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## Exercise 11

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# 1 Tree embedding in cycle [Recommended]

In this exercise, we consider the tree embedding for the simple case where the graph is just a cycle on $n$ vertices. That is, the vertices of our graph are numbers $1, 2, \ldots, n$ and there is an edge between each $i$ ( $1 \leq i \leq n-1$) and $i+1$, as well as between $n$ and 1. Recall that we wish to approximate the metric induced on our graph by shortest paths (that is, distance between $i < j$ is $d_G(i,j) = \min(|j-i|, |n-j+i|)$) by a metric $d_T$ induced by shortest paths on a weighted tree $T$ with the same vertex set as in our cycle.

In the next two parts, first, we argue that we cannot hope for a good deterministic solution. Then, we show that for the case of the cycle we can achieve even a constant stretch *in expectation*.

1. Show that for any tree $T$ with nonnegative lengths of edges that satisfies $\forall i,j : d_G(i,j) \leq d_T(i,j)$, there exist two indices $i, j$ such that $d_T(i,j) \geq (n-1) \cdot d_G(i,j)$.

   Hint 1:

   *Gradually turn $T$ into a path wrapping clockwise or anticlockwise around the cycle.*

   *Each change to $T$ maintains its stretch, while the sum of edge lengths of $T$ drops.*

   Hint 2:

   *Imagine three vertices $u < v < w$ such that $T$ contains an edge $\{u, w\}$*

   *of length $w - u$ and an edge $\{v, w\}$ of length $w - v$.*

   *Change $\{u, w\}$ to $\{u, v\}$ in this particular case.*

2. The tree embedding algorithm from the lecture shows that there is a distribution over trees with average stretch $O(\log n)$, i.e., for any $i, j$ we have $\mathrm{Exp}[d_T(i,j)] \leq O(\log n) \cdot d_G(i,j)$.

   Show that in the case of the cycle there is actually a distribution over trees that achieves the stretch 2.

   Hint:

   *Leaving a random edge out of the cycle gives you a path.*

## 2 Steiner Forest [Recommended]

Given edge weighted graph $G$, and a set of pairs of terminals $(s_1, t_1), \dots, (s_k, t_k)$, consider problem of finding minimal cost $E'$ such that $s_i, t_i$ are connected in $G[E']$. Use the tree embedding algorithm from the lecture to build $\mathcal{O}(\log n)$ approximate algorithm in expectation to this problem.

Hint:

*Sample single tree $T$ and solve the problem in $T$. Project solution back to $G$.*

## 3 Analyze the Ball-Carving with Exponential Clocks

Consider a following process of Ball-Carving:

- every vertex $v$ pick radius $r_v$ according to *exponential distribution*, that is with probability density $\mathrm{EXP}(x) = \beta \cdot e^{-\beta x}$ for $x \geq 0$.
- every vertex $u$ picks as its ball-center the vertex $v = \arg\max_x (r_x - d(u, x))$ (we say that $u \in B_v$)

1. Give a reasonable upper bound on the diameter of each $B_v$ that holds w.h.p.
2. Show that for every $v \in V(G)$ and for all $u \in B_v$, all the vertices on the shortest path from $u$ to $v$ are also in $B_v$.

   Hint:

   *If $w$ lies on the shortest path between $u$ and $v$, but $w \in B_z$, shouldn't $u$ also be in $B_z$?*

3. What is the probability of two neighboring nodes $u$ and $v$ being in different balls, i.e., that they picked different ball centers?