Vapnik-Chervonenkis Dimension and (Pseudo-)Hyperplane Arrangements*

Bernd Gärtner and Emo Welzl
Institut für Informatik, Freie Universität Berlin
Takustraße 9, 14195 Berlin, Germany
e-mail: gaertner@inf.fu-berlin.de, emo@inf.fu-berlin.de

October 15, 1997

Abstract

An arrangement of oriented pseudohyperplanes in affine d-space defines on its set X of pseudohyperplanes a set system (or range space) $(X, \mathcal{R})$, $\mathcal{R} \subseteq 2^X$ of VC-dimension d in a natural way: to every cell c in the arrangement assign the subset of pseudohyperplanes having c on their positive side, and let $\mathcal{R}$ be the collection of all these subsets. We investigate and characterize the range spaces corresponding to *simple* arrangements of pseudohyperplanes in this way; such range spaces are called *pseudogeometric*, and they have the property that the cardinality of $\mathcal{R}$ is maximum for the given VC-dimension. In general, such range spaces are called *maximum*, and we show that the number of ranges $R \in \mathcal{R}$ for which also $X - R \in \mathcal{R}$, determines whether a maximum range space is pseudogeometric. Two other characterizations go via a simple duality concept and 'small' subspaces. The correspondence to arrangements is obtained indirectly via a new characterization of uniform oriented matroids: a range space $(X, \mathcal{R})$ naturally corresponds to a uniform oriented matroid of rank $|X| - d$ if and only if its VC-dimension is d, $R \in \mathcal{R}$ implies $X - R \in \mathcal{R}$ and $|\mathcal{R}|$ is maximum under these conditions.

Keywords: VC-dimension, hyperplane arrangements, oriented matroids, pseudohyperplane arrangements.

1 Introduction and Statement of Results

Set systems of finite VC-dimension have been investigated since the early seventies (starting with [She], [Sau], and [VC1]), and the concept has found numerous applications in statistics (e.g. [Dud1, Vap, Ale, GZ, Dud2, Pol1, Tal, Pol2]), combinatorics (e.g. [Ass, DSW, Hau, KPW, MWW]), learning theory (e.g. [BEHW, Pea, HKS, BH, GJ, LMR, Flo], see also the books [Nat, AB]), and computational geometry (e.g. [CM, CW, HW, Mat]). Although the VC-dimension is a purely combinatorial parameter associated with a set system, it seems that it is mainly applicable to (and naturally occurs in) geometric settings, i.e. when the set system $(X, \mathcal{R})$ is obtained with X as a set of points in d-space, and with $\mathcal{R}$ containing the intersections of X with certain *ranges* in d-space (hyperplanes,

*Part of this work was done while the first author was member of the Graduiertenkolleg "Algorithmische Diskrete Mathematik", supported by the Deutsche Forschungsgemeinschaft, grant We 1265/2-1. Part of this work has been supported by the German-Israeli Foundation for Scientific Research and Development (G.I.F.).

halfspaces, balls, simplices, etc.). That is why we use the terms *range space* for $(X, \mathcal{R})$, and *range* for a set in $\mathcal{R}$.

The goal of this paper is to elaborate on this connection to geometry, in particular to arrangements of (oriented) hyperplanes. We will succeed in characterizing those range spaces – called *pseudogeometric* range spaces – which come from hyperplanes, but we have to respectt the usual frontiers of such combinatorial characterizations (pseudolines [Lev, Grü2, GP2, Rin], circular sequences [Per, GP1], oriented matroids [BL, FL, EM, BLSWZ]): we cannot distinguish between stretchable and non-stretchable pseudoline (or pseudohyperplane) arrangements, so our analogy is actually to simple *pseudohyperplane* arrangements.

Intuitively speaking, arrangements of pseudohyperplanes consist of 'topological' hyperplanes with the same intersection properties as straight hyperplanes, so they differ from the usual arrangements only with respect to the geometric notion of straightness that is not 'recognized' by combinatorial structures like range spaces.

A key concept in our approach is to exploit the structure of range spaces induced by maximality conditions on the number of ranges; an interesting new insight we have to offer in this context is the fact that in order to tell whether a range space $(X, \mathcal{R})$ is pseudogeometric, it suffices to count the number of ranges $R \in \mathcal{R}$ for which the complement $X - R$ is in $\mathcal{R}$; this characterization presumes that $(X, \mathcal{R})$ is *maximum*, i.e. $|\mathcal{R}|$ is maximum for the VC-dimension of $(X, \mathcal{R})$. This is also the basis of another characterization where we show that it suffices to consider 'small' subspaces to decide upon the pseudogeometric nature of the range space.

We consider also range spaces where $|\mathcal{R}|$ is maximum under the additional restriction that $\mathcal{R}$ is *closed*, i.e. $R \in \mathcal{R}$ implies $X - R \in \mathcal{R}$. On the one hand, this class has a close relation to pseudogeometric range spaces and, on the other hand, is already powerful enough to encode uniform oriented matroids. These combinatorial objects are known to have topological representations as arrangements of pseudohyperplanes in projective space. They will form the 'bridge' between pseudogeometric range spaces and the affine arrangements of pseudohyperplanes.

We want to avoid to introduce arrangements of pseudohyperplanes formally in this paper. However, this raises the problem of properly defining pseudogeometric range spaces. Our approach will be to extract just one intuitive property that one 'expects' these arrangements to have, and use it for the definition. Only at the end of the paper we will justify this proceeding by relating the range spaces obtained in this way to oriented matroids. This has the advantage that the paper presents itself at a completely combinatorial level. The correspondence between oriented matroids and actual arrangements will not be dealt with here, but it can be found elsewhere [BLSWZ, EM, FL].

The reader familiar with oriented matroid terminology will discover coincidences in concepts and statements. However, we will avoid to refer to this terminology until the end of the paper. The starting point of this research was the investigation of range spaces, and we feel that the corresponding language is appealing for a first encounter with the subject. A section that shows how our results are related to known facts and imply new statements in the world of oriented matroids will follow our presentation.

In the rest of this section we will formally introduce the crucial concepts and state our results. Proofs and the introduction of further (mainly technical) tools are postponed to the rest of the paper.

Range spaces, VC-dimension, and the fundamental lemma. We start by reviewing the basic definitions and facts about VC-dimension. We will use the term 'range space' rather than 'set system' or 'hypergraph', because of the motivating examples and in order to distinguish from graphs when we use them as tools.

Definition 1 *A* range space *is a pair $\mathcal{S} = (X, \mathcal{R})$, with X a set and $\mathcal{R} \subseteq 2^X$. The elements in X are called* elements *of $\mathcal{S}$, and the sets in $\mathcal{R}$ are called* ranges*. $\mathcal{S}$ is called* finite*, if X is finite.*

For $Y \subseteq X$, the restriction of $\mathcal{S}$ to Y *is defined by $\mathcal{S}|_Y = (Y, \mathcal{R}|_Y)$, $\mathcal{R}|_Y := \{R \cap Y \mid R \in \mathcal{R}\}$. We say that Y is* shattered by $\mathcal{R}$ *if $\mathcal{R}|_Y = 2^Y$.*

The VC-dimension *of S, denoted by* $\dim(\mathcal{S})$*, is the maximum cardinality of a set $Y \subseteq X$ shattered by $\mathcal{R}$; if $\mathcal{R}$ is empty, then we define the VC-dimension to be -1.*

For example, if X is the set of real numbers, and the set $\mathcal{R}$ of ranges is determined by intersecting X with intervals, then no three-element set is shattered: we can never 'cut out' the smallest and largest out of three numbers by an interval. Since any two number set can be shattered, the VC-dimension of this range space is two. Many more examples can be obtained via geometric ranges, some of which we will meet shortly.

Obviously, the number of intervals defined on n real numbers is quadratic in n. The following lemma shows that this – as an upper bound – follows already from the fact that the range space has VC-dimension two. The lemma can be seen as the fundamental lemma and the starting point of investigations of VC-dimension, and it was proved independently (and with different motivations) by Shelah [She], Sauer [Sau] (answering a question of Erdős), and Vapnik and Chervonenkis [VC1]. Although this lemma (and some notions we will use in the sequel) can be formulated for infinite range spaces as well, we will restrict our attention to the finite case, which is the one occurring in our application. Therefore, in all subsequent considerations *any range space is assumed to be finite.*

In the following we will use the integer function

$$\Phi_d(n) := \binom{n}{\leq d} = \sum_{i=0}^{d} \binom{n}{i} = O(n^d)$$

for $d \geq -1$ and $n \geq 0$. Φ is additive in the following sense:

Fact 2

$$\Phi_d(n) = \Phi_d(n-1) + \Phi_{d-1}(n-1), \ \textit{for } d \geq 0, n \geq 1.$$

Lemma 3 *Let $(X, \mathcal{R})$ be a range space of VC-dimension d. Then $|\mathcal{R}| \leq \Phi_d(|X|)$.*

To see that the bound is tight, let X be a finite set of at least d elements and let $\mathcal{R}$ be the set of all subsets of X with at most d elements. Clearly, the resulting range space has VC-dimension d, and indeed $|\mathcal{R}|$ attains the upper bound of the lemma. The above example with intervals is another example for VC-dimension two where the upper bound in Lemma 3 is attained. An interesting implication is that for fixed d, $|\mathcal{R}|$ can only be polynomial rather than exponential.

Maximum range spaces and range spaces from halfspaces. This paper concentrates on range spaces for which the upper bound in Lemma 3 is attained with equality:

Definition 4 *A range space $(X, \mathcal{R})$ of VC-dimension d is called* maximum *if $|\mathcal{R}|$ equals $\Phi_d(|X|)$.*

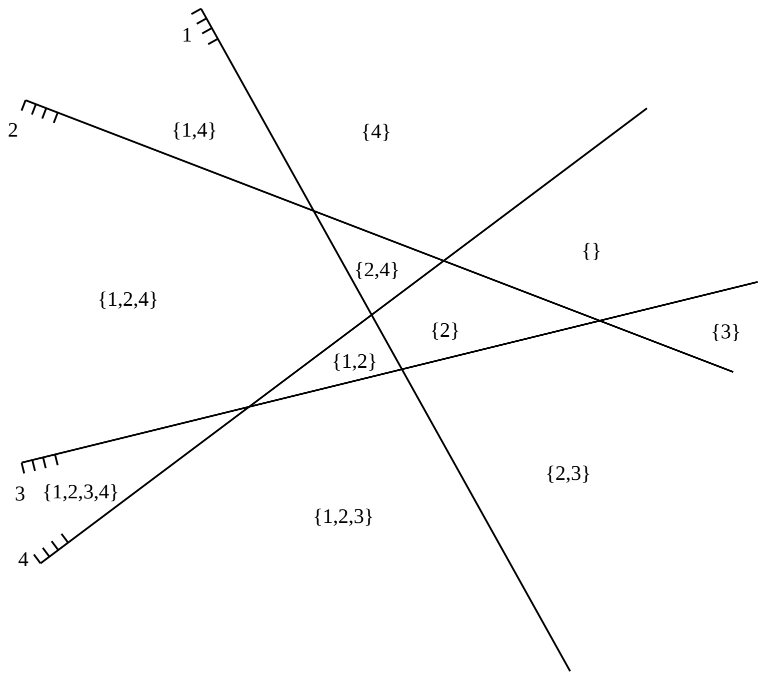

Figure 1: Description of cells of an oriented hyperplane arrangement

An interesting instance of a maximum space can be derived from an arrangement of hyperplanes. Let X be a set of n hyperplanes in affine d-space and let $\mathcal{A}(X)$ denote the arrangement formed by the hyperplanes. We assume X to be in general position, i.e. $n \geq d$, any d hyperplanes meet in a unique vertex, and any $d+1$ have empty intersection (this also excludes parallelities among the hyperplanes). Suppose that for every hyperplane one of the two halfspaces is distinguished as positive. Then each cell (or d-face) c of $\mathcal{A}(X)$ can be labeled with a subset of X, namely the set of hyperplanes which have c in its positive halfspace (Figure 1). If $\mathcal{R}$ denotes the set of all cell labels, then $\mathcal{S} = (X, \mathcal{R})$ is called the *description of cells* of $\mathcal{A}(X)$ [Ass, Dud1] and is maximum of VC-dimension d. This follows from the well-known fact that the number of cells of $\mathcal{A}(X)$ is exactly $\Phi_d(n)$ [Grü1, Zas, Ede].

A range space which stems from a set of oriented hyperplanes (or equivalently, from an arrangement of halfspaces) in this way is called *geometric*, and in the sequel we will assume any arrangement to be *simple*, i.e. in general position.

Pseudogeometric range spaces. A key step in many inductive proofs for arrangements of hyperplanes is to consider (i) the arrangement obtained by removing one of the hyperplanes and (ii) the arrangement (of one dimension smaller) obtained as the intersection of one of the hyperplanes with the remaining hyperplanes. We want corresponding operations for our range spaces. For a geometric range space, removing a hyperplane just means to remove its label from every range. For the other operation, observe that every $(d-1)$-face on a hyperplane x corresponds to two adjacent cells whose label sets differ exactly by x. That is, in the corresponding range space those adjacent cells give rise to pairs of ranges $(R, R \cup \{x\})$. This motivates the following definition for a general range space.

Definition 5 *For a range space $\mathcal{S} = (X, \mathcal{R})$ and $x \in X$, we define*

$$\mathcal{S}\backslash\{x\} = (X - \{x\}, \mathcal{R}\backslash\{x\}), \text{ where } \mathcal{R}\backslash\{x\} := \{R - \{x\} \mid R \in \mathcal{R}\}$$

and

$$\mathcal{S}/\{x\} = (X - \{x\}, \mathcal{R}/\{x\}), \text{ where } \mathcal{R}/\{x\} := \{R \in \mathcal{R} \mid x \notin R, R \cup \{x\} \in \mathcal{R}\}.$$

Since the pairs of ranges which differ in exactly one element seem to be crucial for the structure of a range space, we look at the collection of such pairs which yields a graph on the ranges. (We denote by $A \triangle B$ the symmetric difference of sets A and B.)

Definition 6 *For a range space $\mathcal{S} = (X, \mathcal{R})$, the* distance-1-graph $D^1(\mathcal{S})$ of $\mathcal{S}$ *is the undirected graph on vertex set $\mathcal{R}$ with edge set*

$$E := \{\{R, R'\} \subseteq \mathcal{R} \mid |R \triangle R'| = 1\},$$

where edge $\{R, R'\}$ is labeled with the unique element in $R \triangle R'$.

Let us consider a range space obtained from a 1-dimensional arrangement of hyperplanes, i.e. a set of points on a line. Then the resulting VC-dimension is one, and it is easy to see that the distance-1-graph is simply a path (connecting the cells in the order as they appear on the line). In general, we get the following nice property, proved e.g. in [Dud3, AHW] (for the sake of completeness, we provide a proof in this paper).

Lemma 7 *If $\mathcal{S} = (X, \mathcal{R})$ is a maximum range space of VC-dimension 1, then $D^1(\mathcal{S})$ is a tree, and each $x \in X$ occurs exactly once as an edge label of $D^1(\mathcal{S})$.*

If the edges are directed (each edge pointing towards the respective larger set), any such directed tree determines a maximum range space. Hence, there is a natural one-to-one correspondence between directed trees and maximum range spaces of VC-dimension one. It is quite easy to see that whenever the distance-1-graph is a path, then the range space is geometric (and vice versa). Consequently, geometric range spaces of VC-dimension one are completely characterized.

In order to carry this characterization to higher VC-dimension, we should at least require that in a geometric range space $(X, \mathcal{R})$ the subspace $\mathcal{R}/\{x\}$ (coming from the subarrangement on the hyperplane x) is geometric for all $x \in X$, and apply this property recursively until we reach the just settled one-dimensional case. This should also make sense if the arrangement in question actually consists of pseudohyperplanes (which coincide with hyperplanes in the one-dimensional case); based on this property we will define pseudogeometric range spaces. As mentioned above, the question whether the following definition really describes the range spaces coming from arrangements of pseudohyperplanes, will become an issue only in the last section. For the time being it suffices to have a formal definition we can work with, along with the intuition that it describes arrangements.

Definition 8 *A range space $\mathcal{S} = (X, \mathcal{R})$ of VC-dimension d is called* pseudogeometric *if it is maximum and either*

(i) $d \leq 0$, or
(ii) $d = 1$ and $D^1(\mathcal{S})$ is a path, or
(iii) $d \geq 2$ and $\mathcal{S}/\{x\}$ is pseudogeometric for all $x \in X$.

It is interesting to observe that the first example of a maximum range space we had (take as ranges all sets of up to d elements) is as non-geometric as possible. For example, for $d = 1$ this gives a range space where the distance-1-graph is a star.

We will now proceed by exhibiting (probably easier to grasp) equivalent conditions for a maximum range space to be pseudogeometric. While the necessity of these conditions will be quite obvious (from the geometric intuition), it is somewhat surprising that they are already sufficient.

Duality and characterization via small subspaces.

Definition 9 *For a range space $\mathcal{S} = (X, \mathcal{R})$ the* (complementary) dual[1] $\mathcal{S}^*$ *of $\mathcal{S}$ is defined as*

$$\mathcal{S}^* = (X, \mathcal{R}^*), \text{ where } \mathcal{R}^* := 2^X - \mathcal{R} .$$

We will prove that the dual of a maximum range space of VC-dimension d with n elements is again maximum of VC-dimension $n-d-1$. Similarly, we get for pseudogeometric range spaces:

Theorem 10 *A range space is pseudogeometric if and only if its complementary dual is pseudogeometric.*

In particular, this implies that if $\mathcal{S} = (X, \mathcal{R})$ is pseudogeometric of VC-dimension d, and $|X| = d + 2$, then $\mathcal{S}^*$ is pseudogeometric of VC-dimension 1 and so its structure is completely determined, which – vice versa – implies that the structure of $\mathcal{S}$ is completely determined (we will be more specific about this later). This is in analogy to the fact that – with respect to combinatorial type – there is only one simple d-dimensional arrangement of $d + 2$ (pseudo-)hyperplanes.

We can also prove that for determining whether a maximum range space of VC-dimension d is pseudogeometric, it suffices to look at all the $(d + 2)$-element subspaces.

Theorem 11 *Let $\mathcal{S} = (X, \mathcal{R})$ be maximum of VC-dimension d. The following statements are equivalent:*

(i) $\mathcal{S}$ is pseudogeometric.
(ii) $\mathcal{S}|_Y$ is pseudogeometric for all $Y \subseteq X$, $|Y| = d + 2$.
(iii) $\mathcal{S}|_Y$ is geometric for all $Y \subseteq X$, $|Y| = d + 2$.

Characterization via cardinality of boundary. The number of unbounded cells in a simple hyperplane arrangement of n hyperplanes in d-space is $2\Phi_{d-1}(n - 1)$. This can easily be seen by choosing one of the hyperplanes, call it h, and considering two hyperplanes parallel to h on either side, sufficiently far away so that all unbounded (and only unbounded) cells are intersected. In terms of the corresponding range space, the labels associated with these unbounded cells are those where also the complementary label appears.

Definition 12 *For a range space $\mathcal{S} = (X, \mathcal{R})$ the* (complementary) boundary *is defined as*

$$\partial\mathcal{S} = (X, \partial\mathcal{R}), \text{ where } \partial\mathcal{R} := \{R \in \mathcal{R} \,|\, X - R \in \mathcal{R}\}.$$

[1] This notion of duality is different from the 'standard' one frequently used in computational geometry, where one associates with $(X, \mathcal{R})$ the dual $(\mathcal{R}, \mathcal{X})$ with $\mathcal{X} := \{\mathcal{R}_x \mid x \in X\}$, $\mathcal{R}_x = \{R \in \mathcal{R} \mid x \in R\}$.

Similar as in Lemma 3 we can prove an upper bound for $|\partial\mathcal{R}|$, namely $|\partial\mathcal{R}| \leq 2\Phi_{d-1}(n-1)$ for a range space $(X,\mathcal{R})$ with $|X| = n$ and $\dim(X,\mathcal{R}) = d$. Again simple hyperplane arrangements give rise to range spaces which attain this bound, and actually we get:

Theorem 13 *A maximum range space $(X,\mathcal{R})$ of VC-dimension $d \geq 0$ is pseudogeometric if and only if $|\partial\mathcal{R}| = 2\Phi_{d-1}(|X|-1)$.*

Correspondence to oriented matroids. In order to relate pseudogeometric range spaces to simple arrangements of oriented pseudohyperplanes we exploit the representation theorem of Folkman & Lawrence [FL] that relates such arrangements to oriented matroids. To this end we need to introduce a new class of range spaces, called *pseudohemispherical* range spaces. This is due to the fact that pseudogeometric spaces come from arrangements in *affine* space while oriented matroids correspond to arrangements in *projective* space. The pseudohemispherical property is the 'projective version' of the pseudogeometric one:

Definition 14 *Let $\mathcal{S} = (X,\mathcal{R})$ be a range space. The* (complementary) closure *of $\mathcal{S}$ is the range space*

$$\overline{\mathcal{S}} = (X,\overline{\mathcal{R}}), \text{ where } \overline{\mathcal{R}} := \mathcal{R} \cup \{X - R \mid R \in \mathcal{R}\}.$$

$\mathcal{S}$ is called closed*, if $\mathcal{S} = \overline{\mathcal{S}}$.*

Definition 15 *Let $\mathcal{S} = (X,\mathcal{R})$ be a range space of VC-dimension $d \geq 1$. $\mathcal{S}$ is called* pseudohemispherical *if there exists a pseudogeometric space $\mathcal{T} \neq \mathcal{S}$ with $\mathcal{S} = \overline{\mathcal{T}}$. $\mathcal{T}$ is called an* underlying *space of $\mathcal{S}$.*

In order to get an intuitive idea what this definition means, recall that the d-dimensional projective space can be visualized as the sphere S^d with hyperplanes being great $(d-1)$-spheres, and we can get from an affine hyperplane arrangement to its corresponding projective one as follows. Think of E^d as the tangential hyperplane touching $S^d \subseteq E^{d+1}$ in the north pole. E^d can be mapped bijectively to the open northern hemisphere of S^d using central projection (with the center of the sphere as center of projection). This transformation takes a hyperplane h of E^d to a relatively open great halfsphere of dimension $d-1$. This halfsphere can be continued to a full great $(d-1)$-sphere in S^d, so an arrangement of hyperplanes in E^d induces an arrangement of great spheres in S^d. This is a projective arrangement and the equator plays the role of the 'line at infinity' (Figure 2). Moreover, if we have positive and negative halfspaces associated with the hyperplanes, this information in an obvious way determines positive and negative hemispheres associated with the great spheres, so that we obtain an *arrangement of hemispheres* in S^d. Since an antipodal cell has been generated for every cell in the underlying hyperplane arrangement, the corresponding description of cells (defined in the obvious way as for halfspace arrangements) is the closure of a geometric range space and will be called a *hemispherical* range space. Consequently, we will call the closure of a pseudogeometric range space *pseudohemispherical*.

Under the closure operation we lose information, since different pseudogeometric range spaces can have the same closure. This corresponds to the fact that depending on where the equator is chosen in an arrangement of hemispheres, the underlying affine arrangement changes. However, note that by 'fixing' the equator we get a one-to-one correspondence.

Definition 16 *For a range space $\mathcal{S} = (X,\mathcal{R})$ and e a distinguished element not in X, the range space*

$$\hat{\mathcal{S}} = (X \cup \{e\}, \hat{\mathcal{R}}) \text{ with } \hat{\mathcal{R}} := \mathcal{R} \cup \{(X \cup \{e\}) - R \mid R \in \mathcal{R}\}$$

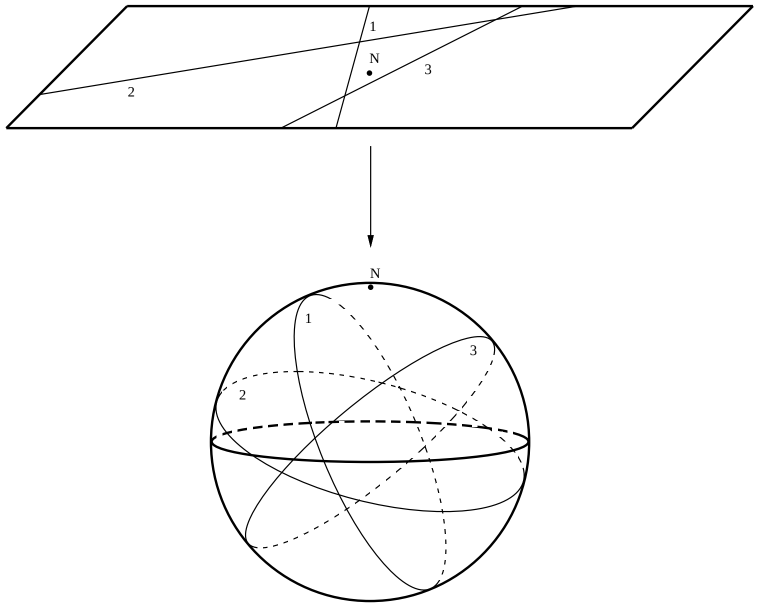

Figure 2: From halfspaces to hemispheres

is called the extended closure *of* $\mathcal{S}$.

It is not surprising from the intuition that the extended closure of a pseudogeometric range space is pseudohemispherical as well.

Theorem 17 *The mapping* $\mathcal{S} \mapsto \hat{\mathcal{S}}$ *forms a bijection between the pseudogeometric range spaces on* X *and the pseudohemispherical range spaces on* $X \cup \{e\}$.

It turns out that a pseudohemispherical space $\mathcal{S} = (X, \mathcal{R})$ of VC-dimension d with $|X| = n$ has $|\mathcal{R}| = 2\Phi_{d-1}(n-1)$ ranges. This is the maximum number of ranges that a closed range space of this VC-dimension can have (the bound of Theorem 13). Moreover, the pseudohemispherical spaces are already characterized by this property, a fact that is not apparent from their rather clumsy definition. As a consequence we obtain a new and simple characterization of uniform oriented matroids. This will finally give us the relation to arrangements – details are given in the last section.

Theorem 18 *For a set* X *of cardinality* n *there exists a natural (one-to-one) correspondence between the uniform oriented matroids of rank* $n - d \geq 0$ *on* X *and the closed range spaces* $(X, \mathcal{R})$ *of VC-dimension* d *with* $|\mathcal{R}| = 2\Phi_{d-1}(n-1)$.

2 Basics and Maximum Range Spaces

This section will make the reader familiar with the necessary range space terminology and it presents basic properties of maximum range spaces. In particular, we will introduce minors (or subspaces) of range spaces and prove the fundamental lemma of VC-dimension theory as well as the related bound on the number of ranges in the boundary of a range space. We give equivalent characterizations of maximum range spaces and discuss the structure of their distance-1-graph.

Basics on range spaces.

Definition 19 *For a range space* $\mathcal{S} = (X, \mathcal{R})$, $Y \subseteq X$, *we define*

$$\begin{aligned} \mathcal{S}\backslash Y &= (X - Y, \mathcal{R}\backslash Y), \text{ where } \mathcal{R}\backslash Y := \{R - Y \mid R \in \mathcal{R}\}, \\ \mathcal{S}/Y &= (X - Y, \mathcal{R}/Y), \text{ where } \mathcal{R}/Y := \{R \in \mathcal{R} \mid R \cap Y = \emptyset, R \cup Y' \in \mathcal{R}\ \forall\ Y' \subseteq Y\}. \end{aligned}$$

$\mathcal{S}\backslash Y$ and $\mathcal{S}/Y$ are the *minors* of $\mathcal{S}$ with respect to Y. $\mathcal{S}\backslash Y$ is said to arise from $\mathcal{S}$ by *deletion* of Y, while $\mathcal{S}/Y$ arises by *contraction* of Y. In a natural way $\mathcal{S}\backslash Y$ and $\mathcal{S}/Y$ generalize $\mathcal{S}\backslash\{x\}$ and $\mathcal{S}/\{x\}$, as introduced in Definition 5. If $\mathcal{S}$ is geometric, $\mathcal{S}\backslash Y$ is obtained by deleting the hyperplanes in Y from the generating arrangement, while $\mathcal{S}/Y$ corresponds to the subarrangement induced by the remaining hyperplanes in the flat $\bigcap_{h \in Y} h$.

If Y is nonempty and $y_1, ..., y_k$ is an arbitrary ordering of the elements of Y, then clearly $\mathcal{S}\backslash Y = \mathcal{S}\backslash\{y_1\}\backslash...\backslash\{y_k\}$. Via an easy induction part (i) of the following lemma also implies $\mathcal{S}/Y = \mathcal{S}/\{y_1\}/.../\{y_k\}$.

Lemma 20 *Let* $\mathcal{S} = (X, \mathcal{R})$ *be a range space,* $x, y \in X, Y \subseteq X$.

(i) $\mathcal{R}/Y/\{x\} = \mathcal{R}/(Y \cup \{x\})$, *for* $x \notin Y$.
(ii) $|\mathcal{R}| = |\mathcal{R}\backslash\{x\}| + |\mathcal{R}/\{x\}|$.
(iii) $\mathcal{R}\backslash Y = \mathcal{R}|_{X-Y}$.
(iv) $\mathcal{R}/\{x\}\backslash\{y\} \subseteq \mathcal{R}\backslash\{y\}/\{x\}$.
(v) $\dim(\mathcal{S}) = d \geq 0$ *implies* $\dim(\mathcal{S}\backslash\{x\}) \leq d$, $\dim(\mathcal{S}/\{x\}) \leq d - 1$.

The proof requires only elementary set manipulations and is omitted for the sake of brevity. Now we are able to show the fundamental lemma of VC-dimension theory that establishes the bound $|\mathcal{R}| \leq \Phi_d(n)$ for any range space $(X, \mathcal{R})$ of VC-dimension d with $|X| = n$ elements.

For the proof of **Lemma 3** we proceed by induction on d and n. The assertion is easily seen to be true for $d \leq 0$ and for $n = d \geq 0$, since in this case $|\mathcal{R}| = 2^d = \sum_{i=0}^{d} \binom{n}{i} = \Phi_d(n)$.

Now assume $d > 0, n > d$. By hypothesis, the bound holds for $\mathcal{S}\backslash\{x\}$ and $\mathcal{S}/\{x\}$, $x \in X$. Using the preceding lemma this immediately yields

$$|\mathcal{R}\backslash\{x\}| \leq \Phi_d(n-1) \text{ and } |\mathcal{R}/\{x\}| \leq \Phi_{d-1}(n-1),$$

so

$$|\mathcal{R}| = |\mathcal{R}\backslash\{x\}| + |\mathcal{R}/\{x\}| \leq \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n)$$

by Fact 2. □

$\mathcal{S} = (X, \mathcal{R})$ is *maximal* if $\dim(X, \mathcal{R} \cup \{R\}) > \dim(X, \mathcal{R})$ for all $R \in 2^X - \mathcal{R}$. By the fundamental lemma every maximum space is maximal, but the converse is not true. As a counterexample consider the range space $(X, \mathcal{R})$ with

$$\begin{aligned} X &= \{1, 2, 3, 4\}, \\ \mathcal{R} &= \{\{1\}, \{2\}, \{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}. \end{aligned}$$

It is straightforward to check that $\mathcal{S}$ is maximal of VC-dimension 2 but not maximum, since $|\mathcal{R}| = 10 < \Phi_2(4) = 11$.

The fundamental lemma helps to prove another bound on the maximum number of ranges in the *boundary* of a range space (Definition 12). We obtain

Theorem 21 *Let $(X, \mathcal{R})$ be a range space of VC-dimension $d \geq 0$. Then*

$$|\partial\mathcal{R}| \leq 2\Phi_{d-1}(|X| - 1)$$

.

Proof. For $d = 0$ the bound is obvious. If $d > 0$ fix $x \in X$ and define $\mathcal{R}' := \{R \in \partial\mathcal{R} \mid x \notin R\}$. It is easily seen that if $Y \subseteq X - \{x\}$ is shattered by $\mathcal{R}'$, then $Y \cup \{x\}$ is shattered by $\partial\mathcal{R}$; so $\dim(X - \{x\}, \mathcal{R}') \leq d - 1$, which by the fundamental lemma implies $|\mathcal{R}'| \leq \Phi_{d-1}(|X| - 1)$. Finally, observe that $|\partial\mathcal{R}| = 2|\mathcal{R}'|$. □

Characterizing maximum range spaces. The extremal property defining maximum range spaces (Definition 4) does not give immediate insights into the structure of these range spaces, so it seems appropriate to look for equivalent characterizations that reveal more of it. For example, one can show that the maximum property is inherited by the minors, a fact that is the basis of many subsequent inductive proofs. Another useful property is that the maximum property is maintained under duality (Definition 9). Before we give a list of equivalent statements most of which characterize maximum range spaces via certain properties of minors, let us briefly discuss the relation between the two minor operations of deleting and contracting elements (Definition 19). The point we want to stress is that although they look like very different operations at first glance, they aren't. On the contrary, they should be considered as having equal rights with respect to all concepts in this paper. The reason is that deletion and contraction change their roles under duality:

Observation 22 *Let $\mathcal{S} = (X, \mathcal{R})$ be a range space, $Y \subseteq X$. Then*

(i) $(\mathcal{R} \backslash Y)^* = \mathcal{R}^*/Y$.
(ii) $(\mathcal{R}/Y)^* = \mathcal{R}^* \backslash Y$.

As it will turn out, we are only concerned with classes of range spaces that are closed under duality, so in any context referring to the structure of a range space the minor operations will appear in a completely symmetric way; if one of them is preferred in an argument, this is merely due to technical convenience. The symmetry already appears in the next theorem which will be the major tool to handle and manipulate maximum range spaces.

Theorem 23 *Let $\mathcal{S} = (X, \mathcal{R})$ be a range space, $d \geq 0$ a natural number with $|X| = n > d$. The following statements are equivalent:*

(i) $\mathcal{S}$ is maximum of VC-dimension d.
(ii) $\mathcal{S} \backslash \{x\}$ and $\mathcal{S}/\{x\}$ are maximum of VC-dimension d and $d-1$, respectively, for all $x \in X$.
(iii) $\dim(\mathcal{S}) = d$*, and $\mathcal{S} \backslash \{x\}$ and $\mathcal{S}/\{x\}$ are maximum of VC-dimension d and $d-1$, respectively, for some $x \in X$.*
(iv) $\dim(\mathcal{S}) = d$ *and $\mathcal{S}/\{x\}$ is maximum of VC-dimension $d-1$, for all $x \in X$.*
(v) $\dim(\mathcal{S}) = d$ *and $|\mathcal{R}/A| = 1$, for all $A \subset X$, $|A| = d$.*
(vi) $\mathcal{S}^*$ is maximum of VC-dimension $n - d - 1$.
(vii) $\dim(\mathcal{S}^*) = n - d - 1$ *and $\mathcal{S} \backslash \{x\}$ is maximum of VC-dimension d, for all $x \in X$.*
(viii) $\dim(\mathcal{S}^*) = n - d - 1$ *and $|\mathcal{R}|_A| = 2^{d+1} - 1$, for all $A \subseteq X$, $|A| = d+1$.*

To see that the the additional dimension requirements in some of the statements are necessary in order to guarantee equivalence with (i), consider $X = \{1, 2, 3\}$ and

$$\begin{aligned} \mathcal{R} &= \{\emptyset, \{1\}, \{2\}, \{1,2,3\}\} \text{ with } x = 1 \text{ for (iii),} \\ \mathcal{R} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\} \text{ for (iv), (v) and} \\ \mathcal{R} &= \{\{1,2\}, \{1,3\}, \{2,3\}\} \text{ for (vii), (viii).} \end{aligned}$$

Such examples exist for arbitrary $|X|$ and d.

Proof. We proceed by showing first the equivalence of statements (i) through (v), then we prove (i) $\Leftrightarrow$ (vi). Together, this yields the missing equivalences.

(i) $\Rightarrow$(ii) let $\mathcal{S}$ be maximum of VC-dimension d, $x \in X$. Then

$$\Phi_d(n) = |\mathcal{R}| = |\mathcal{R}\backslash\{x\}| + |\mathcal{R}/\{x\}| \leq \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n).$$

This yields $|\mathcal{R}\backslash\{x\}| = \Phi_d(n-1)$ and $|\mathcal{R}/\{x\}| = \Phi_{d-1}(n-1)$, so $\mathcal{S}\backslash\{x\}$ and $\mathcal{S}/\{x\}$ are maximum of VC-dimension d and $d-1$, respectively, for all $x \in X$.

(ii) $\Rightarrow$ (iii),(iv) we only need to show that $\dim(\mathcal{S}) = d$. Let $d' \geq d$ denote $\dim(\mathcal{S})$, and let A with $|A| = d'$ be shattered by $\mathcal{R}$. If $|X| > d'$ then there is $y \in X - A$, and A is shattered also by $\mathcal{R}\backslash\{y\}$. Since $\mathcal{S}\backslash\{y\}$ is of VC-dimension d we get $|A| = d$. If $|A| = |X| = d'$ then $\mathcal{R} = 2^A$ which implies $\mathcal{R}/\{x\} = 2^{A-\{x\}}$ for all $x \in X$, so $d-1 = \dim(\mathcal{S}/\{x\}) = |A| - 1$.

(iv) $\Rightarrow$ (i) we proceed by induction on n. If $n = d+1$, let Y be a set of cardinality d shattered by $\mathcal{R}$. Then $\mathcal{R}\backslash\{x\} = \mathcal{R}|_Y = 2^Y$ for x the unique element in $X - Y$, and observing that $2^d = \Phi_d(n-1)$ we obtain

$$|\mathcal{R}| = |\mathcal{R}\backslash\{x\}| + |\mathcal{R}/\{x\}| = 2^d + \Phi_{d-1}(n-1) = \Phi_d(n).$$

Thus S is maximum.

Now assume $n > d+1$ and choose $x \in X$. $\mathcal{S}\backslash\{x\}/\{y\}$ is of VC-dimension at most $d-1$ for all $y \neq x$, and applying Lemma 20(iv) we get

$$\Phi_{d-1}(n-2) \geq |\mathcal{R}\backslash\{x\}/\{y\}| \geq |\mathcal{R}/\{y\}\backslash\{x\}| = \Phi_{d-1}(n-2),$$

which holds because $\mathcal{S}/\{y\}\backslash\{x\}$ is maximum of VC-dimension $d-1$ by implication (i)$\Rightarrow$(ii). But then $\mathcal{S}\backslash\{x\}/\{y\} = \mathcal{S}/\{y\}\backslash\{x\}$, so $\mathcal{S}\backslash\{x\}/\{y\}$ is maximum of VC-dimension $d-1$. Since this holds for all y, $\mathcal{S}\backslash\{x\}$ is maximum of dimension d by the inductive hypothesis. Finally we get

$$|\mathcal{R}| = |\mathcal{R}\backslash\{x\}| + |\mathcal{R}/\{x\}| = \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n),$$

which means that $\mathcal{S}$ is maximum. The last equation also yields implication (iii)$\Rightarrow$ (i).

(i)$\Leftrightarrow$(v) to see that '$\Rightarrow$' holds, iterate implication (i)$\Rightarrow$(iv) d times, starting from $\mathcal{S}$. This shows that $\mathcal{S}/A$ is maximum of VC-dimension 0 for all $|A| = d$, which implies $|\mathcal{R}/A| = \Phi_0(n) = 1$. On the other hand, if $|\mathcal{R}/A| = 1$ then $\mathcal{S}/A$ is maximum of VC-dimension 0, for all $|A| = d$. Using the fact that $\dim(\mathcal{S}) = d$ and Lemma 20(v) we get $\dim(\mathcal{S}/B) = d - k$ for $|B| = k$. Iterative application of (iv)$\Rightarrow$(i) then shows that $\mathcal{S}$ is maximum.

(i)$\Leftrightarrow$ (vi) because of symmetry it suffices to show '$\Rightarrow$'; we have $2^n - \Phi_d(n) = \Phi_{n-d-1}(n)$, so it remains to show that $\mathcal{S}^*$ is of VC-dimension at most $n-d-1$. Assume on the contrary that there is $Y \subseteq X$, $|Y| = n - d$, shattered by $\mathcal{R}^*$. Then $|X - Y| = d$, and from (i)$\Rightarrow$ (v) we get that there is a unique range $R \in \mathcal{R}/(X - Y)$. Since $R \subseteq Y$, there is $R' \in \mathcal{R}^*$, such that $Y \cap R' = R$. This implies $R' \supseteq R$ and $R' - R$ contains no element of Y. But then R' is of the form $R' = R \cup Z, Z \subseteq X - Y$, which is a contradiction, since $R \in \mathcal{R}/(X - Y)$ implies that all the ranges of this form are contained in $\mathcal{R}$.

(vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) these equivalences are obtained by applying the 'dual' equivalences (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) to $\mathcal{S}^*$, together with Observation 22. ◻

Corollary 24 *Let $\mathcal{S} = (X, \mathcal{R})$ be maximum of VC-dimension $d, |X| = n$. Then for all $x, y \in X$*

$$\mathcal{S}/\{x\}\backslash\{y\} = \mathcal{S}\backslash\{y\}/\{x\}.$$

Proof. For $d \leq 0$ the statement is obvious, and for $n = d$ we have $\mathcal{R}/\{x\}\backslash\{y\} = \mathcal{R}\backslash\{y\}/\{x\} = 2^{X-\{x,y\}}$. In any other case the theorem implies

$$|\mathcal{R}/\{x\}\backslash\{y\}| = |\mathcal{R}\backslash\{y\}/\{x\}| = \Phi_{d-1}(n-2).$$

Together with Lemma 20(iv) the claim follows. ◻

The distance-1-graph. We introduce the notion of 'swapping' as a tool to simplify subsequent considerations. In case of geometric range spaces, this operation corresponds to the reorientation of hyperplanes in the generating arrangement.

Definition 25 *For $\mathcal{S} = (X, \mathcal{R})$ and $D \subseteq X$, $\mathcal{S}$* swapped *D is the the range space*

$$\mathcal{S} \triangle D = (X, \mathcal{R} \triangle D) \text{ with } \mathcal{R} \triangle D := \{R \triangle D \mid R \in \mathcal{R}\}.$$

Lemma 26 *For any range space $\mathcal{S} = (X, \mathcal{R}), D \subseteq X$ we have*

(i) $|\mathcal{R} \triangle D| = |\mathcal{R}|$.
(ii) $\dim(\mathcal{S} \triangle D) = \dim(\mathcal{S})$.

We have already indicated that the *distance-1-graph* (Definition 6) captures crucial properties of a range space. In particular, pseudogeometric spaces are defined via a certain property of it (Definition 8). We will conclude this section by exhibiting a basic feature of the D^1-graph in the case of maximum range spaces, and we use the fact that swapping does not change the D^1-graph (strictly speaking, $D^1(\mathcal{S})$ and $D^1(\mathcal{S} \triangle D)$ are isomorphic with corresponding edges having the same labels). For geometric spaces this reflects the fact that reorienting some hyperplanes does not change the combinatorial structure of the arrangement. So whenever we consider some structural property of $D^1(\mathcal{S})$ (isomorphism type, connectivity, etc.) we are free to replace $\mathcal{S}$ with some swapped version $\mathcal{S} \triangle D$, and an appropriate choice of D may result in shorter and more elegant formulations.

The key result on the D^1-graph of a maximum range space is that it is connected. Actually, there holds a stronger property: any two ranges are joined by a path of the shortest possible length which equals the cardinality of their symmetric difference (for a characterization of such graphs see [Djo]). First we need a lemma:

Lemma 27 *Let $\mathcal{S} = (X, \mathcal{R})$ be maximum of VC-dimension $d \geq 1$ and assume $X \in \mathcal{R}$. Then, for all $R \in \mathcal{R}, R \neq X$ there exists $x \in X$ such that $R \cup \{x\} \in \mathcal{R}$.*

Proof. We proceed by induction on $n := |X|$. For $n = d$, any subset of X is a range so the lemma holds in this case. Now assume $n > d$ and consider $R \in \mathcal{R}, R \neq X$. Choose $y \in X$ with $y \notin R$. If $R = X - \{y\}$ then $R \cup \{y\} \in \mathcal{R}$. Otherwise the inductive hypothesis applies to $R \in \mathcal{R}\backslash\{y\}$, so there exists $z \in X - \{y\}$ with $R \in \mathcal{R}\backslash\{y\}/\{z\} = \mathcal{R}/\{z\}\backslash\{y\}$ (Corollary 24). This is equivalent to $R \in \mathcal{R}/\{z\}$ or $R \cup \{y\} \in \mathcal{R}/\{z\}$, which implies $R \cup \{z\} \in \mathcal{R}$ or $R \cup \{y\} \in \mathcal{R}$. ◻

Theorem 28 *Let $\mathcal{S} = (X, \mathcal{R})$ be maximum of VC-dimension $d \geq 1$. For any two ranges $R, R' \in \mathcal{R}$ there is a path of length $\delta(R, R') := |R \triangle R'|$ joining R and R' in $D^1(\mathcal{S})$.*

Proof. By swapping assume $R' = X$ and iterate the lemma. ◻

In case of $\dim(\mathcal{S}) = 1$, $D^1(\mathcal{S})$ is a tree on $\mathcal{R}$ with every element of X occurring exactly once as an edge label. This has been stated in **Lemma 7**, and now it is easy to prove. From the previous theorem we get that $D^1(\mathcal{S})$ is connected. To see that it is acyclic note that $x \in X$ occurs exactly $|\mathcal{R}/\{x\}| = 1$ times as an edge label. On the other hand it is an easy observation that if $x \in X$ occurs as a label in a cycle of edges then it has to occur at least twice in this cycle. It follows that there can be no cycle. ◻

3 Pseudogeometric Range Spaces

In this section we basically prove the characterizations of pseudogeometric spaces via duality (Theorem 10), small subspaces (Theorem 11) and cardinality of boundary (Theorem 13). The latter will be based on a version of Levi's Enlargement Lemma for pseudogeometric spaces. Before this we present a characterization theorem similar to Theorem 23 for maximum spaces.

Let us review the definition of pseudogeometric spaces; the following is just the non-recursive version of Definition 8.

Lemma 29 *A maximum range space $\mathcal{S} = (X, \mathcal{R})$ of VC-dimension d is pseudogeometric if either $d \leq 0$ or $d > 0$ and $D^1(\mathcal{S}/Y)$ is a path for any Y with $|Y| = d - 1$.*

Observe that for $|X| \leq d + 1$, any maximum space is pseudogeometric. As in the maximum case, we can come up with a list of equivalent statements characterizing the pseudogeometric property:

Theorem 30 *Let $\mathcal{S} = (X, \mathcal{R})$ be a range space, $d \geq 2$ a natural number with $|X| = n > d + 2$. The following statements are equivalent:*

(i) $\mathcal{S}$ is pseudogeometric of VC-dimension d.

(ii) $\mathcal{S} \setminus \{x\}$ and $\mathcal{S}/\{x\}$ are pseudogeometric of VC-dimension d and $d - 1$, respectively, for all $x \in X$.

(iii) $\dim(\mathcal{S}) = d$ and $\mathcal{S}/\{x\}$ is pseudogeometric of VC-dimension $d - 1$, for all $x \in X$.

(iv) $\dim(\mathcal{S}) = d$ and $\mathcal{S}/A$ is pseudogeometric of VC-dimension 1, for all $A \subset X$, $|A| = d - 1$.

(v) $\mathcal{S}^*$ is pseudogeometric of VC-dimension $n - d - 1$.

(vi) $\dim(\mathcal{S}^*) = n - d - 1$ and $\mathcal{S} \setminus \{x\}$ is pseudogeometric of VC-dimension d, for all $x \in X$.

(vii) $\dim(\mathcal{S}^*) = n - d - 1$ and $\mathcal{S}|_A$ is pseudogeometric of VC-dimension d, for all $A \subset X$, $|A| = d + 2$.

Note that the equivalence of (i) and (v) yields **Theorem 10**.

Compared with the corresponding Theorem 23 for maximum range spaces, we lose the characterizations via the minors $\mathcal{S}/A$ for $|A| = d$ and $\mathcal{S}|_A$ for $|A| = d + 1$ – they can be pseudogeometric even if $\mathcal{S}$ is not. However, if we consider minors on one element more, i.e. $\mathcal{S}/A$ for $|A| = d - 1$ and $\mathcal{S}|_A$ for $|A| = d + 2$ then we can already recognize the pseudogeometric property.

An analogue of statement (iii) in Theorem 23 cannot be added here. There are cases where $\mathcal{S}\backslash\{x\}$ and $\mathcal{S}/\{x\}$ are pseudogeometric of VC-dimension d and $d-1$, respectively, for some x, but $\mathcal{S}$ itself is not pseudogeometric. To get such an example, let $\mathcal{S}' = (X, \mathcal{R}')$ be a pseudogeometric range space, fix $x \in X$ and define $\mathcal{S} = (X, \mathcal{R})$ by $\mathcal{R} := \mathcal{R}'/\{x\} \cup \{R \cup \{x\} \mid R \in \mathcal{R}' - \mathcal{R}'/\{x\}\}$, i.e. $\mathcal{R}$ arises from $\mathcal{R}'$ by adding x to every range not in $\mathcal{R}/\{x\}$ (this is known as 'shifting' [Hau, Ste]). We get $\mathcal{R}\backslash\{x\} = \mathcal{R}'\backslash\{x\}$ and $\mathcal{R}/\{x\} = \mathcal{R}'/\{x\}$, so these minors of $\mathcal{R}$ will be pseudogeometric. On the other hand it is not hard to show that $\mathcal{S}$ is again maximum, but since for $A \subset X - \{x\}$ we have

$$\mathcal{R}/A = \mathcal{R}/A \cup \{x\} \cup \{R \cup \{x\} \mid R \in \mathcal{R}'/A - \mathcal{R}'/A \cup \{x\}\},$$

by choosing $x \in X$ and $|A| = d-1$ such that $\mathcal{R}/A \neq \mathcal{R}'/A$ (which we can do for $|X| > d$) we see that $\mathcal{S}/A$ is not pseudogeometric. Thus $\mathcal{S}$ cannot be pseudogeometric by definition.

Observe that we need to require $d \geq 2$ – otherwise statement (iii) only implies that $\mathcal{S}$ is maximum; the same holds for the requirement $|X| > d+2$ in connection with statement (vi).

Proof. The equivalence of (i), (iii) and (iv) just repeats Definition 8 and Lemma 29). Furthermore, (ii) immediately implies (iii). Implication (i) $\Rightarrow$ (ii) follows by observing that if $D^1(S/Y)$ is a path, this also holds for $D^1(S/x/Y)$, arising from $D^1(S/Y)$ by contracting the edge labeled with x. Equivalence (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) is dual to (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

We are left to prove equivalence (i) $\Leftrightarrow$ (iv), where because of symmetry one implication suffices. Assume $\mathcal{S}$ is pseudogeometric of VC-dimension d. By Theorem 23 the range space $\mathcal{S}^*$ is already maximum of VC-dimension $n-d-1$, so by Theorem 13 (which we will prove shortly) it suffices to show that $|\partial(\mathcal{R}^*)| = 2\Phi_{n-d-2}(n-1)$, which by an easy computation follows from $|\partial(\mathcal{R}^*)| = |\mathcal{R}^*| - |\mathcal{R} - \partial\mathcal{R}|$. ◽

The characterization of the pseudogeometric property via small subspaces (**Theorem 11**) is now an immediate consequence of the theorem. The requirement '$\dim(\mathcal{S}^*) = n-d-1$' can be omitted since it is already imposed by the maximum property of $\mathcal{S}$, and the fact that for $|Y| = d+2$ any pseudogeometric range space $\mathcal{S}|_Y$ is actually geometric follows by considering the dual range space $(\mathcal{S}|_Y)^* = \mathcal{S}^*/(X-Y)$ which is pseudogeometric of VC-dimension 1. Its D^1-graph is a path connecting all the ranges, so any two pseudogeometric range spaces of VC-dimension 1 on Y are isomorphic, i.e. equal up to swapping and renaming of elements. Of course, this carries over to the primal setting, so any arrangement of $d+2$ hyperplanes in d-spaces has to generate an isomorphic copy of $\mathcal{S}|_Y$, which means that this range space has to be geometric.

We have just mentioned the swap operation (Definition 25) in connection with pseudogeometric spaces, and it is quite clear that swapping does not affect the pseudogeometric property.

Lemma 31 *$\mathcal{S} = (X, \mathcal{R})$ is pseudogeometric if and only if $\mathcal{S} \triangle D$ is pseudogeometric, $D \subseteq X$.*

Levi's Enlargement Lemma. We are approaching the proof of Theorem 13 (the characterization of pseudogeometric range spaces via the number of ranges in the boundary). It will be based on a variant of *Levi's Enlargement Lemma* (Levi's Lemma for short) for pseudogeometric range spaces. The original version states that a pseudoline arrangement in the plane can be enlarged by a new pseudoline containing any two given points (which do not lie already on a common pseudoline). Although this fact is not very hard to prove, it should not be considered trivial: in three dimensions, it is not true that every pseudoplane arrangement can be enlarged by a pseudoplane containing three given points [GP3] (recently, Richter-Gebert [RiG] has shown that there are arrangements that do not even allow a new pseudoplane containing certain two points). However, it is true in all dimensions that any two points can be connected by a pseudoline, i.e. a curve in space which intersects (and crosses) every pseudohyperplane exactly once. In the following we define the range space analogue of such a curve:

Definition 32 *Let $\mathcal{S} = (X, \mathcal{R})$ be a range space. A* segment *in $\mathcal{S}$ is a set of ranges which can be enumerated as $\{R_0, ..., R_k\}$ such that for $1 \leq i \leq k$, $R_{i-1} \triangle R_i = \{x_i\}$, $x_1, ..., x_k$ distinct elements from X. The segment is said to* join *R_0 and R_k. $\mathcal{R}' \subseteq \mathcal{R}$* admits *a segment if there exists a segment containing $\mathcal{R}'$. The segment is a* line *if it joins complementary ranges $R, X - R$.*

Equivalently we could say that a line is a pseudogeometric subspace $(X, \mathcal{L})$, $\mathcal{L} \subseteq \mathcal{R}$, of VC-dimension 1. Note that Theorem 28 states that in a maximum range space any two ranges admit a segment. Using this fact we obtain

Lemma 33 *For $\mathcal{S} = (X, \mathcal{R})$ maximum, ranges R and R' admit a line if and only if there are ranges $T, X - T \in \mathcal{R}$ such that $R - R' \subseteq T$ and $R' - R \subseteq X - T$.*

Theorem 34 (Levi's Lemma) *If $\mathcal{S} = (X, \mathcal{R})$ is pseudogeometric of VC-dimension $d \geq 1$, then any two ranges $R, R' \in \mathcal{R}$ admit a line.*

Proof. We proceed by induction on d and $\delta(R, R') = |R \triangle R'|$.

The assertion is true for $d = 1$, since in this case $\mathcal{R}$ itself is a line. Furthermore, if $\delta(R, R') = 0$, i.e. $R = R'$, then the preceding lemma shows that it is sufficient to find one pair of complementary ranges $T, X - T$. Such a pair always exists, as follows by easy induction on d.

Now let $\mathcal{S} = (X, \mathcal{R})$ be pseudogeometric of VC-dimension $d > 1$, $R, R' \in \mathcal{R}$ with distance $\Delta := \delta(R, R') > 0$ and assume the theorem holds for any pseudogeometric range space of VC-dimension less than d and any pair of ranges with distance less than Δ in $\mathcal{R}$.

Consider a segment joining R and R' and let U be the range adjacent to R' on this segment. After swapping, if necessary, we may assume $R' = U \cup \{x\}$ for some $x \in X$. Since $\delta(R, U) = \Delta - 1$, R and U admit a line $\mathcal{L}$ by hypothesis, so there are ranges $T, X - T$ with

$$R - U \subseteq T, U - R \subseteq X - T.$$

If $x \in X - T$ then we obtain

$$R - R' \subseteq T, R' - R \subseteq X - T,$$

so we are done. Otherwise $x \in T$, and since $x \notin R$, by traversing $\mathcal{L}$ from R to T we encounter a range $S \in \mathcal{R}/\{x\}$. $\mathcal{S}/\{x\}$ is pseudogeometric, so by hypothesis there is a line in $\mathcal{R}/\{x\}$ containing S and U, so we have $T', X - \{x\} - T' \in \mathcal{R}/\{x\}$ with

$$S - U \subseteq T', U - S \subseteq X - \{x\} - T',$$

which yields

$$S - R' \subseteq T', R' - S \subseteq X - T'.$$

Now observe that $R - R' \subseteq S - R'$, $R' - R \subseteq R' - S$, which follows from the fact that S, R and $U = R' - \{x\}$ appear on the original line $\mathcal{L}$ in this order. Consequently, we get

$$R - R' \subseteq T', R' - R \subseteq X - T',$$

and together with the fact that $X - T'$ is a range in $\mathcal{R}$, this shows that R and R' admit a line in $\mathcal{S}$. ◻

For any $d \geq 2$, there are maximum range spaces of VC-dimension d which are not pseudogeometric, with the property that any two ranges admit a line (let $d+2 \leq |X| \leq 2d$ and $\mathcal{R} = \binom{X}{\leq d}$). For $d = 2$, however, the largest such example has 4 elements (see Theorem 37 below). The question whether this generalizes to higher VC-dimension is an interesting open problem.

Problem 35 *Given $d > 2$, does there exist a constant $C(d)$ such that for $\mathcal{S} = (X, \mathcal{R})$, maximum of VC-dimension d with $|X| > C(d)$, Levi's Lemma holds in $\mathcal{S}$ if and only if $\mathcal{S}$ is pseudogeometric? If the answer is yes, is $C(d) = 2d$?*

Here is a characterization that will be useful:

Lemma 36 *Let $\mathcal{S} = (X, \mathcal{R})$ be maximum; Levi's Lemma holds in $\mathcal{S}$ if and only if*

$$(\partial\mathcal{R})\backslash Y = \partial(\mathcal{R}\backslash Y)$$

for all subsets Y of X.

Proof. Observe that for any range space $(\partial\mathcal{R})\backslash Y \subseteq \partial(\mathcal{R}\backslash Y)$ holds for all Y. So it suffices to show the equivalence between Levi's Lemma and $\partial(\mathcal{R}\backslash Y) \subseteq (\partial\mathcal{R})\backslash Y$.

Consider first $\tilde{R} \in \partial(\mathcal{R}\backslash Y)$, and we want to show that $\tilde{R} \in \partial(\mathcal{R})\backslash Y$ follows already from Levi's Lemma. We get that (by definition) $X - Y - \tilde{R} \in \partial(\mathcal{R}\backslash Y)$, and so there must exist R, R' in $\mathcal{R}$ with $\tilde{R} = R - Y$ and $X - Y - \tilde{R} = R' - Y$. Levi's Lemma, which we assume to hold in $(X, \mathcal{R})$, gives us ranges $T, X - T$ in $\mathcal{R}$ (and so in $\partial\mathcal{R}$) with

$$R - R' \subseteq T, \quad R' - R \subseteq X - T\ .$$

Note that this yields

$$\tilde{R} = (R - R') - Y \subseteq T - Y, \quad X - Y - \tilde{R} = (R' - R) - Y \subseteq (X - T) - Y\ .$$

This shows that $\tilde{R} = T - Y$, and so $\tilde{R} \in \partial(\mathcal{R})\backslash Y$.

Now assume $\partial(\mathcal{R}\backslash Y) \subseteq (\partial\mathcal{R})\backslash Y$ for all $Y \subseteq X$. Consider ranges $R, R' \in \mathcal{R}$ and assume by swapping that $R = \emptyset$. Then we have

$$\emptyset = R \in \partial(\mathcal{R}\backslash(X - R')) \subseteq (\partial R)\backslash(X - R'),$$

which means that $\partial\mathcal{R}$ contains ranges $T \subseteq X - R'$ and $X - T \supseteq R'$. This yields $\emptyset = R - R' \subseteq T$ and $R' = R' - R \subseteq X - T$, so Levi's Lemma holds. ◻

We conclude by settling the 2-dimensional case:

Theorem 37 *Let $\mathcal{S} = (X, \mathcal{R})$ be maximum of VC-dimension 2, $|X| \geq 5$. $\mathcal{S}$ is pseudogeometric if and only if Levi's Lemma holds in $\mathcal{S}$.*

Proof. Consider first the case $|X| = 5$, and assume Levi's Lemma holds. From $(\partial\mathcal{R})\backslash Y = \partial(\mathcal{R}\backslash Y)$ it follows that $\partial\mathcal{R}$ shatters any two-element subset of X. With an easy case analysis one can check that this implies $|\partial\mathcal{R}| = 2\Phi_1(4) = 10$, so $\mathcal{R}$ is pseudogeometric by Theorem 13. For $|X| > 5$ observe that if Levi's Lemma holds in $\mathcal{S}$ then it also holds in $\mathcal{S}|_Y$ for any $|Y| = 5$. Consequently $\mathcal{S}|_Y$ is pseudogeometric and from Theorem 30 we obtain that $\mathcal{S}$ itself has to be pseudogeometric. ◻

Characterization via cardinality of boundary. Now we can prove **Theorem 13**, which states that a maximum range space $\mathcal{S} = (X, \mathcal{R})$ of VC-dimension $d \geq 0$ and $|X| = n$ is pseudogeometric if and only if $|\partial\mathcal{R}| = 2\Phi_{d-1}(n-1)$. This holds for $d = 0$, so assume $d > 0$.

First suppose that $\mathcal{S}$ is pseudogeometric. If $d = 1$, $\mathcal{R}$ is a line joining the only two complementary ranges of $\mathcal{R}$, so $|\partial\mathcal{R}| = 2 = 2\Phi_0(n-1)$. If $n = d$, then $|\partial\mathcal{R}| = |\mathcal{R}| = \Phi_d(n) = 2\Phi_{d-1}(n-1)$.

Now let $d > 1, n > d$ and inductively assume that $\partial(\mathcal{R}\backslash\{x\})$ and $\partial(\mathcal{R}/\{x\})$ have the right cardinalities for some $x \in X$. Levi's Lemma holds in $\mathcal{S}$, so we can apply Lemma 36 and obtain

$$\begin{aligned} |\partial\mathcal{R}| &= |(\partial\mathcal{R})\backslash\{x\}| + |(\partial\mathcal{R})/\{x\}| \\ &= |\partial(\mathcal{R}\backslash\{x\})| + |\partial(\mathcal{R}/\{x\})| \\ &= 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) = 2\Phi_{d-1}(n-1). \end{aligned}$$

Now assume $|\partial\mathcal{R}| = 2\Phi_{d-1}(n-1)$. We use induction on d to show that $\mathcal{S}$ is pseudogeometric. If $d = 1$, by Theorem 28 the ranges in $\partial\mathcal{R}$ — there are $2 = 2\Phi_0(n-1)$ of them — are joined by a path of length n in $D^1(\mathcal{S})$. Since $D^1(\mathcal{S})$ itself has only n edges it coincides with this path and so $\mathcal{S}$ is pseudogeometric.

Using Theorem 21 we get for $d > 1$ and $x \in X$

$$\begin{aligned} 2\Phi_{d-1}(n-1) = |\partial\mathcal{R}| &\leq |(\partial\mathcal{R})\backslash\{x\}| + |(\partial\mathcal{R})/\{x\}| \\ &\leq |\partial(\mathcal{R}\backslash\{x\})| + |\partial(\mathcal{R}/\{x\})| \\ &\leq 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) = 2\Phi_{d-1}(n-1), \end{aligned}$$

which, in particular, shows $|\partial(\mathcal{R}/\{x\})| = 2\Phi_{d-2}(n-2)$, so $\mathcal{S}/\{x\}$ is pseudogeometric by hypothesis. Since this holds for all $x \in X$, $\mathcal{S}$ is pseudogeometric (by definition). ◻

4 Pseudohemispherical Range Spaces

We have already introduced pseudohemispherical range spaces (Definition 15) which arise as the closure of pseudogeometric range spaces, and the intuition behind this definition was to have a class of range spaces generated by projective rather than affine arrangements. Theorem 17 states that both classes are in one-to-one correspondence provided we introduce a distinguished 'equator' element. This section will develop the basic properties of pseudohemispherical range spaces; the main statement will be a characterization via the number of ranges.

Let us start by showing that although the pseudogeometric space underlying a pseudohemispherical space is not unique, all underlying spaces have the same VC-dimension.

Lemma 38 *Let $\mathcal{S}$ be pseudohemispherical of VC-dimension $d \geq 1$ with underlying space $\mathcal{T}$. Then $\mathcal{T}$ is of VC-dimension $d-1$.*

Proof. Equivalently we show that if $\mathcal{T} = (X, \mathcal{R})$ with $\mathcal{T} \neq \overline{\mathcal{T}}$ is pseudogeometric of VC-dimension $d - 1 \geq 0$, then $\dim(\overline{\mathcal{T}}) = d$.

If $\mathcal{T} \neq \overline{\mathcal{T}}$, then $|X| \geq d$, so $\mathcal{T}|_Y$ is again pseudogeometric of VC-dimension $d-1$ for $|Y| \geq d$. We obtain

$$|\overline{\mathcal{R}|_Y}| = 2|\mathcal{R}|_Y| - |\partial(\mathcal{R}|_Y)| = 2\Phi_{d-1}(|Y|) - 2\Phi_{d-2}(|Y|-1) = 2\Phi_{d-1}(|Y|-1).$$

Any range space satisfies $\overline{\mathcal{R}|_Y} = \overline{\mathcal{R}}|_Y$, so

$$|\overline{\mathcal{R}}|_Y| = 2\Phi_{d-1}(|Y|-1).$$

For $|Y| = d$ this number equals 2^d, so Y is shattered by $\overline{\mathcal{R}}$, while for $|Y| \geq d+1$ the value is strictly smaller than $2^{|Y|}$, which implies that $\dim(\overline{\mathcal{T}}) = d$. □

From the lemma it follows that a pseudohemispherical space of VC-dimension d has $2\Phi_{d-1}(|X|-1)$ ranges, and from Theorem 21 we know that this number is maximum for closed range spaces (Definition 14). In analogy to the maximum spaces that attain the bound of Lemma 3 we define the concept of *c-maximum* spaces ('c' stands for 'closed').

Definition 39 *$\mathcal{S} = (X, \mathcal{R})$ closed of VC-dimension $d \geq 1$ with $|X| = n$ is called* c-maximum *if $|\mathcal{R}| = 2\Phi_{d-1}(n-1)$.*

Corresponding to Theorem 23 for maximum spaces we obtain similar characterizations also for c-maximum spaces (where only some numbers have to be adjusted):

Theorem 40 *Let $\mathcal{S} = (X, \mathcal{R})$ be a closed range space, $d \geq 2$ a natural number with $|X| = n > d$. Then the following statements are equivalent:*

(i) $\mathcal{S}$ is c-maximum of VC-dimension d.
(ii) $\mathcal{S}\backslash\{x\}$ and $\mathcal{S}/\{x\}$ are c-maximum of VC-dimension d and $d-1$, respectively, for all $x \in X$.
(iii) $\dim(\mathcal{S}) = d$, and $\mathcal{S}\backslash\{x\}$ and $\mathcal{S}/\{x\}$ are c-maximum of VC-dimension d and $d-1$, respectively, for some $x \in X$.
(iv) $\dim(\mathcal{S}) = d$ and $\mathcal{S}/\{x\}$ is c-maximum of VC-dimension $d-1$, for all $x \in X$.
(v) $\dim(\mathcal{S}) = d$ and $|\mathcal{R}/A| = 2$, for all $A \subseteq X$, $|A| = d-1$.
(vi) $\mathcal{S}^*$ is c-maximum of VC-dimension $n-d$.
(vii) $\dim(\mathcal{S}^*) = n-d$ and $\mathcal{S}\backslash\{x\}$ is c-maximum of VC-dimension d, for all $x \in X$.
(viii) $\dim(\mathcal{S}^*) = n-d$ and $|\mathcal{R}|_A| = 2^{d+1}-2$, for all $A \subseteq X$, $|A| = d+1$.

The proof is completely similar to the one of Theorem 23, so we do not repeat the arguments.

We also get

Theorem 41 *Let $\mathcal{S} = (X, \mathcal{R})$ be c-maximum of VC-dimension $d \geq 2$. For any two ranges $R, R' \in \mathcal{R}$ there is a path of length $\delta(R, R') = |R \triangle R'|$ joining R and R' in $D^1(\mathcal{S})$.*

Again the proof is almost literally the same as that of Theorem 28.

Pseudohemispherical spaces are c-maximum. The surprising fact is that the converse is also true:

Theorem 42 *Let $\mathcal{S} = (X, \mathcal{R})$ be closed of VC-dimension $d \geq 1, |X| = n$. $\mathcal{S}$ is pseudohemispherical if and only if $\mathcal{S}$ is c-maximum.*

Proof. We need to show that if $\mathcal{S}$ is a c-maximum space then $\mathcal{S}$ is pseudohemispherical, and we proceed by induction on d. If $\mathcal{S}$ is of VC-dimension 1 with $|\mathcal{R}| = 2 = 2\Phi_0(n-1)$ then $\mathcal{S} = (X, \{R, X-R\}), R \subseteq X$. Now $\mathcal{T} = (X, \{R\})$ is of VC-dimension 0 and hence pseudogeometric with $\mathcal{S} = \overline{\mathcal{T}}$.

Now suppose $d > 1$, $x \in X$. $\mathcal{S}/\{x\}$ is c-maximum, so $\mathcal{S}/\{x\}$ is pseudohemispherical of VC-dimension $d-1$ by hypothesis. Let $\mathcal{S}' = (X - \{x\}, \mathcal{R}')$ be a pseudogeometric space (of VC-dimension $d-2$) underlying $\mathcal{S}/\{x\}$ and consider the range space

$$\mathcal{T} = (X, \mathcal{R}' \cup \mathcal{R}''),$$

where $\mathcal{R}'' := \{R \in \mathcal{R} \mid x \in R\}$. Obviously $\mathcal{S} = \overline{\mathcal{T}}$, so to see that $\mathcal{S}$ is pseudohemispherical it remains to show that $\mathcal{T}$ is pseudogeometric of VC-dimension $d-1$.

The number of ranges of $\mathcal{T}$ is

$$|\mathcal{R}'| + |\mathcal{R}''| = \Phi_{d-2}(n-1) + \Phi_{d-1}(n-1) = \Phi_{d-1}(n).$$

Furthermore, $\mathcal{T}$ has $2|\mathcal{R}'| = 2\Phi_{d-2}(n-1)$ ranges in the boundary. If we can show that $\mathcal{T}$ is of VC-dimension at most $d-1$, then $\mathcal{T}$ is maximum and therefore pseudogeometric by Theorem 13. To this end consider $A \subseteq X$, such that A is shattered by $\mathcal{R}' \cup \mathcal{R}''$; we show that this implies $|A| \leq d-1$. There are two cases:

(a) $x \notin A$: For $R \in \mathcal{R}'$ we have $R \cup \{x\} \in \mathcal{R}''$, and since $A \cap R = A \cap (R \cup \{x\})$ we know that A is already shattered by $\mathcal{R}''$. This implies that $A \cup \{x\}$ is shattered by $\mathcal{R}$, so $|A \cup \{x\}| \leq d$, i.e. $|A| \leq d-1$.

(b) $x \in A$: By intersecting A with the ranges in $\mathcal{R}''$ we only get subsets of A that contain x. This means, $A - \{x\}$ is shattered by $\mathcal{R}'$. We get $|A - \{x\}| \leq d-2$, so $|A| \leq d-1$. □

We conclude with a proof of **Theorem 17** which states that the pseudogeometric spaces on X and the pseudohemispherical spaces on $X \cup \{e\}$, $e \notin X$ are in one-to-one correspondence via the *extended closure* that takes a range space $\mathcal{S} = (X, \mathcal{R})$ to $\hat{\mathcal{S}} = (X \cup \{e\}, \hat{\mathcal{R}})$ with

$$\hat{\mathcal{R}} := \mathcal{R} \cup \{(X \cup \{e\}) - R \mid R \in \mathcal{R}\}.$$

We show that $\mathcal{S}$ is pseudogeometric if and only if $\hat{\mathcal{S}}$ is pseudohemispherical.

First, let $\mathcal{S}$ be pseudogeometric of VC-dimension d. Then $|\hat{\mathcal{R}}| = 2|\mathcal{R}| = 2\Phi_d(|X|)$. Furthermore, $\dim(\mathcal{S}) = d+1$. To see this consider A shattered by $\hat{\mathcal{R}}$. If $e \in A$ then $A - \{e\}$ is shattered by $\mathcal{R}$, so $|A| \leq d+1$. Otherwise A is shattered already by $\hat{\mathcal{R}} \backslash \{e\} = \overline{\mathcal{R}}$, and we have $|A| \leq d+1$ also in this case by Lemma 38. It follows that $\hat{\mathcal{S}}$ is c-maximum and hence pseudohemispherical by Theorem 42.

If $\hat{\mathcal{S}}$ is pseudohemispherical (and hence c-maximum) of VC-dimension $d+1$, then $|\mathcal{R}| = |\hat{\mathcal{R}}|/2 = \Phi_d|X|$ (Theorem 40), and $\mathcal{S}$ is of VC-dimension at most d. From this it follows that $\mathcal{S}$ is maximum. Furthermore, $\partial\mathcal{R} = \hat{\mathcal{R}}/\{e\}$, so $|\partial\mathcal{R}| = 2\Phi_{d-1}(|X|-1)$ (again by Theorem 40), and $\mathcal{S}$ is pseudogeometric by Theorem 13. □

5 The Correspondence to Oriented Matroids.

The characterizations of pseudohemispherical range spaces developed in the previous section will form the basis of our proof that these spaces correspond to oriented matroids. These combinatorial objects have been independently introduced by Bland and Las Vergnas [BL] and Folkman and Lawrence [FL]. A comprehensive treatment of the known theory can be found in [BLSWZ].

It was first shown in [FL] that oriented matroids have natural representations as arrangements of pseudohemispheres, and vice versa. The oriented matroid approach can handle arbitrary arrangements, while we are only talking about *simple* arrangements in this paper; so we restrict our attention to *uniform* oriented matroids.

As in the case of ordinary matroids, there exist several equivalent axiomatizations of oriented matroids, most of which are abstractions of intuitive properties one observes by studying objects like directed graphs or hyperplane arrangements. We will choose the axiomatization in terms of *covectors*, which will turn out to be most suitable for our purposes. Chapter **3** of [BLSWZ] discusses the different axiomatizations and proves their equivalence; the terminology and 'background facts' are taken from this source.

Let X be a finite set. A *signed vector* on X is a mapping $F : X \to \{+,-,0\}$ (this will also be written as $F \in \{+,-,0\}^X$). The image of $x \in X$ under F is denoted by F_x. The *support* of F is defined as the set $\underline{F} := \{x \in X \mid F_x \neq 0\}$. Denote by F^i the set $\{x \in X \mid F_x = i\}$, for $i \in \{+,-,0\}$. F^0 is the *zero set* of F, F^+ and F^- the *positive* and *negative* sets, respectively.

$\mathbf{0}$ is the vector satisfying $\mathbf{0}_x = 0$ for all $x \in X$. $-F$ is defined by $(-F)_x := -(F_x)$. The restriction of F to $Y \subseteq X$ is denoted by $F|_Y$.

We say that $x \in X$ *separates* F and G if $F_x = -G_x \neq 0$.

A partial order $\leq$ is defined on signed vectors as follows:

$$F \leq G :\Leftrightarrow \forall x \in X : F_x = 0 \text{ or } F_x = G_x,$$

i.e. F can be obtained from G by switching some entries to zero.

The *composition* of signed vectors is defined by

$$(F_1 \circ F_2 \circ \ldots \circ F_k)_x := \begin{cases} (F_i)_x \text{ if } i = \min\{j \mid (F_j)_x \neq 0\} \text{ exists} \\ 0 \text{ otherwise} \end{cases}.$$

Definition 43 *Let X be a finite set, $\mathcal{L}$ a set of signed vectors on X. The pair $\mathcal{M} = (X, \mathcal{L})$ is called an* oriented matroid *if*

(V0) $\mathbf{0} \in \mathcal{L}$,
(V1) $\mathcal{L} = -\mathcal{L}$, *(symmetry)*
(V2) for all $F, G \in \mathcal{L}$ *we have* $F \circ G \in \mathcal{L}$, *(composition)*
(V3) for all $F, G \in \mathcal{L}$ *and* x *separating them there is* $H \in \mathcal{L}$ *with* $H_x = 0$ *and* $H_y = (F \circ G)_y = (G \circ F)_y$ *for all* y *not separating* F *and* G. *(elimination)*

$\mathcal{L}$ is the set of covectors *of the oriented matroid. The covector H in (V3) is said to* eliminate x between F *and* G.

The set $\mathcal{C} \subseteq \mathcal{L}$ of non-zero covectors which are *minimal* with respect to the partial order $\leq$ are called *cocircuits* or *vertices* of $\mathcal{M}$, and they already determine the oriented matroid: F is a covector if and only if it is the composition of cocircuits. Therefore $\mathcal{L}$ is also referred to as the *cocircuit span* of $\mathcal{M}$.

Definition 44 $\mathcal{M} = (X, \mathcal{L})$ *is called* uniform of rank r *if exactly all subsets of X with $r+1$ elements occur as support sets of cocircuits. (if $\mathcal{L} = \{\mathbf{0}\}$, the rank is defined to be $|X|$).*

The set $\mathcal{T} \subseteq \mathcal{L}$ of covectors which are *maximal* with respect to $\leq$ are called *topes* of $\mathcal{M}$. As in the case of cocircuits, the topes already determine the oriented matroid: F is a covector if and only if its composition with any tope is a tope.

Minors. Let $\mathcal{M} = (X, \mathcal{L})$ be an oriented matroid, $Y \subseteq X$. The pairs

$$\mathcal{M}\backslash Y := (X - Y, \mathcal{L}\backslash Y) \text{ with } \mathcal{L}\backslash Y := \{F|_{X-Y} \mid F \in \mathcal{L}, F_y = 0 \text{ for all } y \in Y\}$$

and

$$\mathcal{M}/Y := (X - Y, \mathcal{L}/Y) \text{ with } \mathcal{L}/Y := \{F|_{X-Y} \mid F \in \mathcal{L}\}$$

are the *minors* of $\mathcal{M}$ with respect to Y. $\mathcal{M}\backslash Y$ arises by *deletion* of Y, while $\mathcal{M}/Y$ is obtained by *contraction* of Y. The minors of a (uniform) oriented matroid are (uniform) oriented matroids again, and in the uniform case their ranks easily follow from the above definition; one gets

Fact 45 *Let $\mathcal{M}$ be an oriented matroid on X, uniform of rank $r \geq 0$, $Y \subseteq X$. Then $\mathcal{M}\backslash Y$ and $\mathcal{M}/Y$ are oriented matroids, uniform of rank r (for $|Y| \leq |X| - r - 1$) and $r - |Y|$ (for $|Y| \leq r$), respectively.*

The main correspondence. Now we are prepared to prove the correspondence between pseudohemispherical spaces and uniform oriented matroids. Let Ψ be the canonical bijection between 2^X and $\{+, -\}^X$, i.e.

$$\Psi(R)_x := \begin{cases} + \text{ if } x \in R, \\ - \text{ otherwise} \end{cases}.$$

Theorem 46 *Let $\mathcal{M} = (X, \mathcal{L})$ be an oriented matroid, uniform of rank r with set of topes $\mathcal{T}$. Then $\mathcal{S} = (X, \mathcal{R})$ with $\mathcal{R} = \Psi^{-1}(\mathcal{T})$ is a pseudohemispherical range space of VC-dimension $|X| - r$.*

Proof. We start by showing

$$\dim(\mathcal{S}) \leq |X| - r.$$

To this end consider $Y \subset X$, $|Y| = |X| - r + 1$. It is clear that $\mathcal{R}|_Y$ corresponds to the topes of $\mathcal{M}/(X - Y)$; by Fact 45 this minor has rank 1, and its tope set cannot equal $\{+, -\}^Y$, because then every vector in $\{+, -, 0\}^Y$ would have to be a covector of $\mathcal{M}/(X - Y)$, implying rank 0. Consequently, $\mathcal{R}|_Y \neq 2^Y$, so Y is not shattered by $\mathcal{R}$.

On the other hand it is not hard to see that in the uniform case $\mathcal{R}/Y$ corresponds to the topes of $\mathcal{M}\backslash Y$, and again by Fact 45 this minor has rank r for $|Y| = |X| - r - 1$ and is in particular nontrivial, so

$$|\mathcal{R}/Y| \geq 2 \text{ for any } Y \text{ of cardinality } |X| - r - 1.$$

Both properties together imply $\dim(\mathcal{S}) = |X| - r$ and $|\mathcal{R}/Y| = 2$, which via Theorem 40 proves the claim. ◻

Theorem 47 *Let $\mathcal{S} = (X, \mathcal{R})$ be a pseudohemispherical range space of VC-dimension d. Then $\mathcal{T} = \Psi(\mathcal{R})$ is the set of topes of a uniform oriented matroid of rank $n - d$.*

Proof. We explicitly construct the oriented matroid by obtaining its covectors from the *faces* of $\mathcal{S}$, which is the set $\mathcal{F}$ of all the pairs (R, A) with $R, A \subseteq X$ and $R \in \mathcal{R}/A$. Let Γ be the canonical bijection between pairs of disjoint subsets of X and $\{+, -, 0\}^X$, i.e.

$$\Gamma(R, A)_x := \begin{cases} 0 \;\text{ if } x \in A, \\ + \text{ if } x \in R, \\ - \text{ otherwise} \end{cases}.$$

We will show that $\mathcal{L} = \{\mathbf{0}\} \cup \Gamma(\mathcal{F})$ is the set of covectors of an oriented matroid on X. It will then be uniform of rank $n-d$, because Theorem 40 implies that for every A with $|A| = d-1$ there are two cocircuits with zero set A, hence support set $X - A$ of size $n-d+1$. Furthermore, its set of topes will be $\Gamma(\{(R, \emptyset) \mid R \in \mathcal{R}\}) = \Psi(\mathcal{R})$, as required.

(V0) is satisfied by definition and (V1) follows from the fact that $\mathcal{S}/A$ is closed for any $A \subseteq X$. To establish (V2) we observe that the following stronger property holds (and this is due to the uniform case):

(V2') for all $F \in \mathcal{L}, F \neq \mathbf{0}$ we have $F \circ U \in \mathcal{L}$ for any $U \in \{+,-,0\}^X$.

To see this consider some $F = \Gamma(R, A)$ and any signed vector U. Then

$$F \circ U = \Gamma(R \cup U^+, A \cap U^0),$$

where $R \in \mathcal{R}/A$ by definition of range space contraction implies

$$R \cup U^+ \in \mathcal{R}/(A - U^+) \subseteq \mathcal{R}/(A \cap U^0),$$

so $F \circ U$ is the image of a face of $\mathcal{S}$ under Γ.

To show (V3), choose $F, G \in \mathcal{L}$, separated by $x \in X$ and construct a covector H eliminating x between F and G as follows: we may assume $F^0 = G^0$ by replacing F and G with $F \circ G$ and $G \circ F$, respectively. This means, $F = \Gamma(R, A), G = \Gamma(R', A)$ for some $A \subseteq X$, $R, R' \in \mathcal{R}/A$. $\mathcal{S}/A$ is pseudohemispherical, so Theorem 41 ensures that R and R' are joined by a shortest possible path in $D^1(\mathcal{S}/A)$ (unless $\dim(\mathcal{S}/A) = 1$ in which case $R = X - A - R', F = -G$ and $H = \mathbf{0}$ is the required covector). Since x separates R, R', on the path there must be ranges $T, T \triangle \{x\}$. Assume $x \notin T$. Then $T \in \mathcal{R}/(A \cup \{x\})$ and $H := \Gamma(T, A \cup \{x\})$ eliminates x between F and G. ◻

Both theorems together give the main characterization **Theorem 18** : $\mathcal{S} = (X, \mathcal{R})$ is pseudohemispherical of VC-dimension d if and only if $\Psi(\mathcal{R})$ is the set of topes of an oriented matroid, uniform of rank $n-d$.

Affine oriented matroids. As a corollary of the main correspondence we also obtain a one-to-one correspondence between pseudogeometric range spaces and so-called *affine* uniform oriented matroids, which can be shown to correspond to affine arrangements of pseudohyperplanes [EM].

Definition 48 *Let $\mathcal{L}'$ be a set of signed vectors on X, $e \notin X$. The pair*

$$\mathcal{M}' := (X, \mathcal{L}')$$

is an affine (uniform) oriented matroid on X if there exists a (uniform) oriented matroid $\mathcal{M} = (X \cup \{e\}, \mathcal{L})$ with

$$\mathcal{L}' = \{F|_X \mid F \in \mathcal{L}, F_e = -\}.$$

So there is a one-to-one correspondence between affine (uniform) oriented matroids on X and (uniform) oriented matroids on $X \cup \{e\}$ with e not a loop (i.e. $F_e \neq 0$ for some covector). This reflects exactly the relation between pseudogeometric range spaces on X and pseudohemispherical range spaces on $X \cup \{e\}$ as stated in Theorem 17. Moreover, $\Psi(\mathcal{R})$ is the 'tope set' of $\mathcal{M}'$ if and only if $\Psi(\hat{\mathcal{R}})$ is the tope set of $\mathcal{M}$, and we get

Theorem 49 $\mathcal{S} = (X, \mathcal{R})$ *is a pseudogeometric range space if and only if* $\Psi(\mathcal{R})$ *is the set of topes of an affine uniform oriented matroid on X.*

6 Discussion and Relations to Oriented Matroid Theory

In this paper we have introduced three classes of range spaces, each of which is characterized by a simple extremal property. We have shown in the previous section that the pseudohemispherical range spaces are in one-to-one correspondence with the uniform oriented matroids (and thus the pseudogeometric range space correspond to uniform affine oriented matroids). In view of this correspondence we obtain a new characterization of uniform (affine) oriented matroids via counting arguments; on the other hand, many statements which we have shown to hold in the range space environment, by our correspondence reduce to known facts about oriented matroids. For some of these, however, we give independent proofs in the uniform case which are substantially easier than the original proofs in the generic situation. This means, a reader particularly interested in the uniform case might still benefit from our techniques. Other statements already hold for maximum range spaces (which properly generalize affine uniform oriented matroids) and thus embed known facts into a broader context. In this concluding section we discuss the interplay between our concepts and the corresponding oriented matroid theory.

Range space results revisited. Most concepts we have defined on range spaces have an obvious interpretation in the (affine) oriented matroid setting when specialized to the pseudohemispherical or pseudogeometric case. These are the minor operations (Definition 19), swapping (Definition 25, reorientation), the distance-1-graph (Definition 6, tope graph) and duality (Definition 9). The defining fact that dual pairs $\mathcal{S}, \mathcal{S}^*$ of range spaces have complementary sets of ranges specializes to the statement that dual pairs of uniform oriented matroids have complementary sets of topes, which is an easy characterization of duality in the uniform case and follows e.g. from the well-known fact that dual tope sets are disjoint together with counting formulas for topes, see below.

The characterization of pseudogeometric range spaces by small subspaces (Theorem 11) states that a maximum range space of VC-dimension d is pseudogeometric if and only if all restrictions to $d+2$ elements are geometric (or *realizable* in the terminology of oriented matroids), i.e. a characterization in terms of *local realizability* is obtained. A general result of similar flavor is known for oriented matroids, where cocircuit signatures of matroids are shown to be oriented matroids if and only if all contractions to rank 2 are realizable. This means, for a set of signed vectors with underlying structure being a matroid, the axiomatics can be substantially relaxed (or equivalently, the 3-term Grassmann-Plücker relations are sufficient to characterize oriented matroids). This has first been shown by Las Vergnas [LaV], see also Section 3.6 of [BLSWZ]. In the case of pseudogeometric spaces we have maximum range spaces as underlying structure, which is somewhat stronger than a (uniform) matroid.

The uniform case via counting arguments. We can define the VC-dimension of a set $\mathcal{U} \subseteq \{+,-\}^X$ as the cardinality of the largest subset $Y \subseteq X$ that is *shattered* by $\mathcal{U}$, i.e.

$$\mathcal{U}(Y) := \{F|_Y \mid F \in \mathcal{U}\} = \{+,-\}^X.$$

Our main new result then reads as follows:

Theorem 50 *$\mathcal{T} \subseteq \{+,-\}^X$ of VC-dimension d is the set of topes of a uniform oriented matroid M on X if and only if $\mathcal{T} = -\mathcal{T}$ and $|\mathcal{T}| = 2\Phi_{d-1}(|X|-1)$.*

The 'only if' part is well known – the number of topes of $\mathcal{M}$ depends only on the matroid underlying $\mathcal{M}$ and can be computed from it (this generalizes face counting formulas for arrangements of hyperplanes). In the uniform case the matroid has a particularly trivial structure, and one arrives at the above number. To the knowledge of the authors the 'if' part is new, showing that the uniform case has a very simple structure which can be described just by counting (and can therefore be understood without considering it as a specialization of the generic situation).

The corresponding theorem for the affine case is similar:

Theorem 51 *$\mathcal{T} \subseteq \{+,-\}^X$ of VC-dimension d is the set of topes of a uniform affine oriented matroid M on X if and only if $|\mathcal{T}| = \Phi_d(|X|)$ and $|\mathcal{T}'| = 2\Phi_{d-1}(|X|-1)$, where*

$$\mathcal{T}' := \{F \in \mathcal{T} \mid -F \in \mathcal{T}\}$$

is the set of unbounded topes.

Again the 'only if' part is a consequence of known counting results. Section **4.6** of [BLSWZ] gives an overview on these results for (affine) oriented matroids. Note that we obtain an intrinsic characterization of affine uniform oriented matroids which does not depend on an ambient oriented matroid. Karlander [Kar] has obtained such a characterization for arbitrary affine oriented matroids.

Other characterizations in terms of topes. The problem of finding a non-recursive axiomatization of oriented matroids in terms of topes has been solved by da Silva [Sil2], generalizing a result of Lawrence [Law] for the uniform case; prior to this, Bienia and Cordovil [BC], da Silva [Sil1] and Handa [Han] had given recursive characterizations.

The result of Handa is particularly interesting. It is known that the tope graph of an oriented matroid (the distance-1-graph of the topes) is an isometric subgraph of the hypercube on vertex set $\{+,-\}^X$ – i.e. any two topes are joined by a path whose length equals the number of elements separating them (for simplicity we assume that there are no parallel elements). By considering all the sets $\mathcal{U} \subseteq \{+,-\}^X$, $\mathcal{U} = -\mathcal{U}$ with this *reorientation* property one arrives at the notion of *acycloids* [Tom], which properly generalize tope sets of oriented matroids. By Theorem 41, pseudohemispherical range spaces can be regarded as acycloids in this sense after interpreting them as signed vector systems in the obvious way. Handa's theorem now states that an acycloid $\mathcal{U}$ is the tope set of an oriented matroid if and only if every contraction (suitably defined) is an acycloid; this is the case for a pseudohemispherical space, which gives an alternative proof for the fact that it forms the tope set of an oriented matroid.

Note that the isometric subgraph property already holds for maximum range spaces (Theorem 28). The proof directly uses the fact that the operations of deleting and contracting single elements are interchangeable (Corollary 24, and this of course generalizes to arbitrary minors); the recursive axiomatizations by Bienia, Cordovil and by da Silva are based on the analogous fact for oriented matroids, namely that the deletion and contraction on topes (suitably defined) are interchangeable, and this is already sufficient to characterize tope sets.

Lawrence's characterization of the uniform case also has an interesting relation to our work. It is based on the notion of lopsided subsets of $\{+,-\}^X$ and states that $\mathcal{T} \subseteq \{+,-\}^X$ is the tope set of a uniform oriented matroid if and only if for all $x \in X$ the set

$$\mathcal{T}^x := \{F \in \mathcal{T} \mid F_x = +\}$$

is lopsided, where $\mathcal{U} \subseteq \{+,-\}^X$ is called *lopsided* if for all $Y \subseteq X$

- Y is not shattered by $\mathcal{U}$,
- or $X - Y$ is not shattered by $\{+,-\}^X - \mathcal{U}$.

One immediately sees that $\mathcal{U}$ is lopsided if and only if $\{+,-\}^X - \mathcal{U}$ is lopsided, i.e. lopsided sets are closed under (complementary) duality, a feature shared with all the classes of range spaces we have introduced. Moreover, from Theorem 23(vi) it is easy to see that lopsided sets generalize maximum range spaces (and this generalization is proper), so Lawrence's characterization brings together two concepts at different levels of generality, with maximum range spaces wedged between them in a certain sense.

Acknowledgement. The authors thank David Haussler for many discussions which have significantly contributed to the material developed in this paper. Special thanks to Günter Ziegler and Jürgen Richter-Gebert for reading several versions of the manuscript, pointing us to related results and helping to improve the presentation.

References

[AHW] N. Alon, D. Haussler and E. Welzl, Partitioning and geometric embedding of range spaces of finite Vapnik-Chervonenkis dimension, *Proc. 3rd Ann. ACM Symp. on Computational Geometry* (1987), 331–340.

[Ale] K. S. Alexander, Probability-inequalities for empirical processes and a law of the iterated logarithm, *Annals of Probability* **6** (1984), 1041–1067.

[AB] M. Anthony and N. Biggs, *Computational Learning Theory*, Cambridge Tracts in Theoretical Computer Science (30), Cambridge University Press (1992).

[Ass] P. Assouad, Densité et dimension, *Ann. Inst. Fourier* **33** (1983), 233–282.

[BH] E. Baum and D. Haussler, What Size Net Gives Valid Generalization?, *Neural Computation*, **1** (1989), 151–160.

[BC] W. Bienia and R. Cordovil, An axiomatic of non-Radon partitions of oriented matroids, *Europ. J. Combin.* **8** (1987), 1–4.

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented matroids*, Encyclopedia of Mathematics, Cambridge University Press (1992).

[BL] R. G. Bland and M. Las Vergnas, Orientability of matroids, *J. Combin. Theory Ser. B* **24** (1978), 94–123.

[BEHW] A. Blumer, A. Ehrenfeucht, D. Haussler and M. Warmuth, Learnability and the Vapnik-Chervonenkis dimension, *J. Assoc. Comput. Mach.* **36** (1989), 929–965.

[CM] B. Chazelle and J. Matoušek, On linear-time deterministic algorithms for optimization problems in fixed dimension, *Proc. 4th Ann. ACM-SIAM Symp. Discr. Algor.* (1993), 281–290.

[CW] B. Chazelle and E. Welzl, Quasi-optimal range searching in spaces of finite VC-dimension, *Discr. Comp. Geometry* **4** (1989), 467–489.

[Djo] D. Z. Djoković, Distance-preserving subgraphs of hypercubes, *J. Combin. Theory, Ser. B* **14** (1973), 263–267.

[DSW] G. Ding, P. Seymour and P. Winkler, Bounding the vertex cover number of a hypergraph, *Combinatorica*, to appear.

[Dud1] R. M. Dudley, Central limit theorems for empirical measures, *Annals of Probability* **6** (1978), 899–929.

[Dud2] R. M. Dudley, A course on empirical measures, *Lecture Notes in Math.* **1097** (1984), 2–142.

[Dud3] R. M. Dudley, The structure of some Vapnik-Chervonenkis classes, *Proc. of Berkeley Conference in honor of Jerzy Neyman and Jack Kiefer*, Volume II, (LeCam and Olshen, Eds.) Wadsworth (1985), 495–507.

[Ede] H. Edelsbrunner, *Algorithms in combinatorial geometry*, EATCS Monographs in Theoretical Computer Science, Springer-Verlag, Berlin (1987).

[EM] J. Edmonds and A. Mandel, *Topology of oriented matroids*, Ph. D. thesis, University of Waterloo, Ontario (1981).

[FL] J. Folkman and J. Lawrence, Oriented matroids, *J. Combin. Theory Ser. B* **25** (1978), 199–236.

[Flo] S. Floyd, *On Space-bounded learning and the Vapnik-Chervonenkis dimension*, Ph. D. thesis, University of California at Berkeley (1989).

[GJ] P. Goldberg and M. Jerrum, Bounding the Vapnik-Chervonenkis dimension of concept classes parameterized by real numbers, *Proceedings of the Sixth Annual ACM Conference on Computational Learning Theory* (1993), 361–369.

[GP1] J. E. Goodman and R. Pollack, On the combinatorial classification of nondegenerate configurations in the plane, *J. Combin. Theory Ser. A* **29** (1980), 220–235.

[GP2] J. E. Goodman and R. Pollack, Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines, *J. Combin. Theory Ser. A* **29** (1980), 385–390.

[GP3] J. E. Goodman and R. Pollack, Three points do not determine a (pseudo-)plane, *J. Combin. Theory Ser. A* **31** (1981), 215–218.

[Grü1] B. Grünbaum, Convex polytopes, *Ser. Pure and Appl. Math.***16**, Interscience Publishers (1967).

[Grü2] B. Grünbaum, Arrangements and Spreads, *CBMS Regional Conference Series in Mathematics*, No.**10**, *Amer. Math. Soc., Providence, R. I.* (1972).

[GZ] E. Giné and J. Zinn, Some limit theorems for empirical processes, *Annals of Probability* **12** (1984), 929–989.

[Han] K. Handa, A Characterization of oriented matroids in terms of topes, *Europ. J. Combin.* **11** (1990), 41–45.

[Hau] D. Haussler, Sphere-Packing numbers for subsets of the boolean n-cube with bounded Vapnik-Chervonenkis dimension, *J. Combin. Theory, Ser. A*, to appear.

[HKS] D. Haussler, M. Kearns and R. Schapire, Bounds on the sample complexity of Bayesian learning using information theory and the VC dimension, *Machine Learning*, **14**, (1994) 84–114.

[HW] D. Haussler and E. Welzl, ϵ-nets and simplex range queries, *Discr. Comp. Geometry* **2** (1987), 127–151.

[Kar] J. Karlander, A characterization of affine sign vector systems, Preprint, KTH Stockholm (1992)

[KPW] J. Komlós, J. Pach and G. Woeginger, Almost tight bounds for ϵ-nets, *Discr. Comp. Geometry* **7** (1992), 163–173.

[LaV] M. Las Vergnas, Oriented matroids as signed geometries real in corank 2, in: *Finite and Infinite Sets* (Proc. 6th Hungarian Combinatorial Conf. Eger 1981), North-Holland (1984), 555–565.

[Law] J. Lawrence, Lopsided sets and orthant-intersection by convex sets, *Pac. J. Math.* **104** No. 1 (1983), 155–173.

[Lev] F. Levi, Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade, *Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig* **78** (1926), 256–267.

[LMR] N. Linial, Y. Mansour and R. L. Rivest, Results on Learnability and the Vapnik-Chervonenkis Dimension, *Inform. Comput.* **90** (19910, 33–49.

[Mat] J. Matoušek, Approximations and optimal geometric divide and conquer, *Proc. 23rd Ann. ACM Symp. on Theory of Computing* (1991), 506–511.

[MWW] J. Matoušek, E. Welzl and L. Wernisch, Discrepancy and approximations for bounded VC-dimension, *Combinatorica* **13** (1993), 455-466.

[Nat] B. K. Natarajan, *Machine Learning: A Theoretical Approach*, Morgan Kaufmann, San Mateo, California (1991)

[Pea] J. Pearl, Capacity and Error Estimates for Boolean Classifiers with Limited Complexity, *IEEE Trans. Pattern Anal. Mach. Intell.* **1** (1979), 350–355.

[Per] R. Perrin, Sur le problème des aspect, *Bull. Soc. Math. France* **10** (1881/82), 103-127.

[Pol1] D. Pollard, *Convergence of stochastic processes*, Springer-Verlag, Berlin (1984).

[Pol2] D. Pollard, *Empirical processes: theory and applications, volume 2 of NSF-CBMS Regional Conference Series in Probability and Statistics*, Institute of Math. Stat. and Am. Stat. Assoc. (1990).

[RiG] J. Richter-Gebert, Oriented Matroids with few Mutations, *Discr. Comp. Geometry*, to appear.

[Rin] G. Ringel, Teilungen der Ebene durch Geraden oder topologische Geraden, *Math. Z.* **64** (1956), 79-102.

[Sau] N. Sauer, On the density of families of sets, *J. Combin. Theory Ser. A* **13** (1972), 145–147.

[She] S. Shelah, A combinatorial problem; stability and order for models and theories in infinitary languages, *Pac. J. Math.* **41** (1972), 247-261.

[Sil1] I. P. da Silva, An axiomatic for the set of maximal vectors of an oriented matroid, based on symmetry properties of the set of vectors, Preprint (1988).

[Sil2] I. P. da Silva, Axioms for maximal vectors of an oriented matroid, a combinatorial characterization for the regions of an arrangement of pseudohyperplanes, *Europ. J. Combin.*, to appear.

[Ste] J. M. Steele, Existence of sub-matrices with all possible columns, *J. Combin. Theory Ser. A* **24** (1978), 84–88.

[Tal] M. Talagrand, Donsker classes of sets, *Prob. Theory and Related Fields* **78** (1988), 169–191.

[Tom] N. Tomizawa, Theory of acycloid and holometry, *RIMS kokyuroku* **534** (*Graph Theory and Applications*) (1984), 91–138 (in Japanese).

[Vap] V. N. Vapnik, *Estimation of dependences based on empirical data*, Springer-Verlag, Berlin (1982).

[VC1] V. N. Vapnik and A. Y. Chervonenkis, On the uniform convergence of relative frequencies of events to their probability, *Theory Probab. Appl.* **16** (1971), 264–280.

[Zas] T. Zaslavsky, Facing up to arrangements: face count formulas for partitions of space by hyperplanes, *Memoirs Amer. Math. Soc.* **1** (1975), 154.