1 MAX-SAT

Consider a conjunctive normal form (CNF) formula on Boolean variables \(x_1, x_2, \ldots, x_n\), that is, a formula defined as AND of a number \(m\) of clauses, each of which is the OR of some literals appearing either positively as \(x_i\) or negated as \(\overline{x_i}\). Suppose that each clause \(c_j\) has some weight \(w_j\). The goal is to find an assignment of TRUE/FALSE values to the literals with the objective of maximizing the total weight of the satisfied clauses.

(A) Consider a clause that has \(k \geq 1\) literals. Argue that the simple randomized algorithm that sets each variable at random (true or false, each with probability half) satisfied this clause with probability \((1 - 2^{-k})\).

(B) Consider the linear program with objective

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\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{m} w_j z_j \\
\text{subject to} & \quad \forall j \in \{1, 2, \ldots, m\} : \sum_{i \in S^+_j} y_i + \sum_{i \in S^-_j} (1 - y_i) \geq z_j \\
& \quad \forall j \in \{1, 2, \ldots, m\} : z_j \in [0, 1] \\
& \quad \forall i \in \{1, 2, \ldots, n\} : y_i \in [0, 1]
\end{align*}
\]

Here, \(S^+_j\) denotes the set of variables that appear in the \(j^{th}\) clause positively, and \(S^-_j\) denotes the set of variables that appear in the \(j^{th}\) clause in a negated form. Explain how this is a relaxation of an integer linear program for our objective. Moreover, let \((y^*, z^*)\) denote the optimal solution of this LP. Show that the natural randomized rounding algorithm that sets each \(x_i\) to be True with probability \(y^*_i\) provides the following guarantee: if the \(j^{th}\) clause has \(k\) literals, it is satisfied with probability at least \((1 - (1 - \frac{1}{k})^k))z^*_j\).

(C) Notice that the first algorithm handles well large clauses and the second algorithm handles well the smaller clauses. Put the two together to get a 3/4 approximation algorithm for the MAX-SAT problem.

(D) In part (B), we considered a linear randomized rounding process. Consider a non-linear round which rounds variable \(x_i\) to be true with probability \(f(y^*_i)\) where \(f : [0, 1] \rightarrow [0, 1]\) is an arbitrary function such that \(f(y) \in [1 - 4^{-y}, 4^{y-1}]\). Prove that this non-linear rounding directly gives a 3/4 approximation algorithm.

(E) Find a CNF such that there is a 3/4 gap between the value of the solution of LP described in part (B) and the optimal Boolean assignment to the variables. Hint: find a CNF with 4 clauses, each of weight 1, such that the LP has value 4 but any assignment satisfies at most 3 clauses. This implies that the 3/4 factor is the integrability gap of this LP formulation and no rounding technique for it will give an approximation better than 3/4.