = R\triangle\{\{1,2\},\{2,3,4\}\}$

7.2 Characterizing Elementary Transformations

Local swaps of simplices are not the only possible elementary transformations, and we will give a characterization of the pairs of ranges $\{r, r'\}$, which define an elementary transformation.

To begin with, we need one more

7.2.1 Definition

Let X be fixed and consider $R \subset 2^X$, $r \in R$. The *incidence set* of r with respect to R is defined by

$$I_R(r) := \{\lambda \in X \mid r\triangle\{\lambda\} \in R\}.$$

If $|I_R(r)| = k$, then r is called a k-range.
If r is a $(d+1)$-range, then r is called a *simplex*, iff $r\triangle B \in R$, for all $B \subsetneq I_R(r)$.

As an example let $S = (X, R)$ be the description of cells of the original arrangement in figure 7.1, $r = \{1,2\}$, $r' = \{2,3,4\}$, $R' = R\triangle\{r, r'\}$. Then $I_R(r) = I_{R'}(r') = \{1,3,4\}$. Furthermore, r is a simplex in R as well as r' is a simplex in R'. $\{1,2,3\}$ is a 3-range but not a simplex.

The fact that $I_R(r) = I_{R'}(r')$ is not at all accidental. We will show that this is a necessary and sufficient condition for r, r' to determine an elementary transformation.

7.2.2 Lemma

Let $S = (X, R)$ be a range space of dimension $d \geq 0$, $r' \notin R$.
If $|I_{R\cup\{r'\}}(r')| > d+1$, then $S \cup \{r'\}$ is of dimension d.

Proof:
Assume on the contrary that $S \cup \{r'\}$ is of dimension at least $d+1$, and let $A \subset X$, $|A| = d+1$ be shattered in $R \cup \{r'\}$. A is not shattered in R, and this implies $A \cap r' \neq A \cap r$, for all $r \in R$. Especially, $A \cap r' \neq A \cap (r'\triangle\{\lambda\})$, which shows that $\lambda \in A$, for all $\lambda \in I_{R\cup\{r'\}}(r')$.

Hence $I_{R\cup\{r'\}}(r') \subset A$, so $|I_{R\cup\{r'\}}(r')| \leq d+1$, a contradiction to the assumption. It follows that $S \cup \{r'\}$ is again of dimension d. ■

Now we are able to establish the characterization that we have already announced above:

7.2.3 Theorem

Let $S = (X, R)$ be complete of dimension $d \geq 0$, $r \in R, r' \notin R$.
$S \to S' := S\triangle\{r, r'\}$ is an elementary transformation, if and only if

$$I_R(r) = I_{R'}(r'),$$

where $R' = R\triangle\{r, r'\}$.

Proof:
If $S \to S'$ is an elementary transformation, then clearly $|R^{\{x\}}| = |R'^{\{x\}}|$ for all $x \in X$. This immediately implies $I_R(r) = I_{R'}(r')$.

Now assume $I_R(r) = I_{R'}(r')$. We show that $S' = (X, R')$ is of dimension d, which proves the theorem.

Assume on the contrary, there is $|A| > d$ shattered in $R' \subset R \cup \{r'\}$. From the proof of the lemma it follows that then

$$I_R(r) = I_{R'}(r') \subset I_{R\cup\{r'\}}(r') \subset A.$$

By swapping assume $r = \emptyset$. We know that $A \cap r' \neq A \cap r = \emptyset$, and every range in $R - \{r\}$ also has nonempty intersection with A. To see this, recall that a range s from $R - \{r\}$ is connected with r by a shortest possible path in $D^1(S)$ (theorem 2.3.5). Clearly, the label of the edge incident to r on this path must be some λ contained in $I_R(r) \subset A$, and because of $r = \emptyset$ this implies $\lambda \in s$; therefore $A \cap s \neq \emptyset$.

Together this shows that $A \cap s \neq \emptyset$, for all $s \in R'$, which means that A is not shattered in R', a contradiction.

It follows that S' must be of dimension d. ■

As a corollary we get the following result, which shows that we have to search only among the ranges with small incidence set to find the ones that can be replaced in an elementary transformation:

7.2.4 Corollary

Let $S = (X, R)$ be complete of dimension d, $r \in R$. If there exists $r' \notin R$, such that $S \to S\triangle\{r, r'\}$ is an elementary transformation, then

$$d \leq |I_R(r)| \leq d + 1.$$

Proof:
Lemma 2.3.7 shows that $|I_R(r)|$ is at least d for any range in R, which proves the lower bound on $|I_R(r)|$.

Now consider $r' \notin R$, such that the replacement of r by r' is an elementary transformation. S is complete, so $S \cup \{r'\}$ is of dimension more than d. Then lemma 7.2.2 shows

$$d + 1 \geq |I_{R\cup\{r'\}}(r')| \geq |I_{R\triangle\{r,r'\}}(r')| = |I_R(r)|,$$

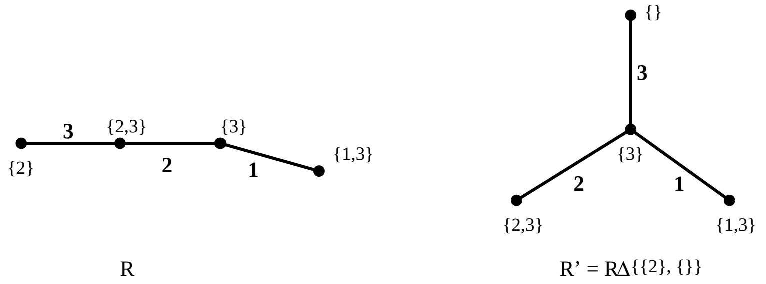

Figure 7.2: elementary transformation $R \to R'$ destroys the PG-property

and this gives the desired upper bound. ■

7.3 Simplex Transformations

If S is a pseudogeometric space, one may ask whether this property is invariant under elementary transformations. In general, this is not the case – in figure 7.2 we present a one-dimensional counterexample. The reader can check that also the original space in figure 7.1 can be transformed into a non-PG-space by applying the elementary transformation $S \to S\triangle\{\{1,2,3,4\},\{2,3,4\}\}$.

On the other hand, there is one important type of elementary transformations that maintains the pseudogeometric property, namely *simplex transformations*. An example of a simplex transformation has already been given in figure 7.1.

7.3.1 Theorem

Let $S = (X, R)$ be complete of dimension d, $r \in R$ a simplex of S. Then

(i) $r' := r\triangle I_R(r) \notin R$
(ii) $S \to S' := S\triangle\{r, r'\}$ is an elementary transformation
(iii) r' is a simplex in S'

$S \to S'$ is called a *simplex transformation*.

Proof:

(i) By swapping we may assume $r = \emptyset$. The simplex property implies $A \in R$ for all $A \subsetneqq I_R(r)$. Assume $r' \in R$. Then we have $A \in R$ for all subsets of $I_R(r)$, so $I_R(r)$ is shattered in R, which cannot be the case, since S is of dimension d, while $|I_R(r)| = d + 1$.

(ii) We have to show that $I_R(r) = I_{R'}(r')$. Again assume $r = \emptyset$. Since $A \in R'$ for all $A \subset I_R(r)$, $|A| = d$, we know that $I_{R'}(r') \supset I_R(r)$. From lemma 7.2.2 it follows that $|I_{R'}(r')| \leq d+1$, so $I_{R'}(r') = I_R(r)$.

(iii) If $r = \emptyset$, then $A \in R'$ for all $\emptyset \subsetneqq A \subset I_{R'}(r')$, which is the simplex property for $r' = I_{R'}(r')$ in S'. ■

In order to show that the pseudogeometric property is invariant under simplex transformations, it will be useful to have an equivalent, more intuitive characterization of a simplex in this case:

7.3.2 Theorem

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$. $r \in R$ is a simplex, if and only if

(i) r is a $(d+1)$-range
(ii) $r \notin \delta R$

Proof:
First assume, r is a simplex. (i) is satisfied by definition. To see that (ii) holds, by swapping assume $r = \emptyset$. If $r \in \delta R$, then $X \in R$, and similar to the proof of 7.3.1 (i) this implies that $I_R(r)$ is shattered in R, a contradiction.

Now let r satisfy (i) and (ii); by swapping $r = \emptyset$. It is an easy observation that the simplex condition is equivalent to $r \in R^A$ for all $A \subset I_R(r)$, $|A| = d$.

Assume there exists such an A with $r \notin R^A$. Because of $|R^A| = 1$, there must be some $r' \neq \emptyset$ with $r' \in R^A$.

Let a denote the unique element in $I_R(r) - A$. We claim that r' contains a. To see this, consider a shortest path from r' to r in $D^1(S)$. $r' \in R^A$ implies $A \cap (r \triangle r') = \emptyset$, so the labels of the edges on the path must be from $X - A$. It follows that the label of the edge incident to $r = \emptyset$ is a, which shows that $a \in r'$.

From $r' \in R^A$ we furthermore conclude $r' \cup A \in R$, so $I_R(r) = \{a\} \cup A \subset r' \cup A$, which means $(r' \cup A) \cap I_R(r) = I_R(r)$. Hence $I_R(r) \in R|_{I_R(r)}$.

On the other hand we have $r = \emptyset \in R|_{I_R(r)}$, which implies $r \in \delta(R|_{I_R(r)})$. But this is a contradiction to $r \notin \delta R$ – to show this it suffices to prove that $r \notin \delta R$ implies $r \notin \delta(R - \{x\})$ for $x \notin I_R(r)$; by iterating we obtain the desired contradiction.

So assume $x \notin I_R(r)$, which is equivalent to $\{x\} \notin R$. Hence $r = \emptyset \notin (\delta R) - \{x\} = \delta(R - \{x\})$ due to lemma 3.4.3.

It follows that r must be the unique range in R^A for all $A \subset I_R(r)$, $|A| = d$, so r is a simplex. ■

Now it is very easy to see that the PG-property is invariant under simplex transformations:

7.3.3 Theorem

Let $S = (X, R)$ be pseudogeometric of dimension $d \geq 1$, $S \rightarrow S \triangle \{r, r'\}$ a simplex transformation. Then $S \triangle \{r, r'\}$ is again pseudogeometric, and $\delta S = \delta(S \triangle \{r, r'\})$.

Proof:
The simplex r is not a boundary range, so $\delta R = \delta(R - \{r\}) \subset \delta(R \triangle \{r, r'\})$. By theorem 3.5.3 δR is maximal for a space of dimension d, so $\delta R = \delta(R \triangle \{r, r'\})$, and by the same theorem $S \triangle \{r, r'\}$ is pseudogeometric. ■

Using a result of Ringel [Ri], we can prove a theorem that demonstrates the power of simplex transformations in the 2-dimensional case:

7.3.4 Theorem

Given two pseudogeometric spaces S and T of dimension $d = 2$, there is a finite number of simplex transformations that take S to a space S', which – up to swapping and relabelling the elements – is equal to T.

Proof:
δS and δT are the closures of PG-spaces of dimension 1 (corollary 4.1.3), which are equal up to swapping and relabelling (cf. corollary 3.2.3), so we may assume that $\delta S = \delta T$.

Ringel has defined set systems similar to PG-spaces of dimension 2, which we will call *Ringel-schemes* here. Translated to our terminology, a Ringel-scheme is a finite range space $S = (X, R)$ with the following properties:

(i) $|R|_Y| = 7$ for all $Y \subset X$, $|Y| = 3$
(ii) $|(\delta R)|_Y| = 6$ for all $Y \subset X$, $|Y| = 3$
(iii) S is maximal with respect to property (i), i.e. adding one more range to R destroys this property
(iv) If $|X| = 4$, S is the description of cells of an arrangement of 4 halfplanes (and is therefore unique up to swapping and relabelling, as shown again in 3.2.3)

Ringel has shown that these schemes characterize the simple arrangements of pseudohalfplanes, where he needs property (iv) to rule out one type of scheme that satisfies (i) through (iii) but is not the description of cells of any arrangement.

If S is a Ringel-scheme, a *triangular cell* is a range $r \in R$ with $r \notin \delta R$ and $|I_R(r)| = 3$. Ringel shows that a triangular cell r can be replaced by $r \triangle I_R(r)$, again resulting in a valid Ringel-scheme (he calls such an operation a *triangle transformation*), and he derives the following result:

Let S and T be Ringel-schemes with $\delta S = \delta T$. Then S can be transformed into T by a finite number of triangle transformations.

To make this result valid for PG-spaces of dimension 2, it remains to show that these spaces are Ringel-schemes. Clearly then, triangle transformations by theorem 7.3.2 correspond to our simplex transformations.

To see that a 2-dimensional PG-space satisfies (i) and (ii) is easy by using the counting results for complete and pseudogeometric spaces from chapters 2 and 3.

Property (iii) follows from completeness: Adding one more range causes a set $Y \subset X$, $|Y| = 3$ to be shattered, which means that $|R|_Y| = 8$.

Property (iv) finally is simply characterization (v) of theorem 3.2.1. ■

This theorem does not generalize to higher dimensions. If $d \geq 3$, then δS and δT are the closures of 2-dimensional PG-spaces, which are not necessarily equal up to swapping and relabelling. Since simplex transformations do not affect the boundary, S and T cannot obey the theorem in this case.

Even if $\delta S = \delta T$, it is not clear whether there is a theorem of this kind. Ringel's methods are limited to dimension 2, and we cannot even prove the existence of a simplex in a PG-space of dimension more than 2.

A crucial feature of the planar case seems to be the following property that is best explained in geometric terms: if a triangle is cut by a line, then one of the two pieces is again a triangle. Ringel uses an equivalent to this property for his schemes to perform an inductive proof that deduces the existence of a triangular cell in the scheme S from the existence of such a cell in the subscheme $S - \{x\}$.

For $d \geq 3$, however, this property is lost. It is always possible to cut a tetrahedron (or a general simplex) with a plane (or a hyperplane) in such a way that none of the two resulting pieces is a tetrahedron (or a general simplex).

So we doubt, whether simplex transformations in higher dimensions are as basic as in the 2-dimensional case.

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