

## Exercise 10

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## 1 Cut Counting [Recommended]

- (A) Use Karger's Random Contraction Algorithm to prove that in each graph with edge connectivity  $k$ , for any  $\alpha \geq 1$ , the number of cuts of size at most  $\alpha k$  is at most  $(2n)^{\lceil 2\alpha \rceil}$ .
- (B) A result of Tutte and Nash-Williams from 1960s shows that every graph with edge connectivity  $k$  contains at least  $k/2$  edge-disjoint spanning trees. Use this result to argue that the number of cuts of size at most  $\alpha k$  is at most  $\Theta(kn^{\lceil 2\alpha \rceil})$ .

Hint:

Assign each cut to one of the trees.

## 2 Random Sampling in Graphs with Good Expansion [Recommended]

Consider an undirected unweighted graph  $G = (V, E)$ , for each subset  $S \subset V$  of vertices, let  $E(S, V \setminus S)$  denote set of edges connecting  $S$  to  $V \setminus S$ , i.e., the edges with exactly one endpoint in  $S$ . Suppose that for some  $\alpha = \Omega(\log n)$ , we have that  $\frac{|E(S, V \setminus S)|}{\min(|S|, |V \setminus S|)} \geq \alpha$ , for each subset  $S \subset V$ .

Prove that if we subsample the edges of  $G$  with probability  $p = \Omega(\frac{\log n}{\alpha \varepsilon^2})$ , then all cuts are concentrated around their expectation, with high probability. That is, with probability  $1 - 1/n$ , for each cut  $(S, V \setminus S)$ , the number of sampled edges of this cut is in  $(1 \pm \varepsilon)p|E(S, V \setminus S)|$ .

Hint:

You do not need the cut-counting arguments that we saw in the class. What is the failure probability of a cut with  $k$  nodes in one side and  $n - k$  nodes on the other side?

## 3 Sparsification for Hypergraphs

In this exercise, we derive a sparsification for hypergraphs of rank  $r$ . The rank of a hypergraph is the maximum number of vertices in one hyperedge, that is, in a hypergraph of rank  $r$  each hyperedge contains at most  $r$  endpoints. We prove that we can sparsify each such hypergraph to merely  $O(nr \log n / \varepsilon^2)$  weighted hyperedges<sup>1</sup>, while maintaining all cut sizes up to  $(1 \pm \varepsilon)$ . Notice that a priori, such a hypergraph might have up to  $\sum_{i=2}^r \binom{n}{i} \gg nr \log n$  hyperedges. For a non-trivial partition of the vertices  $(S, V \setminus S)$ —where  $S \neq \emptyset$  and  $S \neq V$ —the corresponding cut in the hypergraph is defined as the set of all hyperedges that have at least one endpoint in each side of the cut.

- (A) Using an extension of the contraction algorithm to hypergraphs, prove that the number of  $\alpha$ -min cuts is at most  $n^{O(r\alpha)}$ .

<sup>1</sup>This is  $\tilde{O}(\frac{n^2}{\varepsilon^2})$  edges, in the worst case of  $r = n$ . This was the best result until the very recent work of Chen et al. [1] in which a sparsifier with  $\tilde{O}(\frac{n}{\varepsilon^2})$  edges is constructed in polynomial-time.

- (B) Prove that a uniform sampling of each hyperedge with probability  $p = \Omega(\frac{r \log n}{\varepsilon^2 k})$  preserves the size of each cut around its expectation, up to a  $(1 \pm \varepsilon)$  factor. Here,  $k$  denotes the size of the minimum cut.
- (C) Follow the steps of what we did in the class for graphs to construct a non-uniform sampling that produces a sparsifier with  $O(nr \log n / \varepsilon^2)$  edges.

## References

- [1] Yu Chen, Sanjeev Khanna, and Ansh Nagda. *Near-linear Size Hypergraph Cut Sparsifiers*, <https://arxiv.org/abs/2009.04992>, 2020.