Sampling Vertices Uniformly from a Graph

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With subsets of
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Ravi Kumar *Google MTV*
Tamás Sarlós *Google MTV*

Social Networks

- Social Networks are “large”
- We would like to study their properties
- We need to be able to sample from them

Learning Average Opinions

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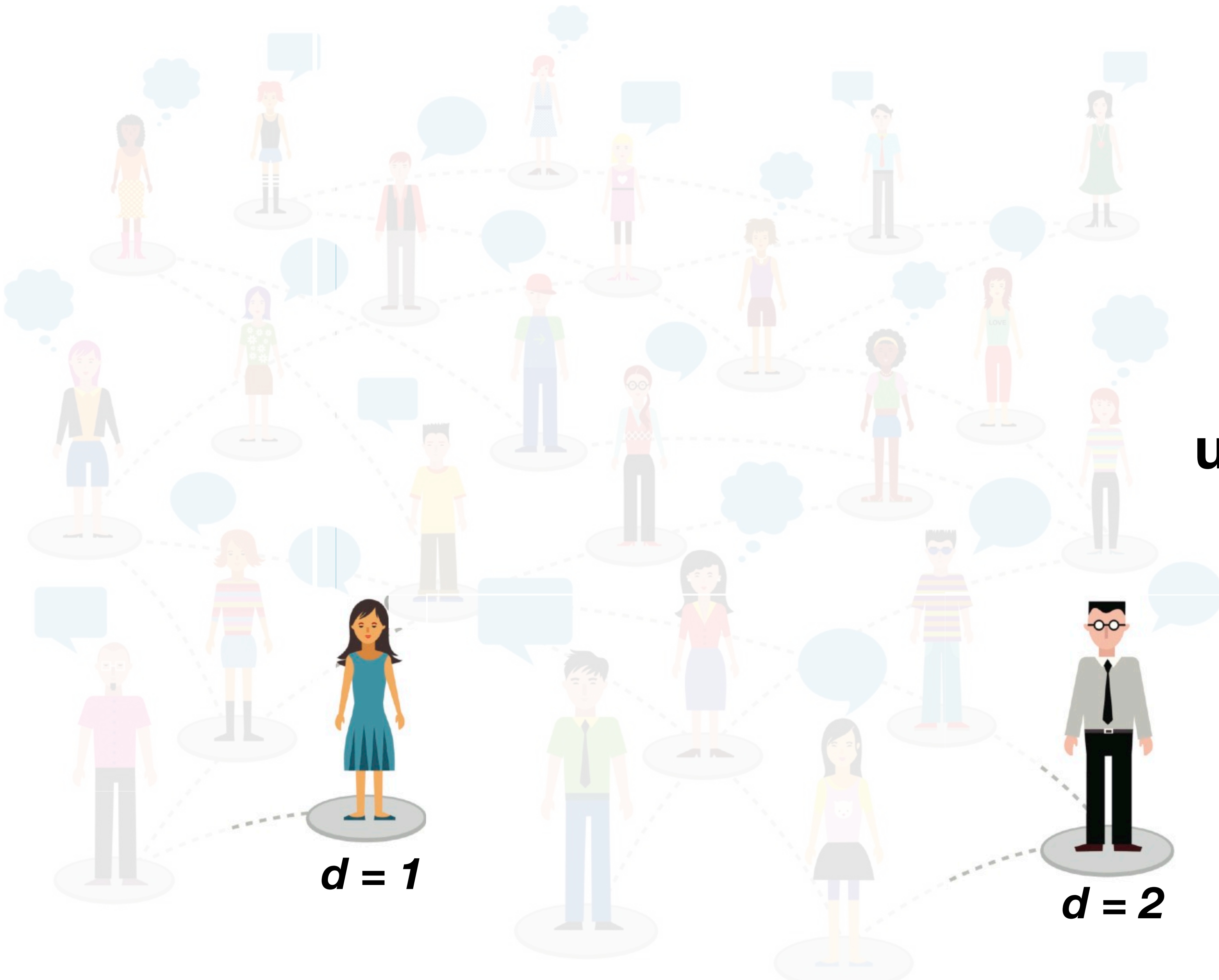
Asking all the users is too costly!

Learning Average Opinions

Select some people **uniformly-at-random** and ask them their opinion

Learning Average Opinions



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The empirical average will be close to the real average

Learning Average Opinions

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What is the fraction of  ?

Learning Average Opinions

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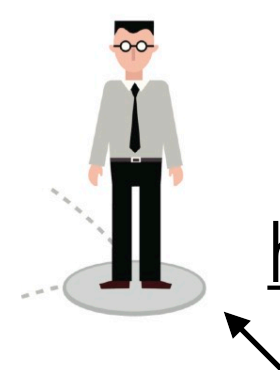
Learning Average Opinions

Select some people **uniformly-at-random** and ask them their opinion

The empirical fraction of will be close to the real fraction

How do we select uniform-at-random profiles in a Social Network?

- We can access the SN through a crawling process.



http://s-n.com/001.html

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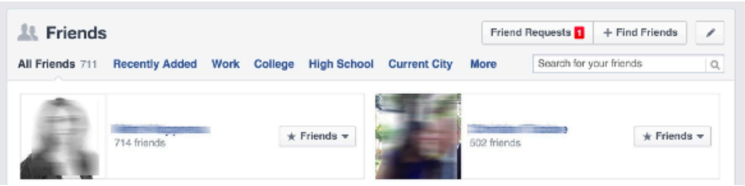

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How do we select uniform-at-random profiles in a Social Network?

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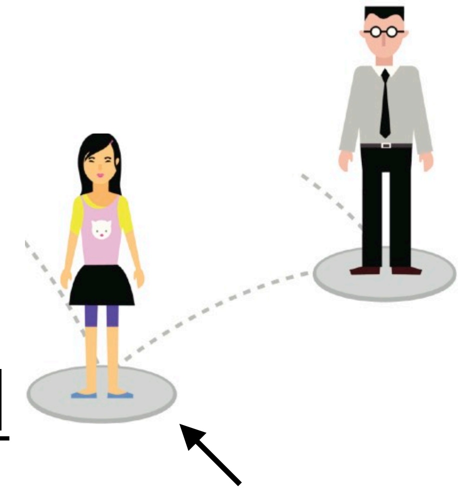
http://s-n.com/005.html



How do we select uniform-at-random profiles in a Social Network?

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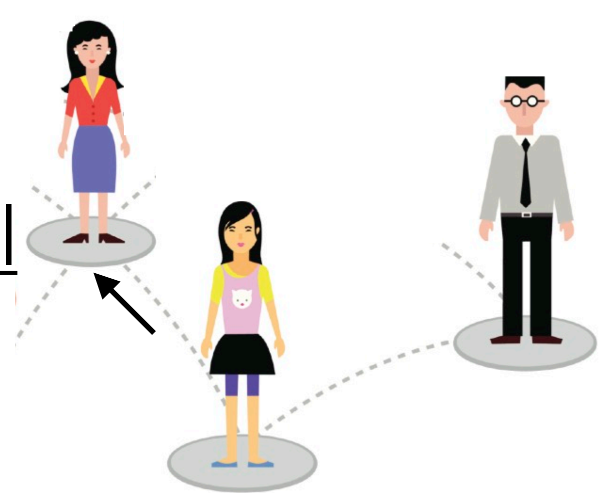
http://s-n.com/011.html



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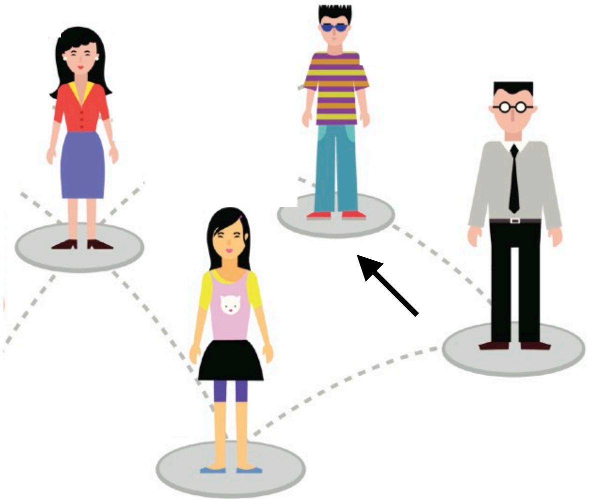
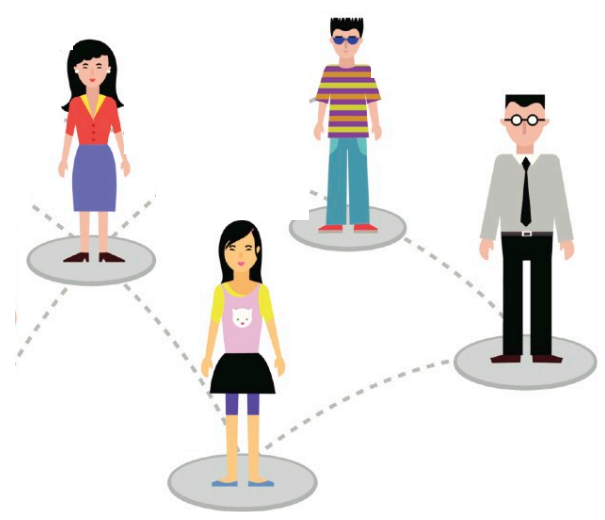
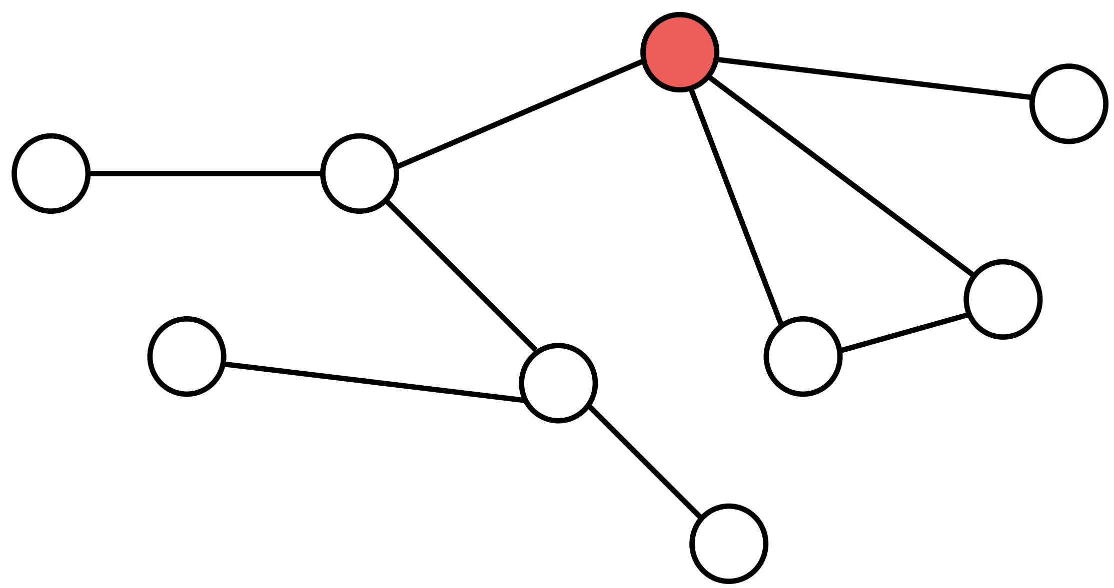


How do we select uniform-at-random profiles in a Social Network?

- We can access the SN through a crawling process.
- We **cannot** crawl the whole network.



Random Walks



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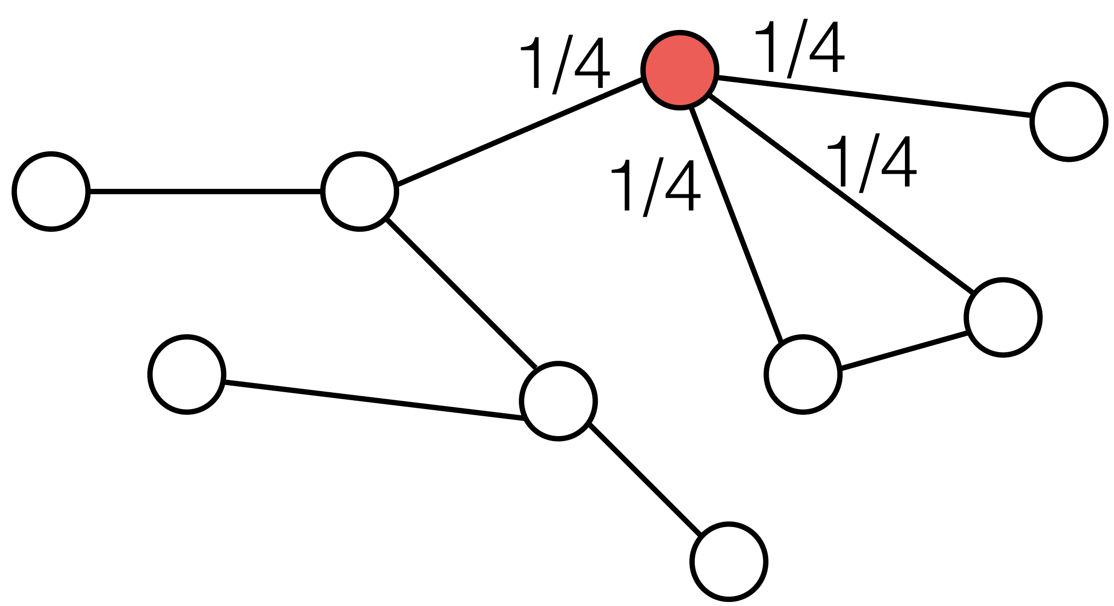

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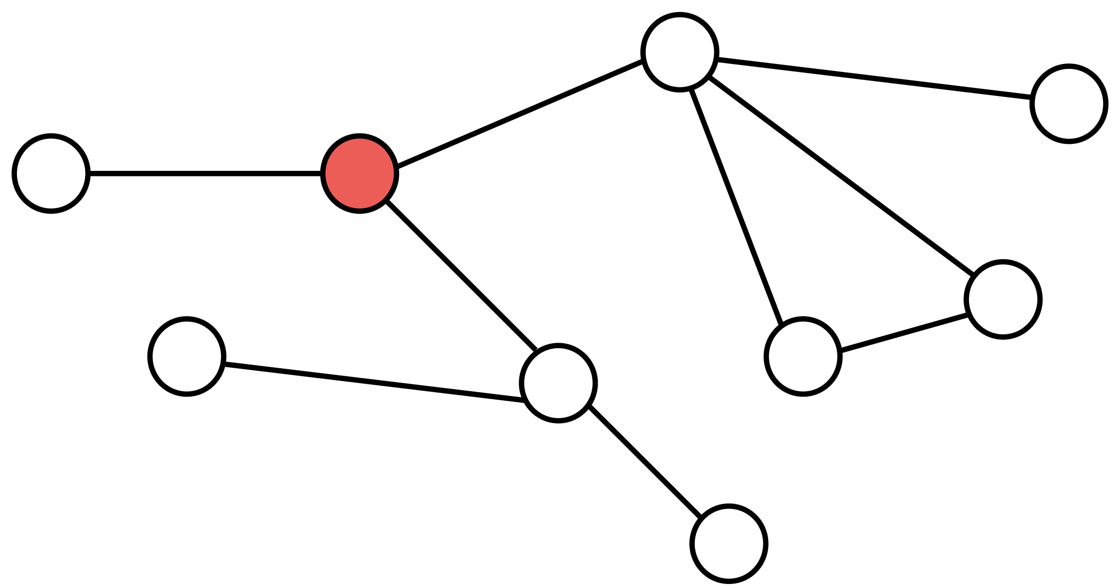

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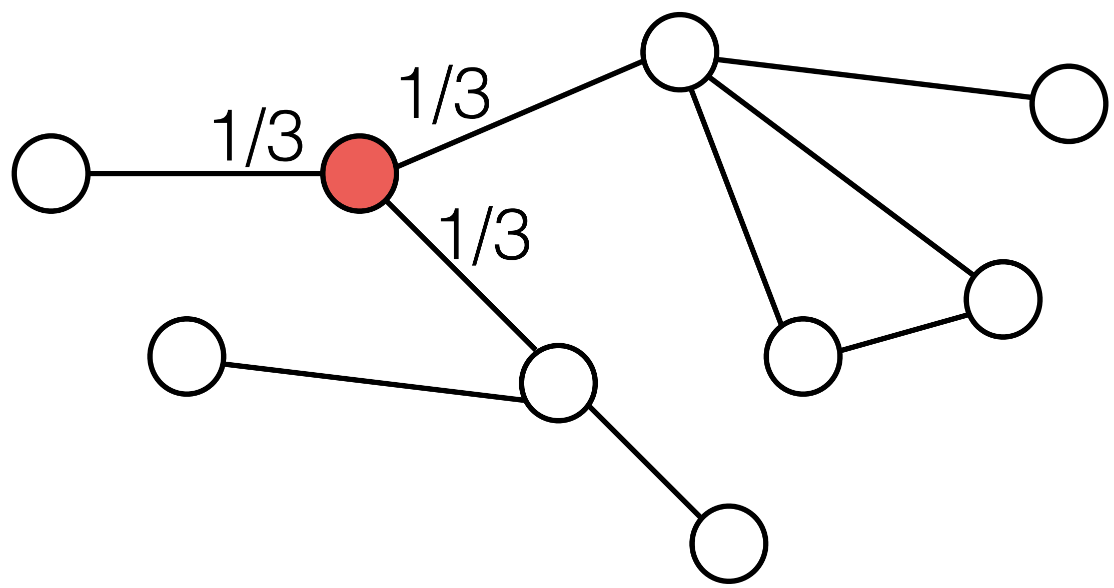

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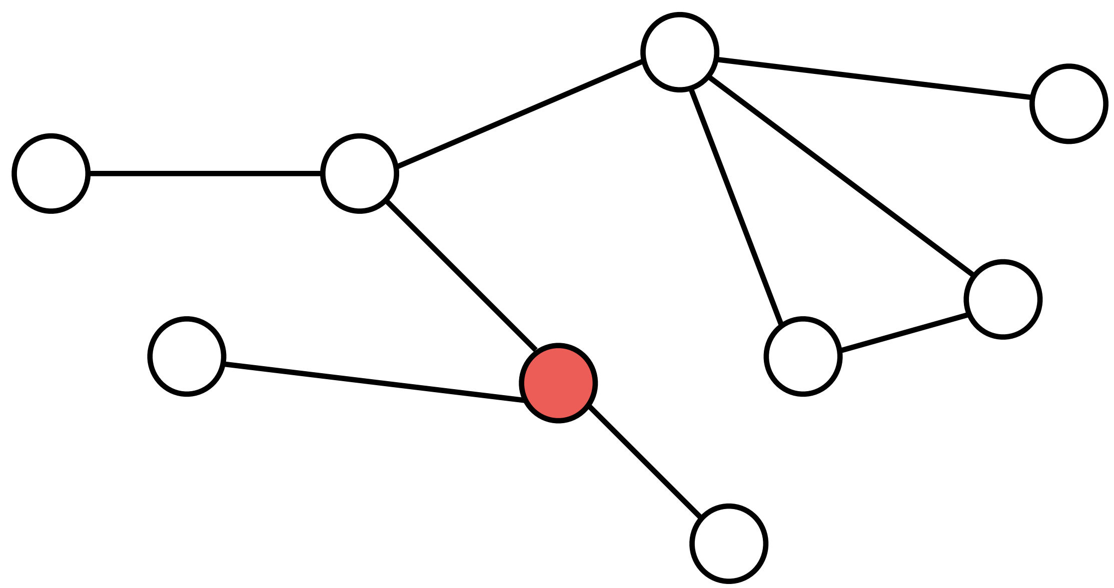

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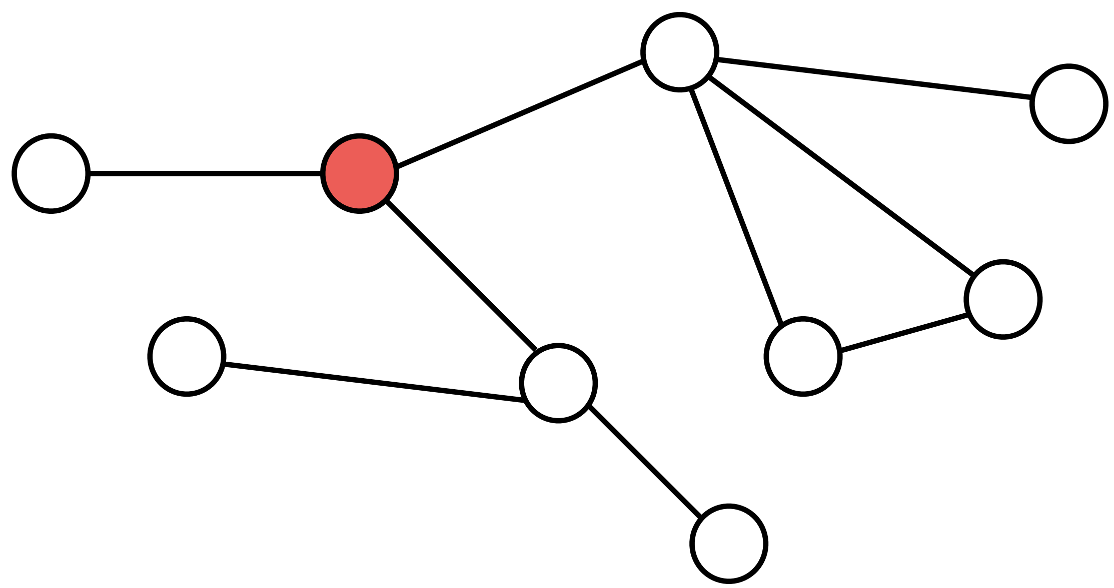

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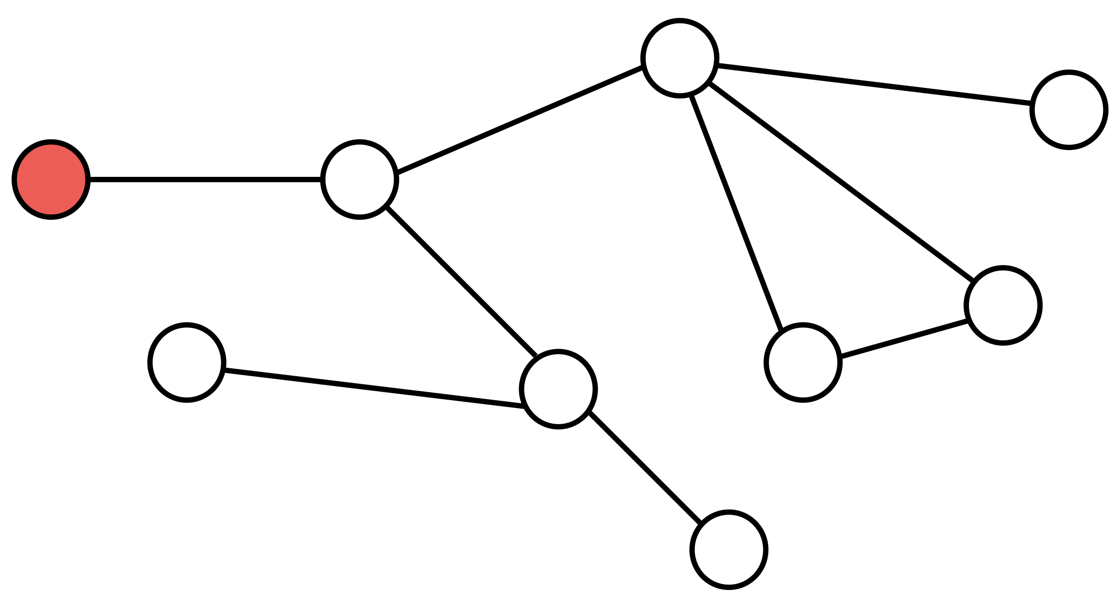

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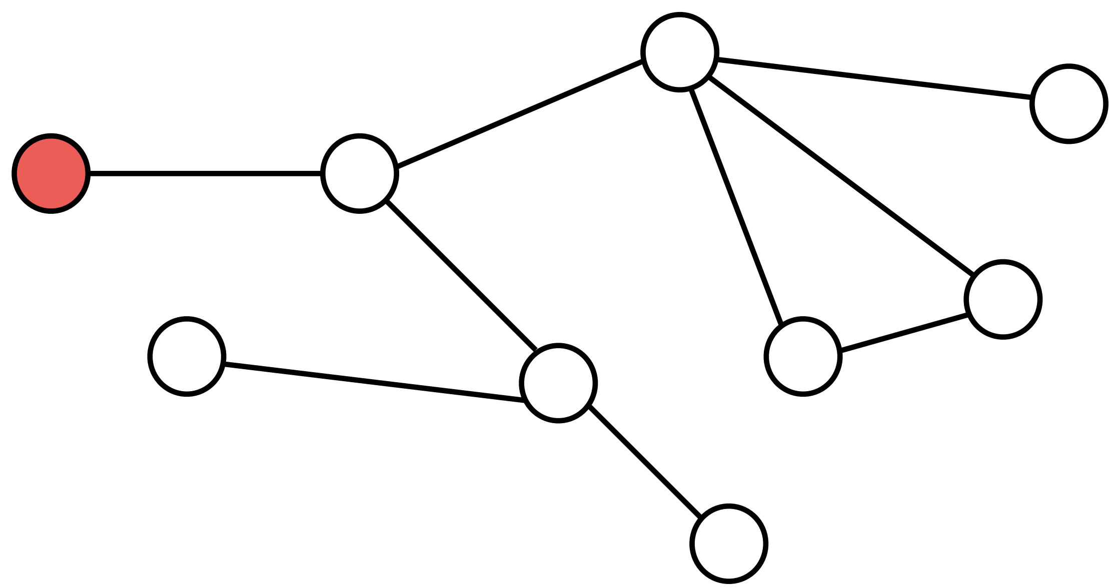
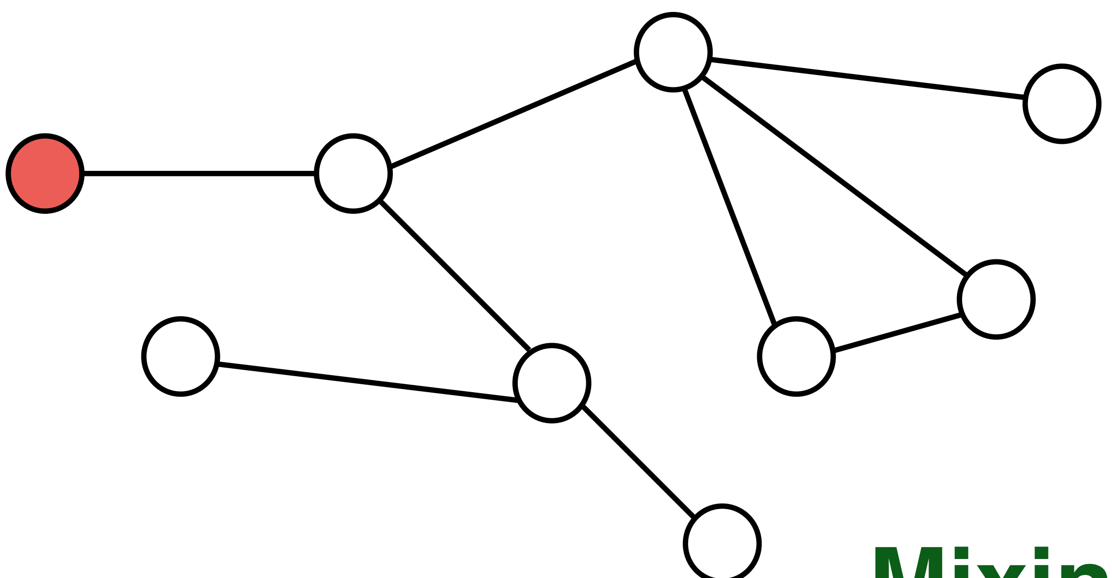


If the process goes on for enough many steps,
the random node it ends up on will be “random”

Random Walks



Mixing Time T(G)

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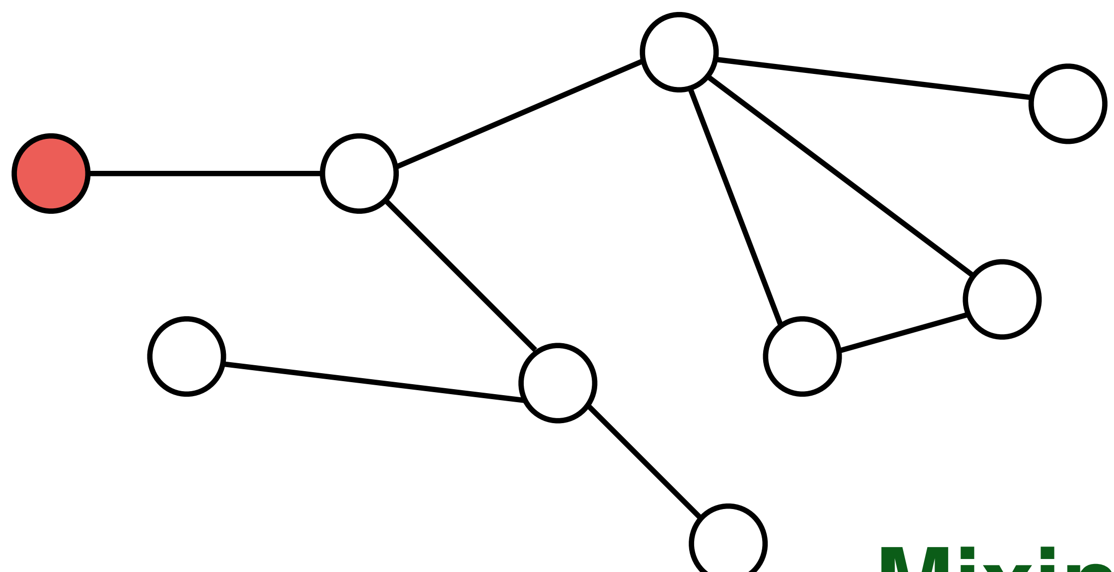
Random Walks

The Mixing Times of many "Social Networks" are small
[Leskovec et al, '08]



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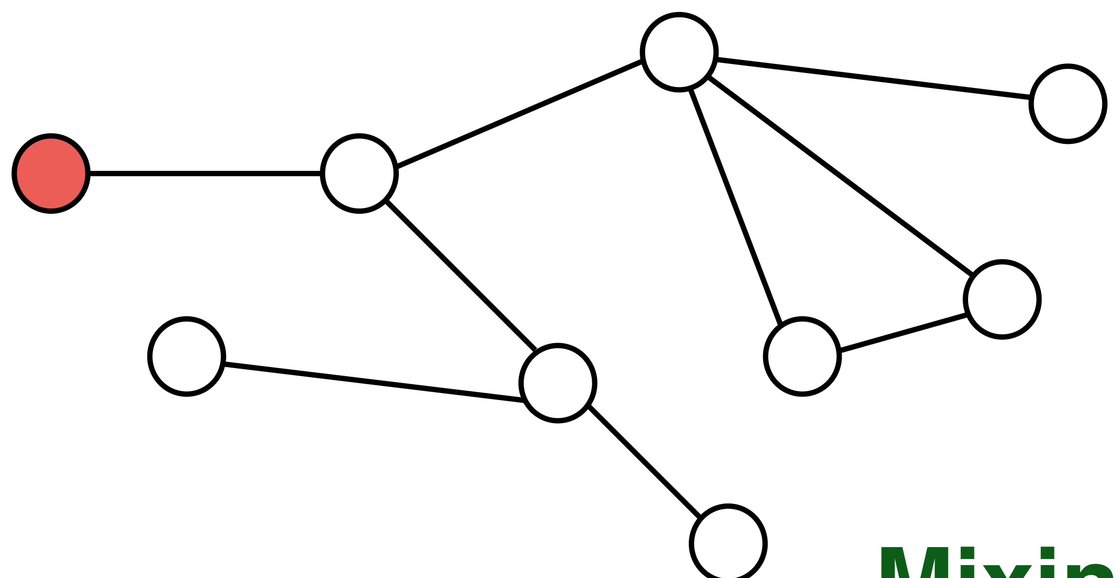

Mixing Time T(G)

If the process goes on for enough many steps,
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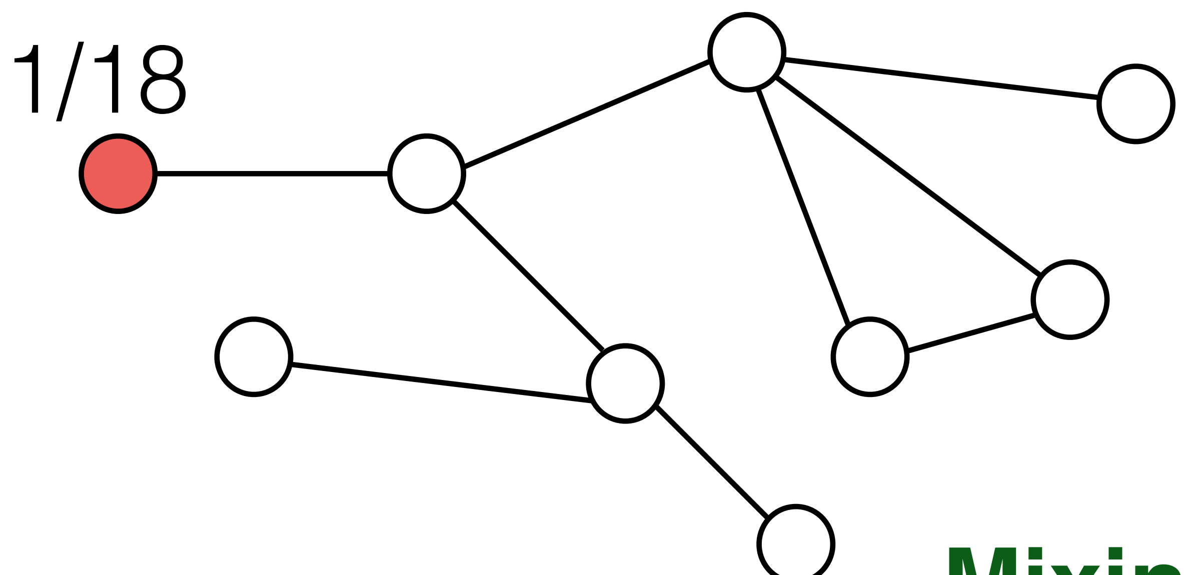
chosen with probability proportional to its degree

Random Walks



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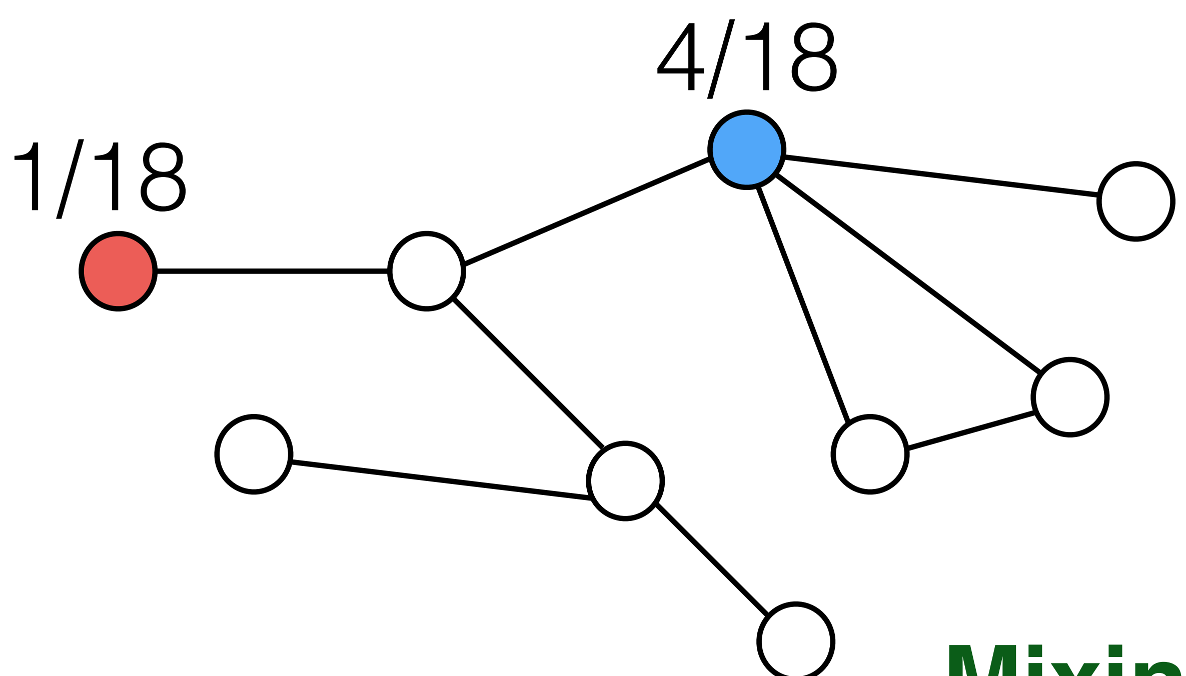

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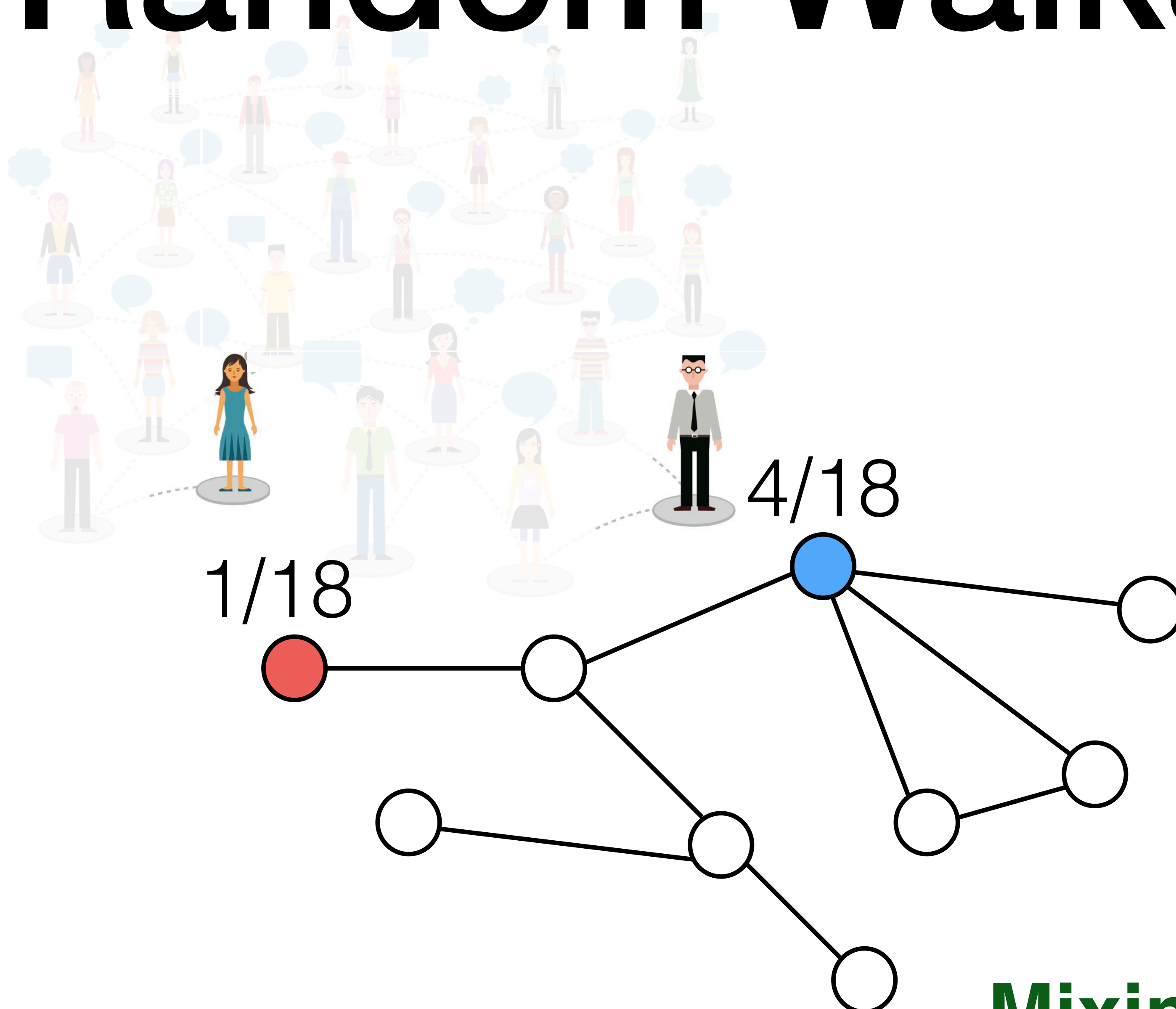

Mixing Time T(G)

If the process goes on for enough many steps, the random node it ends up on will be “random”, *chosen with probability proportional to its degree*

A Folklore Algorithm

- **While** True:
  - run the random walk for T(G) steps;
  - suppose it ends on the node v;
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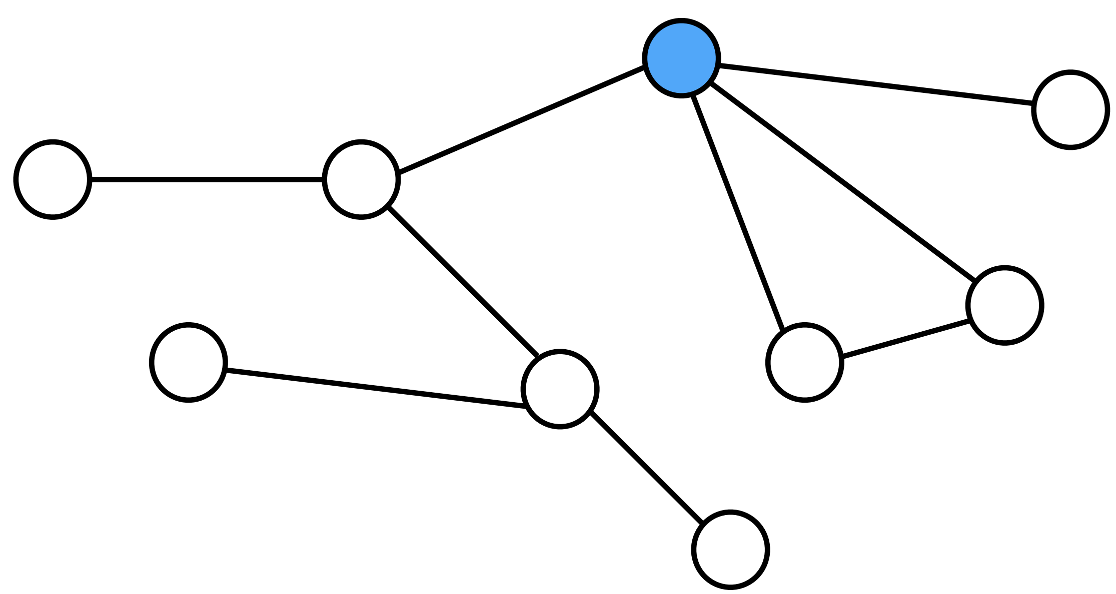

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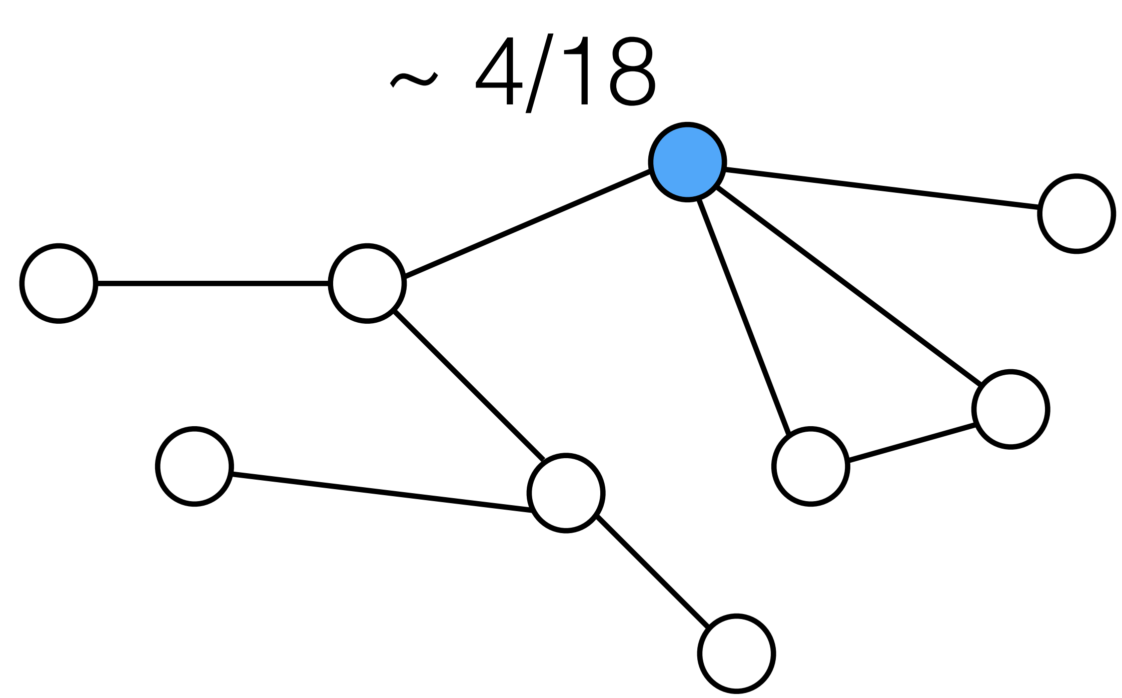

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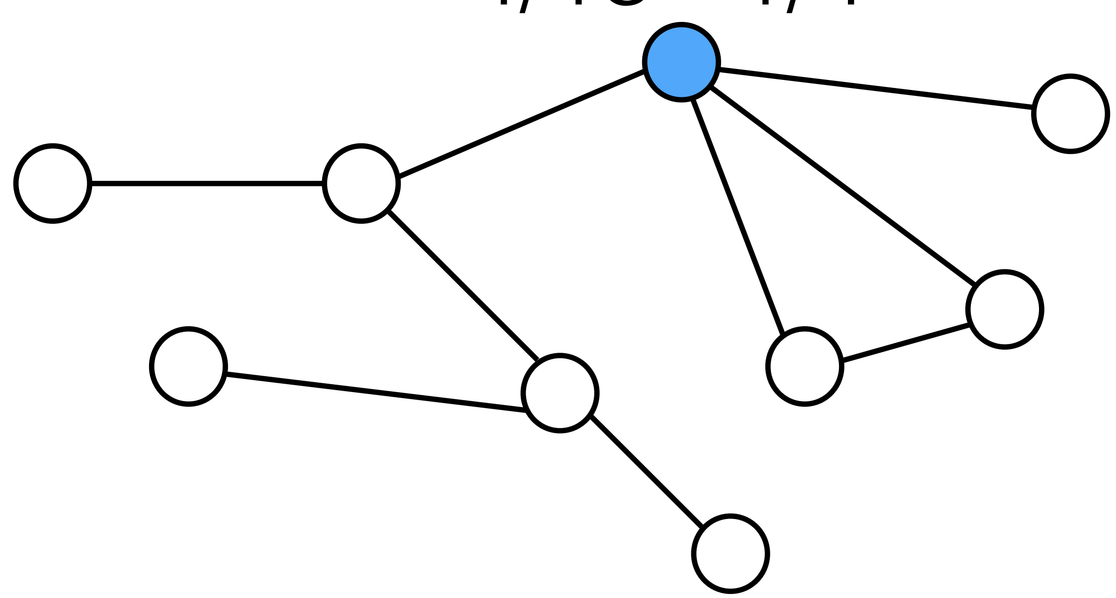

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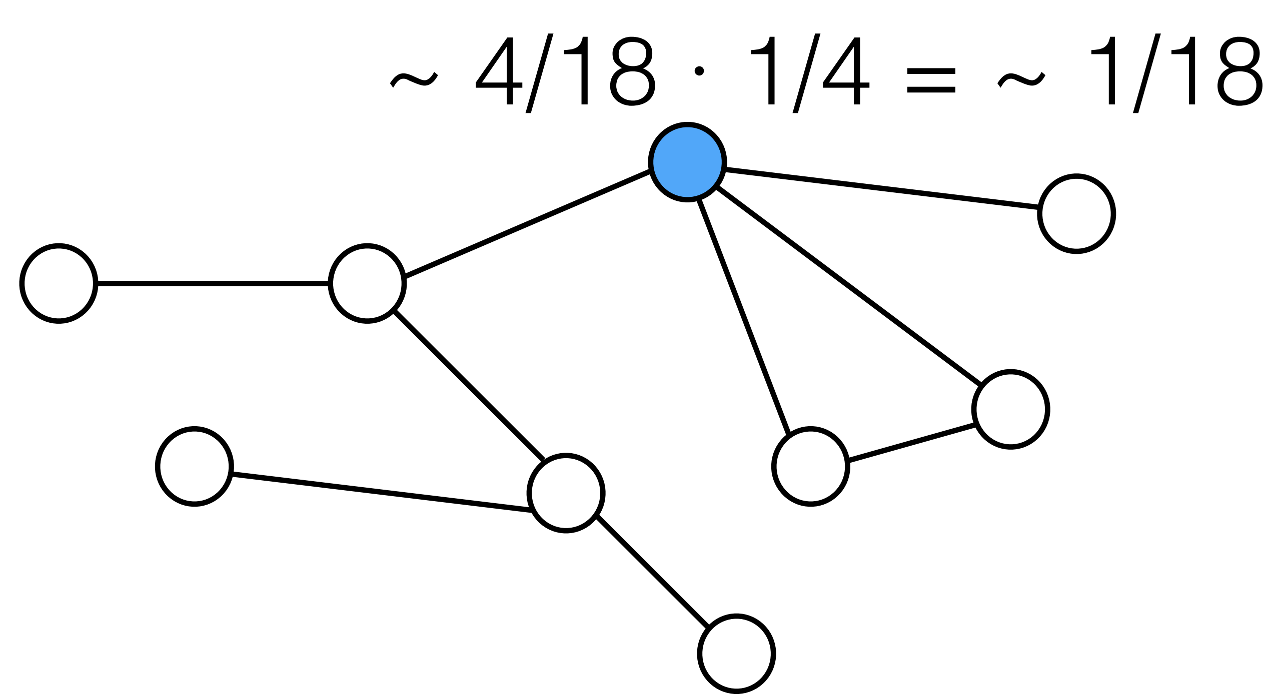

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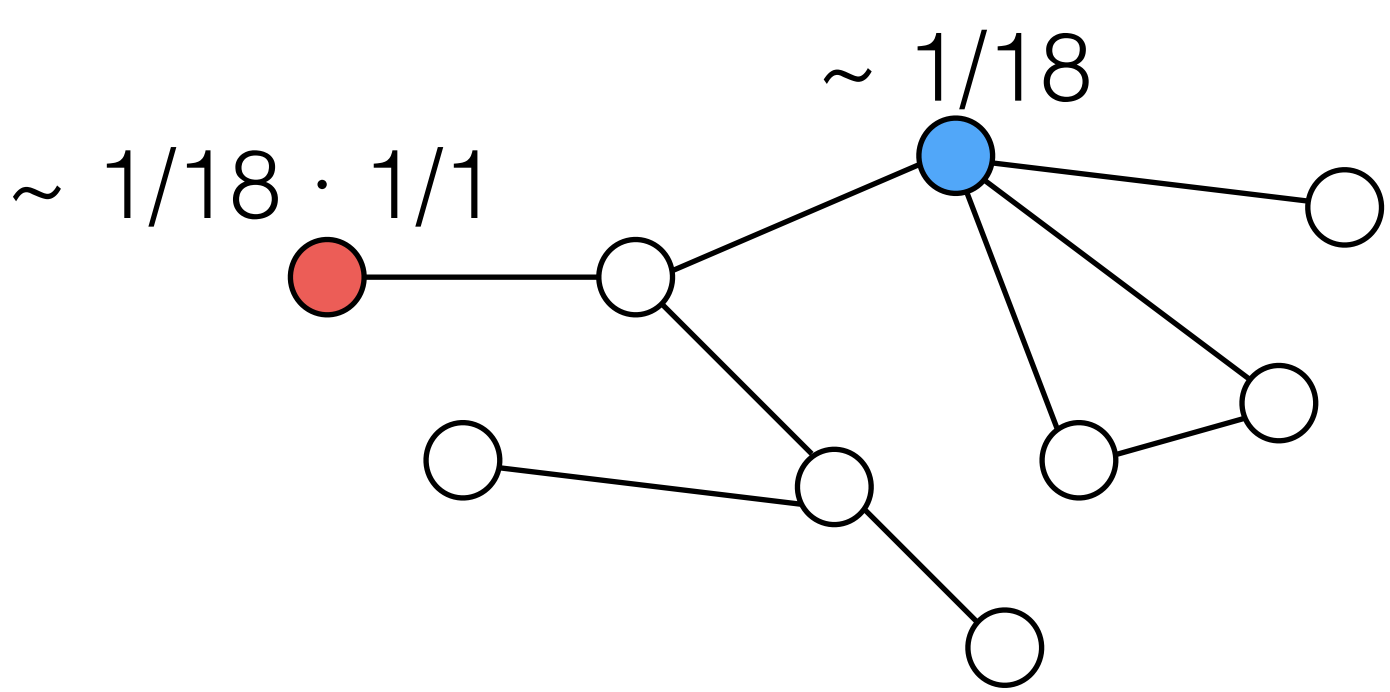

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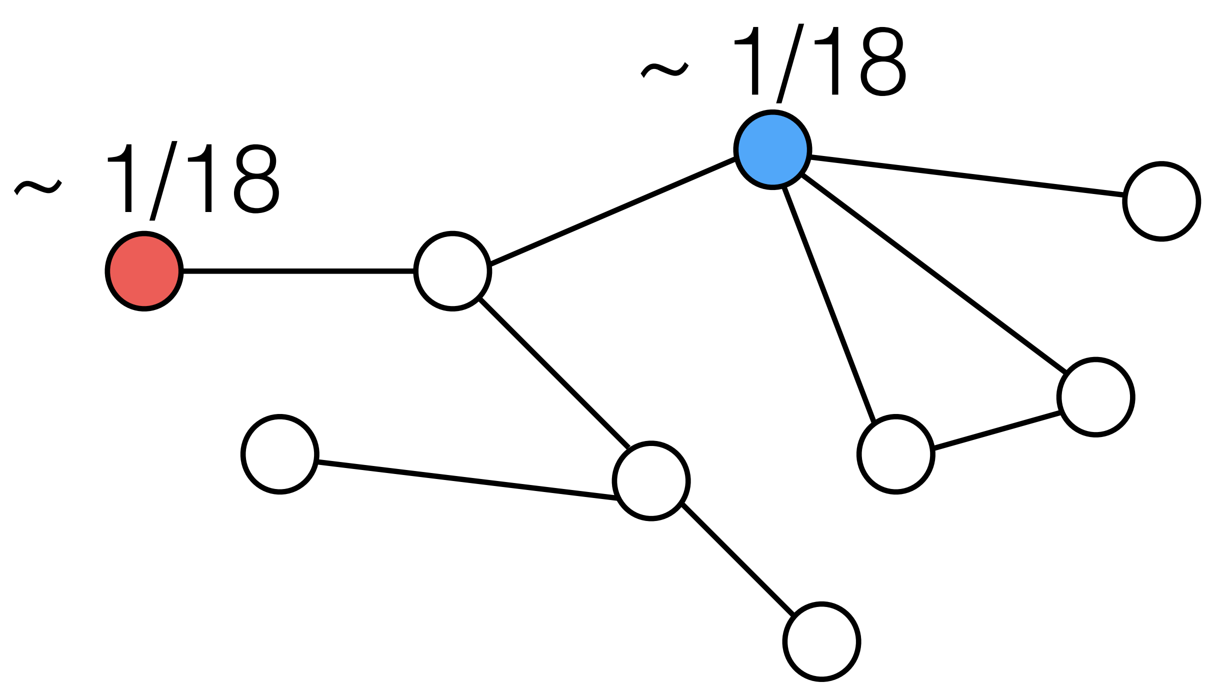

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This algorithm returns a node chosen *(arbitrarily close to)* **uniformly at random**

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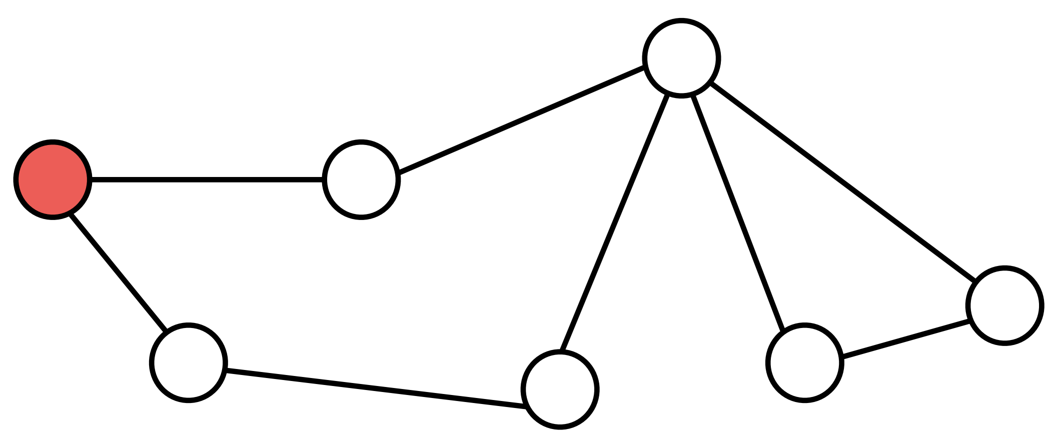
One can easily show that this algorithm **downloads**, with high probability, at most *O(T(G) · AvgDeg(G))* nodes from the network

The Max-Degree Algorithm

- Let D be the max-degree of G.
- Add self-loops to G in order to make it D-regular.
- Run the random walk for $D \cdot T(G)$ steps.
- **return** the node on which it ends.



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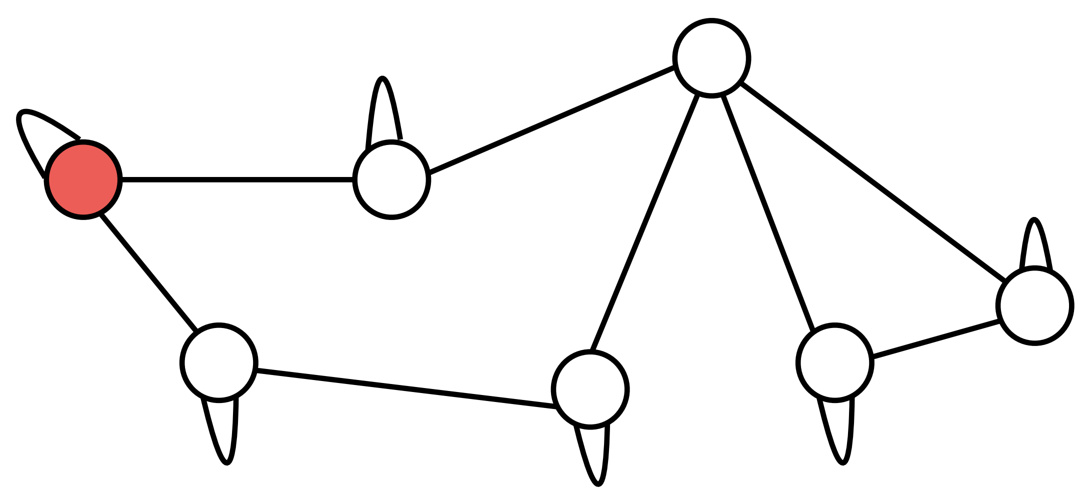

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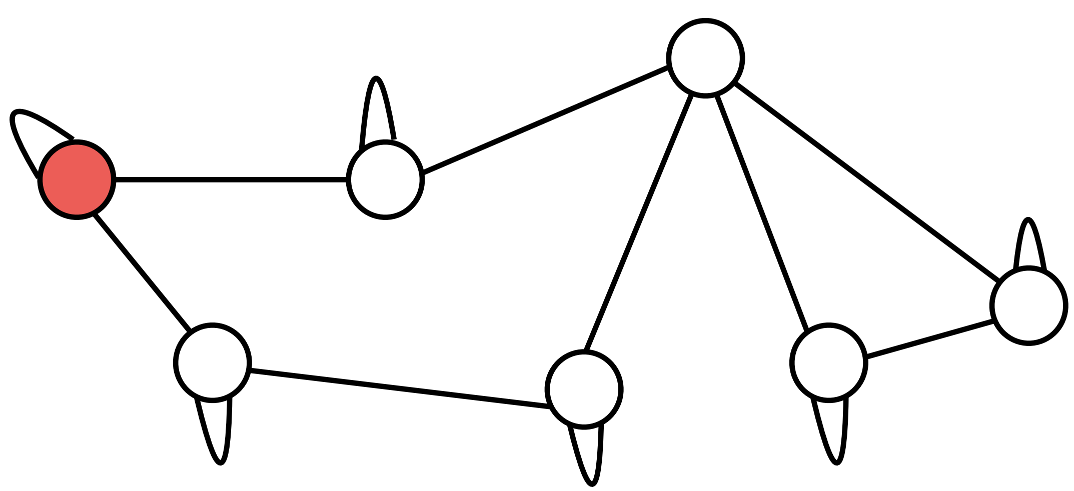

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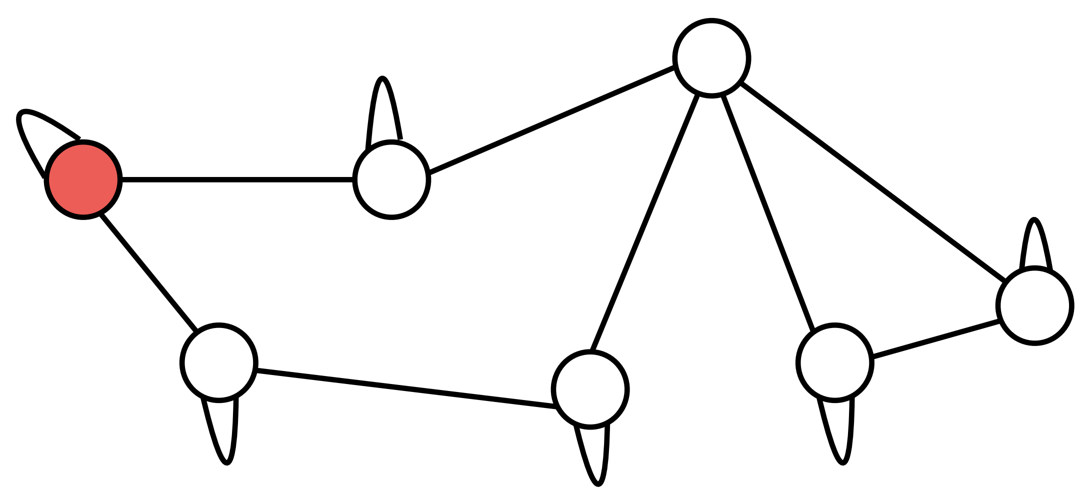
# of Downloaded Vertices ≤ AvgDeg(G) · T(G)



Can one do better?

- In *[C., Dasgupta, Kumar, Lattanzi, Sarlós,’16]* we analyzed various algorithms for selecting a UAR node.
- Some of them were on-par with the Folklore Algorithm, some of them were worse.

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- Some of them were on-par with the Folklore Algorithm, some of them were worse.

- In *[C., Haddadan, '18],* we show that if an algorithm downloads < *o(T(G) AvgDeg(G))* nodes from the network, then it cannot return anything close to a uniform-at-random node.

- That is, the Folklore algorithm is optimal.

Two Main Ingredients

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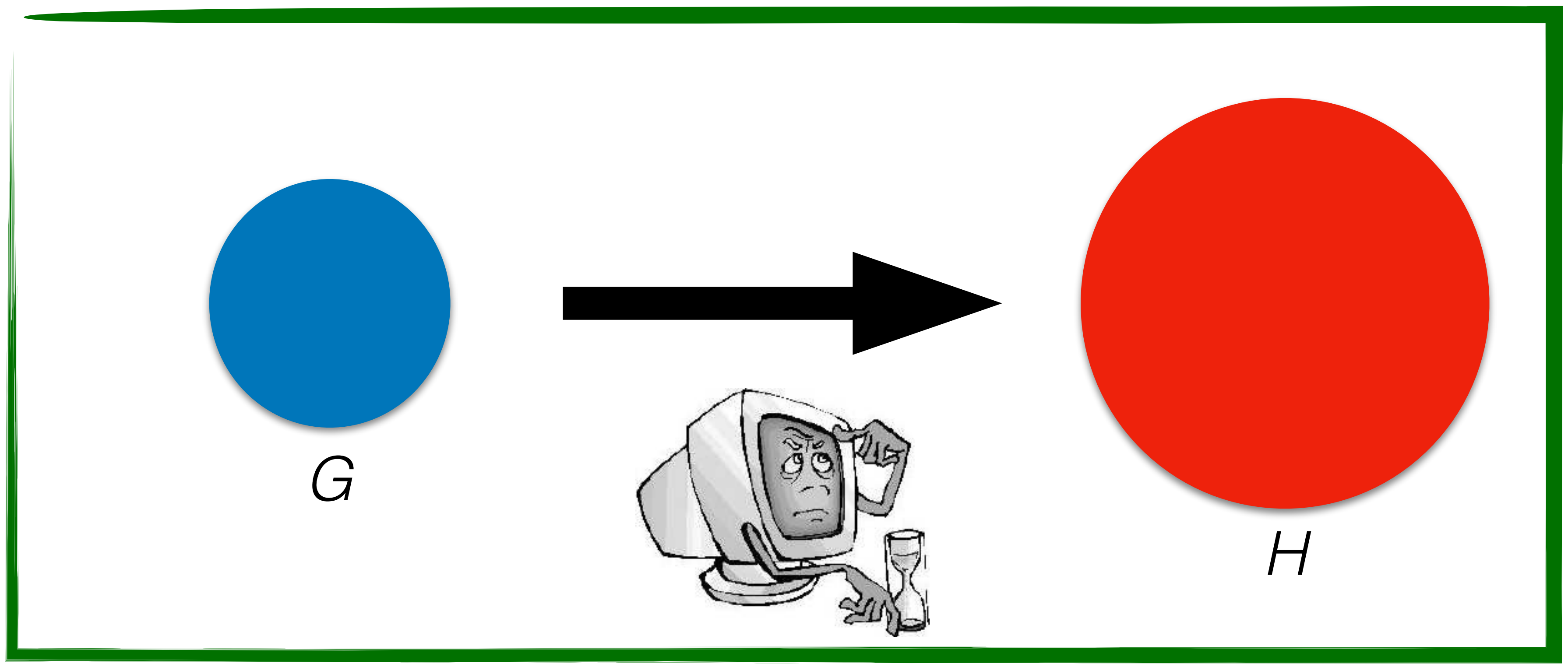

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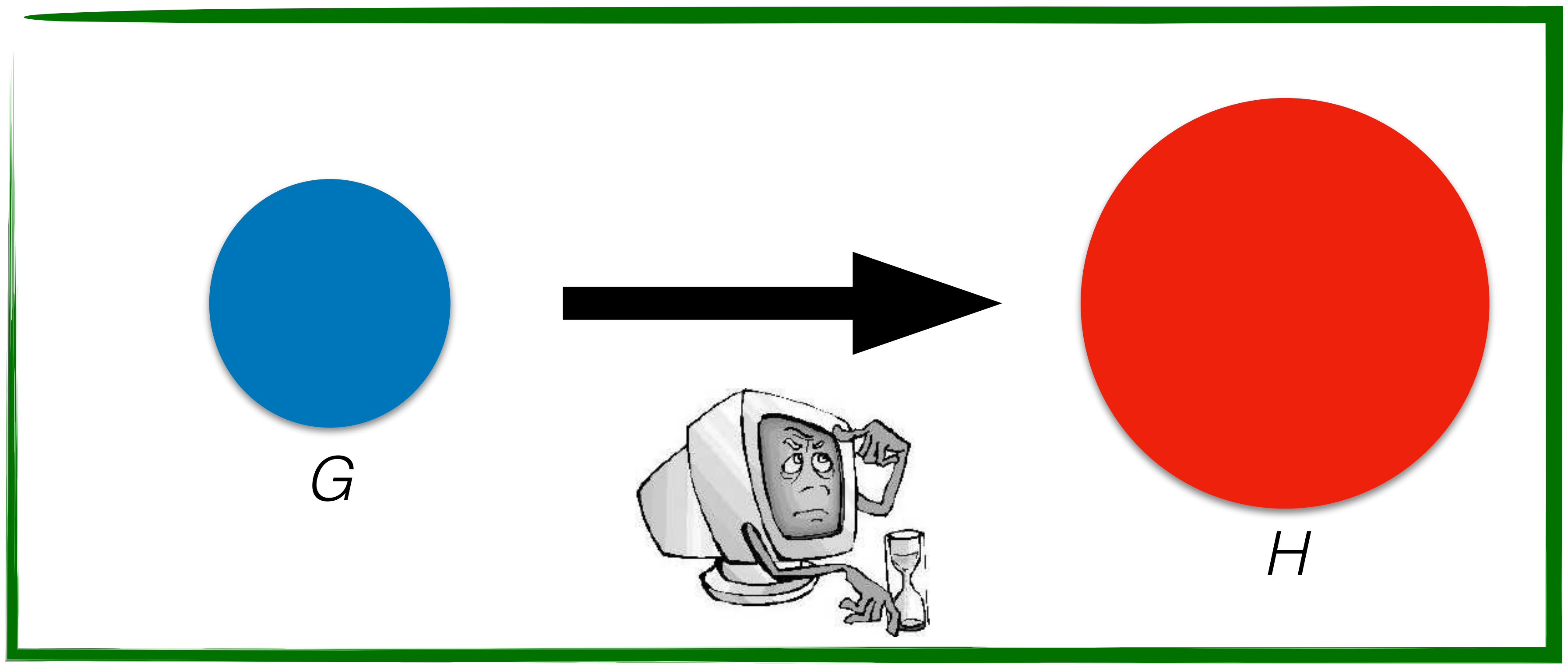

A distribution over graphs *G*

Decoration Construction

[C., Haddadan,'18]

- Let $G = (V,E)$ be a graph, with mixing time T.
- The (random) decoration of G is a super-graph H of G constructed as follows:
  - for each v in V, flip an iid coin: with probability $1/T$,
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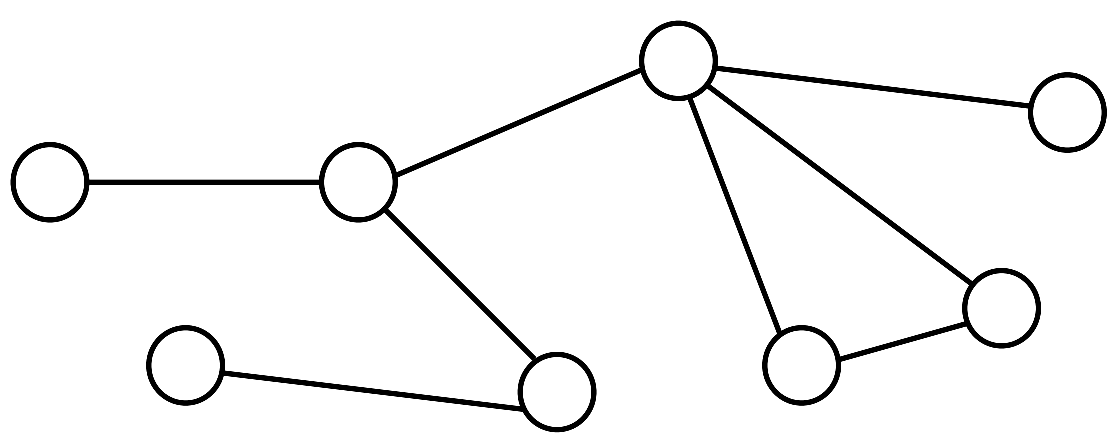

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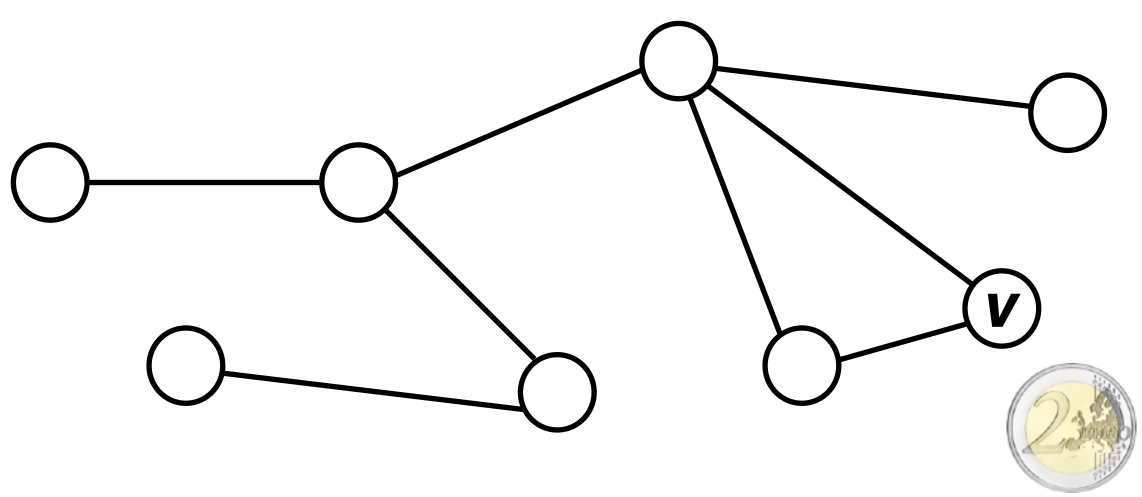

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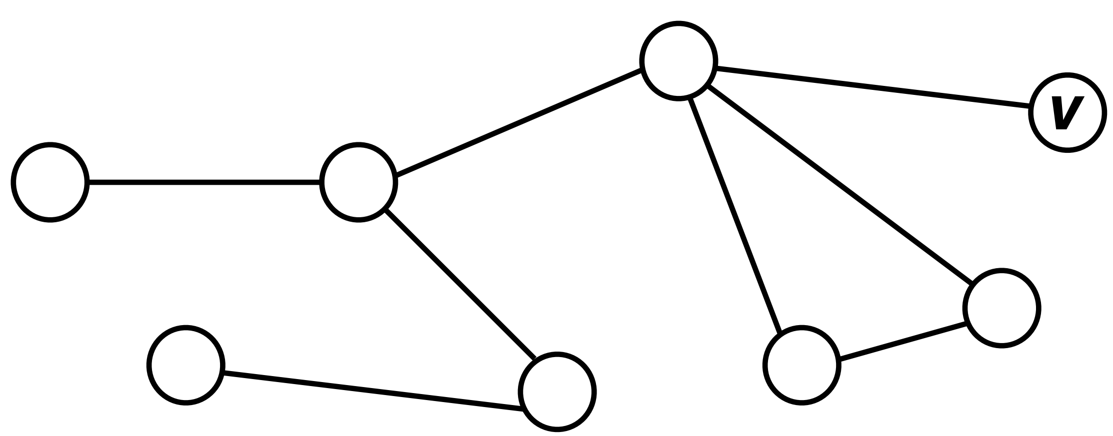

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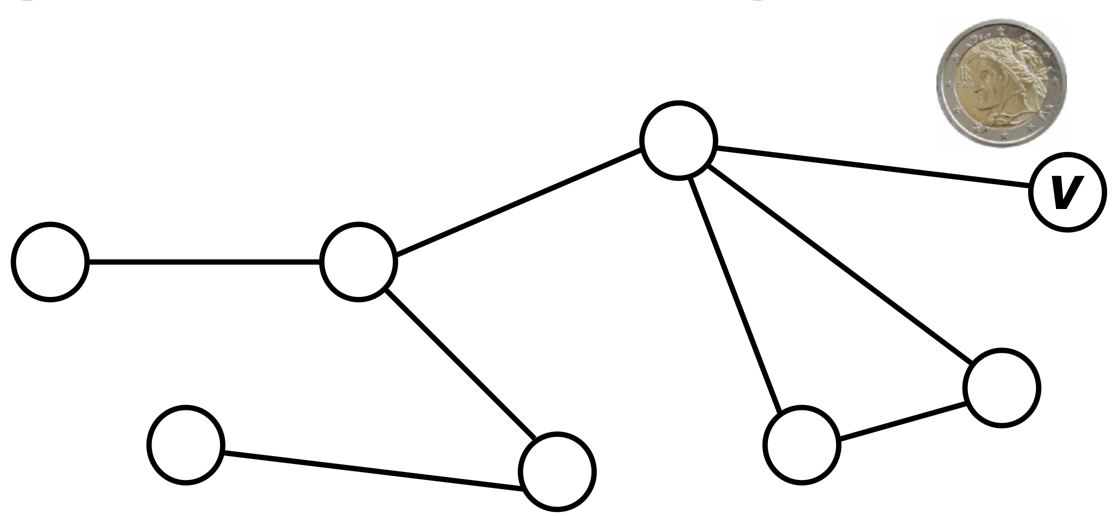

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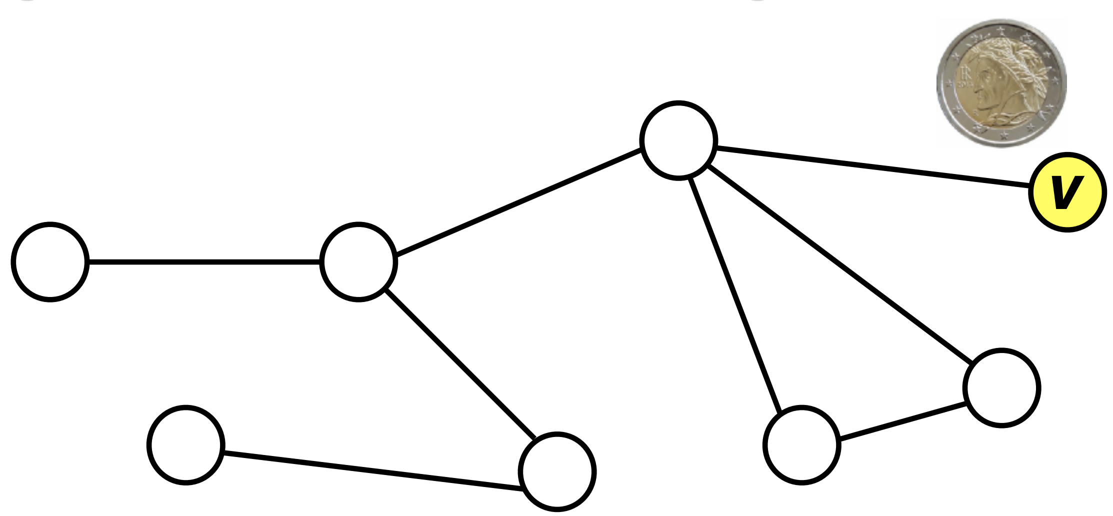

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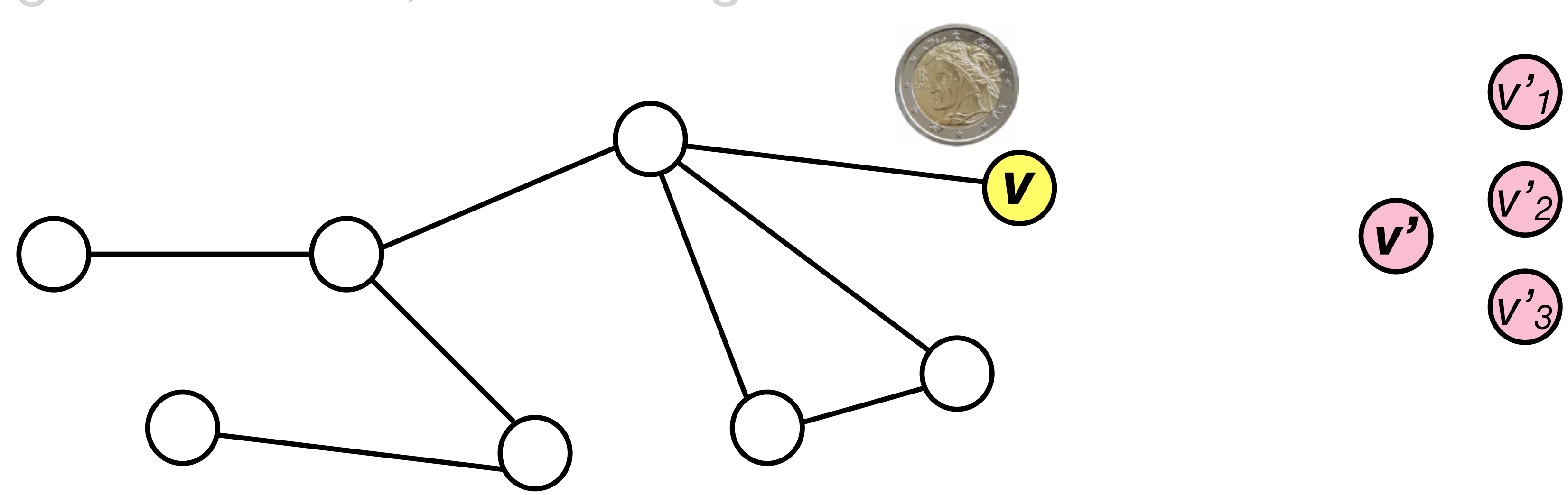

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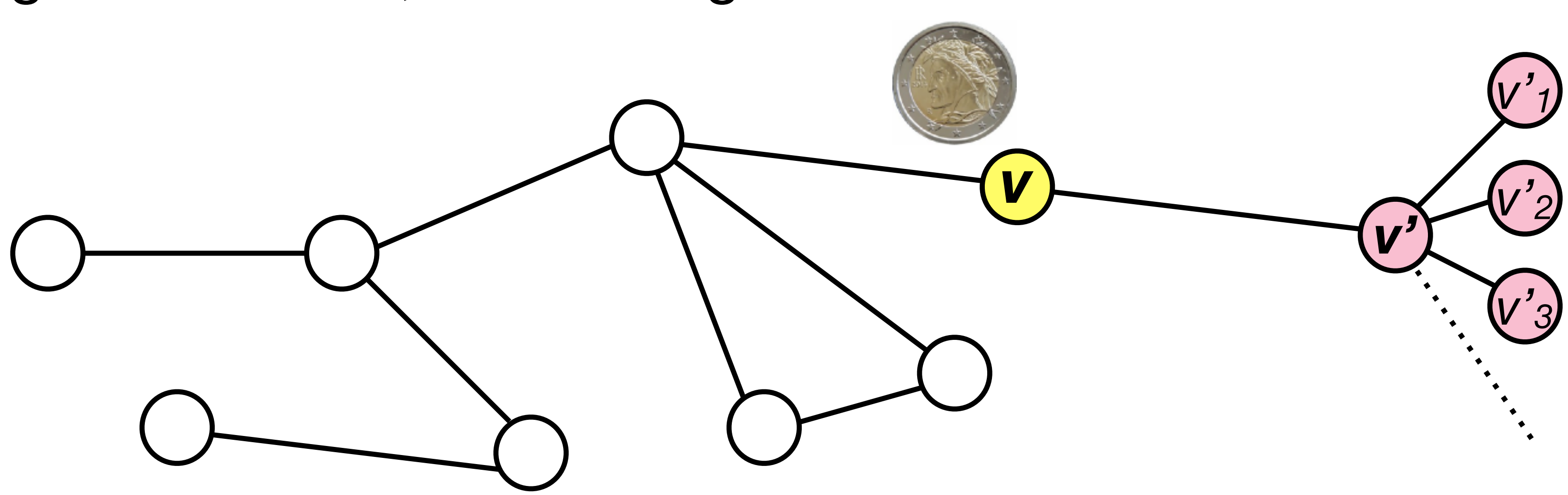

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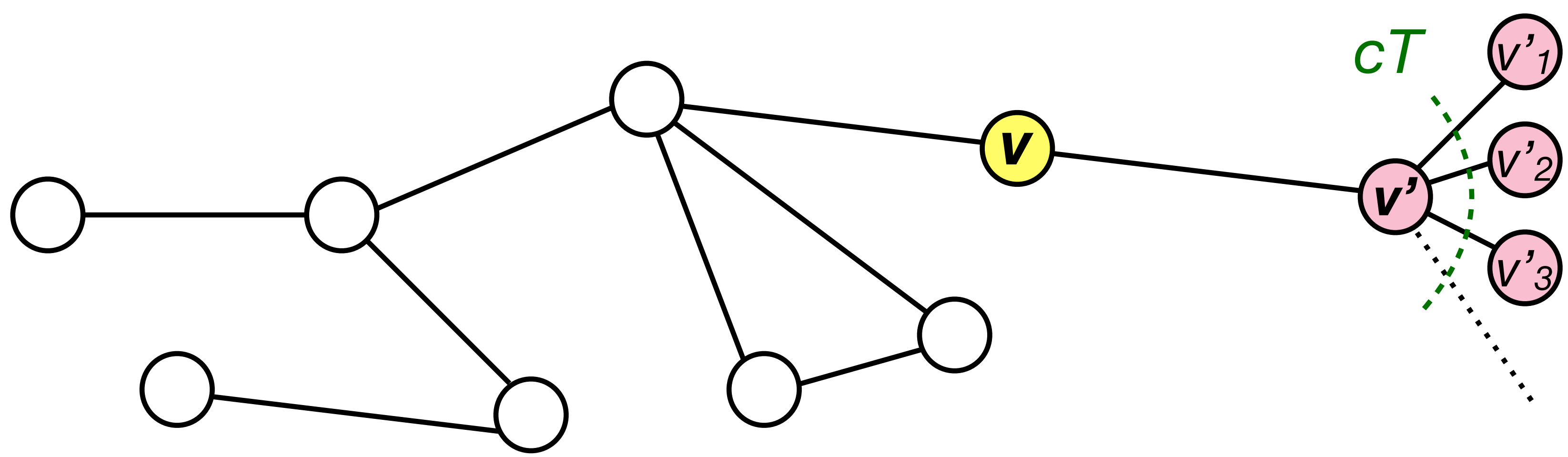

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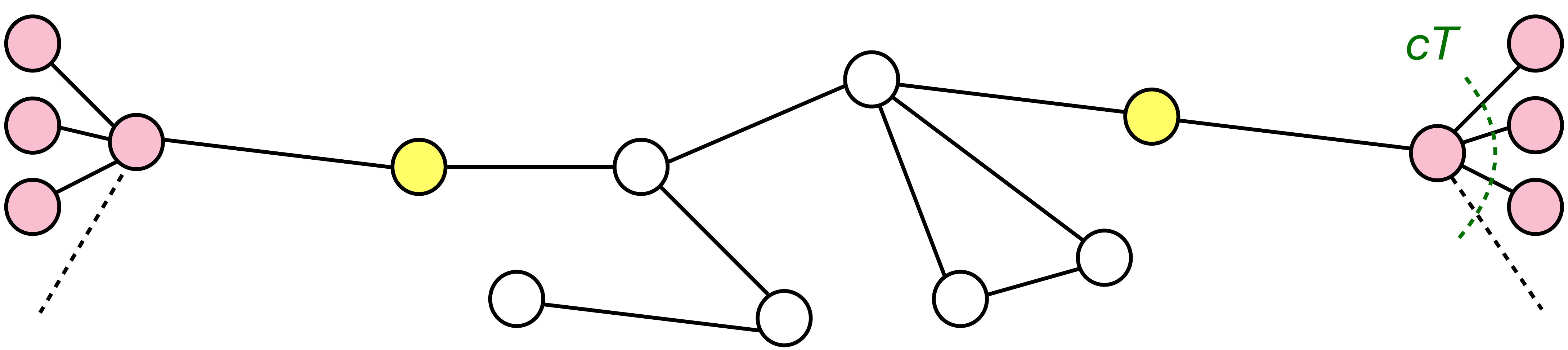

Decoration Construction

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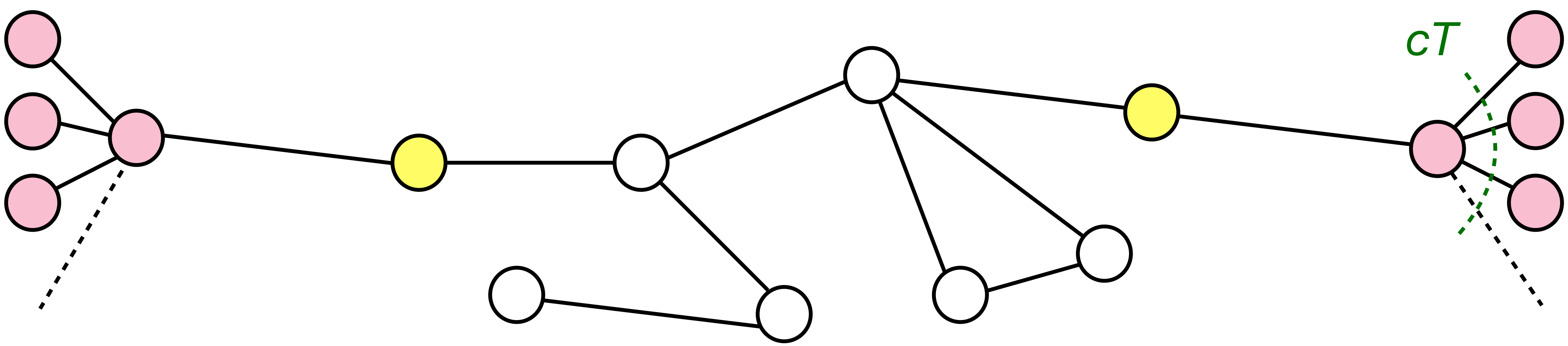
- Let $G = (V,E)$ be a graph, with mixing time $T < o(|V|)$ and average degree $d > \omega(1)$.
- Let H be a random decoration of G.



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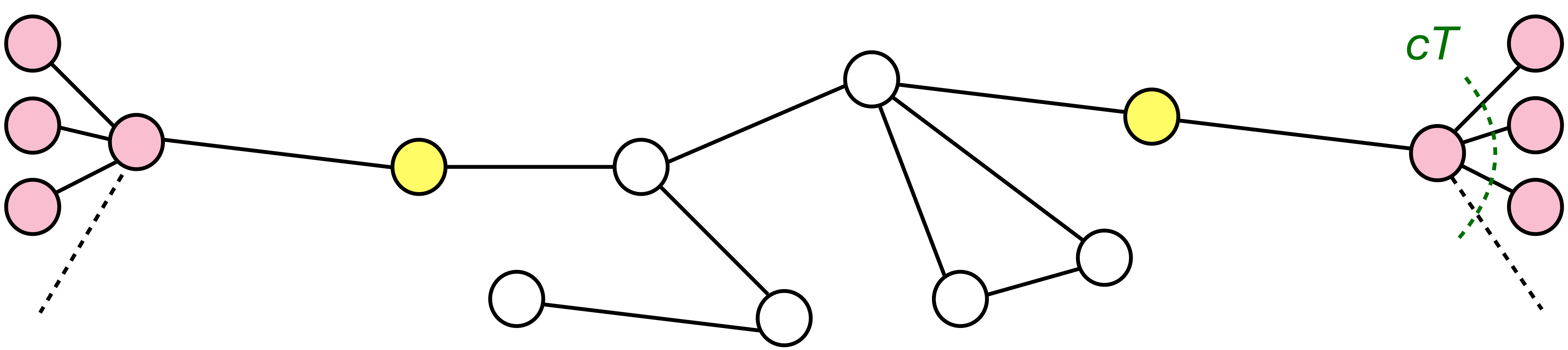
- Let H be a random decoration of G.
- Then, with probability $1\text{-}o(1)$, the mixing time S of H satisfies $\alpha\, T < S < \alpha'\, T$, for constants $\alpha = \alpha(c)$ and $\alpha'=\alpha'(c)$.



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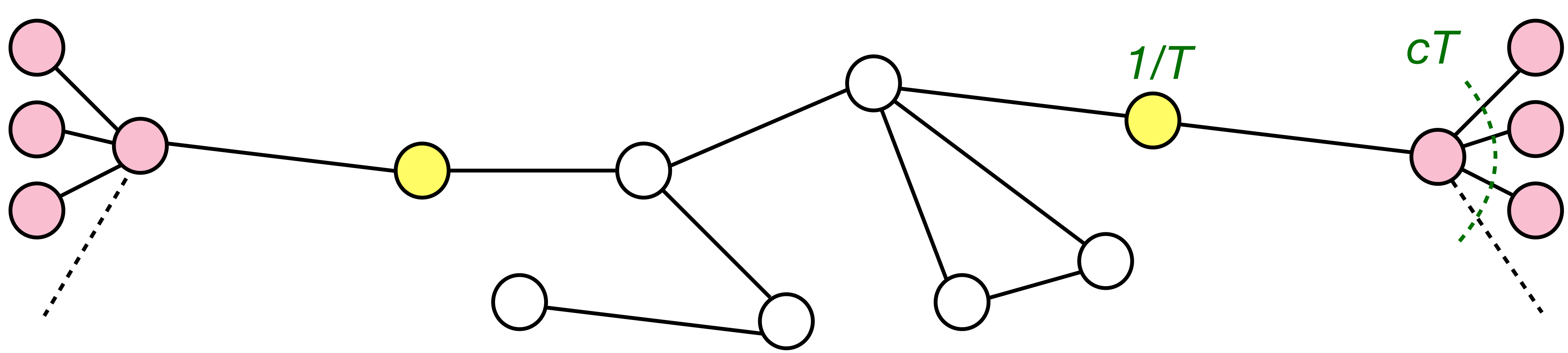
- Moreover, with probability $1 - o(1)$, the number of nodes increases by a factor of $1 + \Theta(c)$



Decoration Construction

[C., Haddadan,'18]

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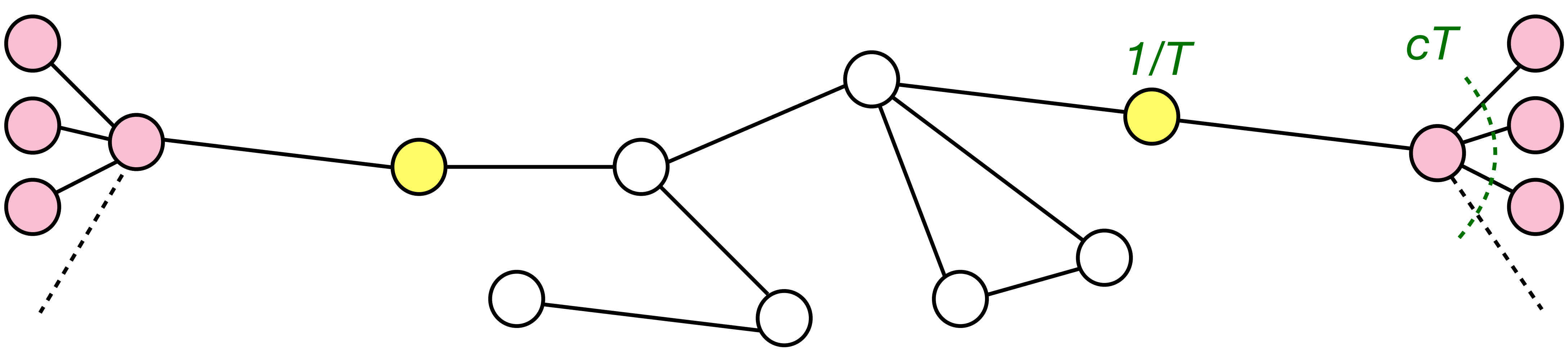

Decoration Construction

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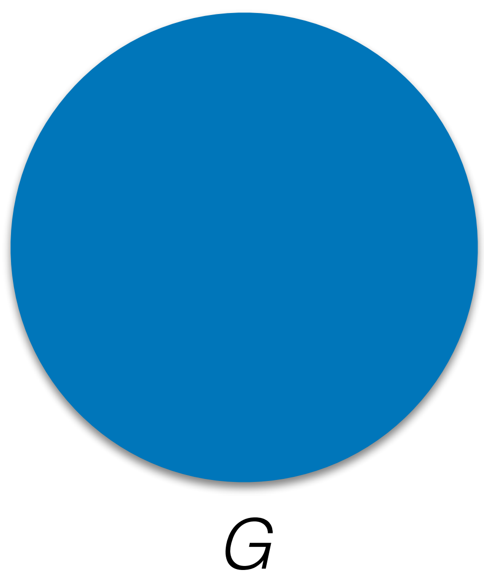
- Let $G = (V,E)$ be a graph, with mixing time $T < o(|V|)$ and average degree $d > \omega(1)$.
- Let H be a random decoration of G.
- Then, with probability $1 - o(1)$:
  - the mixing time S of H satisfies $S = \Theta(T)$,
  - the number of nodes increases by a factor of $1 + \Theta(c)$,
  - the average degree decreases by a factor of $1 + \Theta(c)$.

How to Use The Lemma



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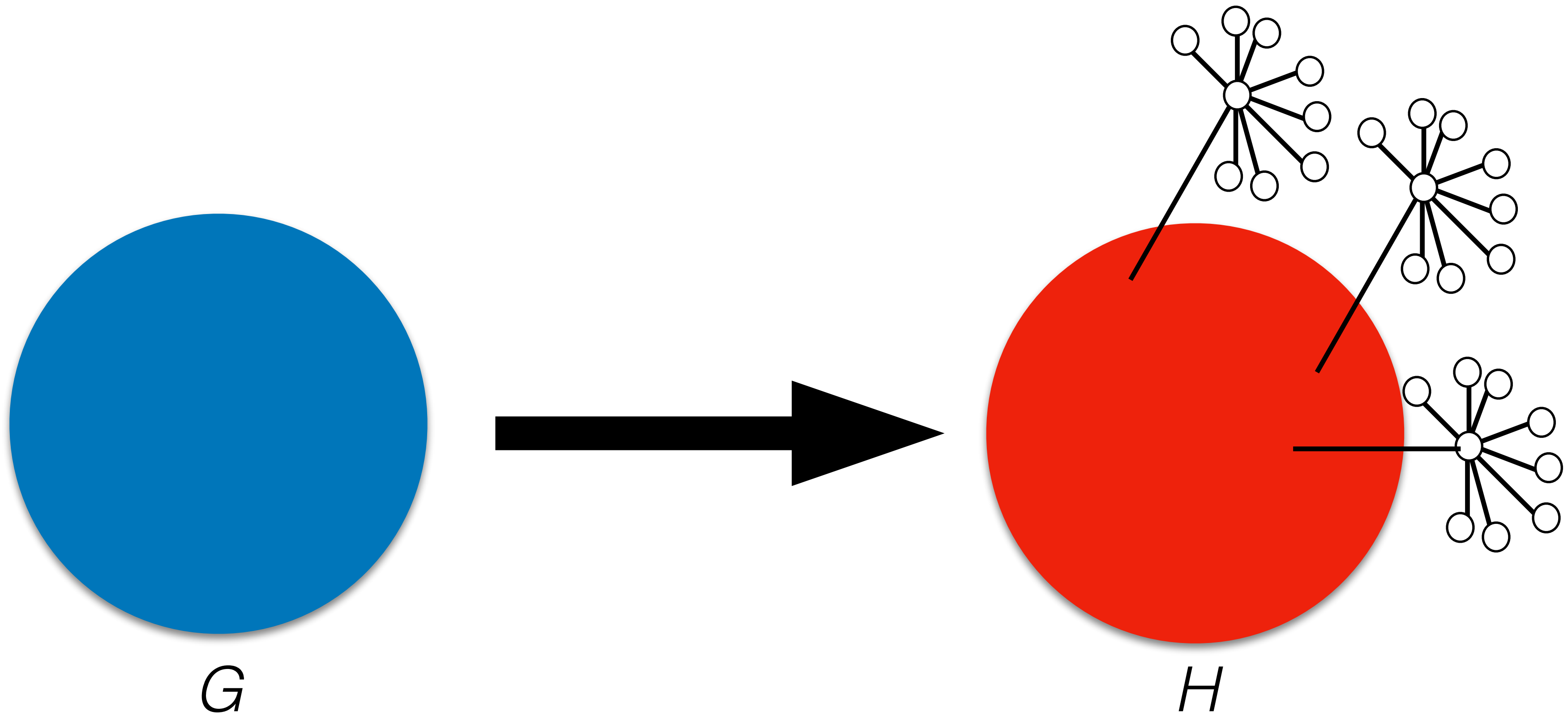

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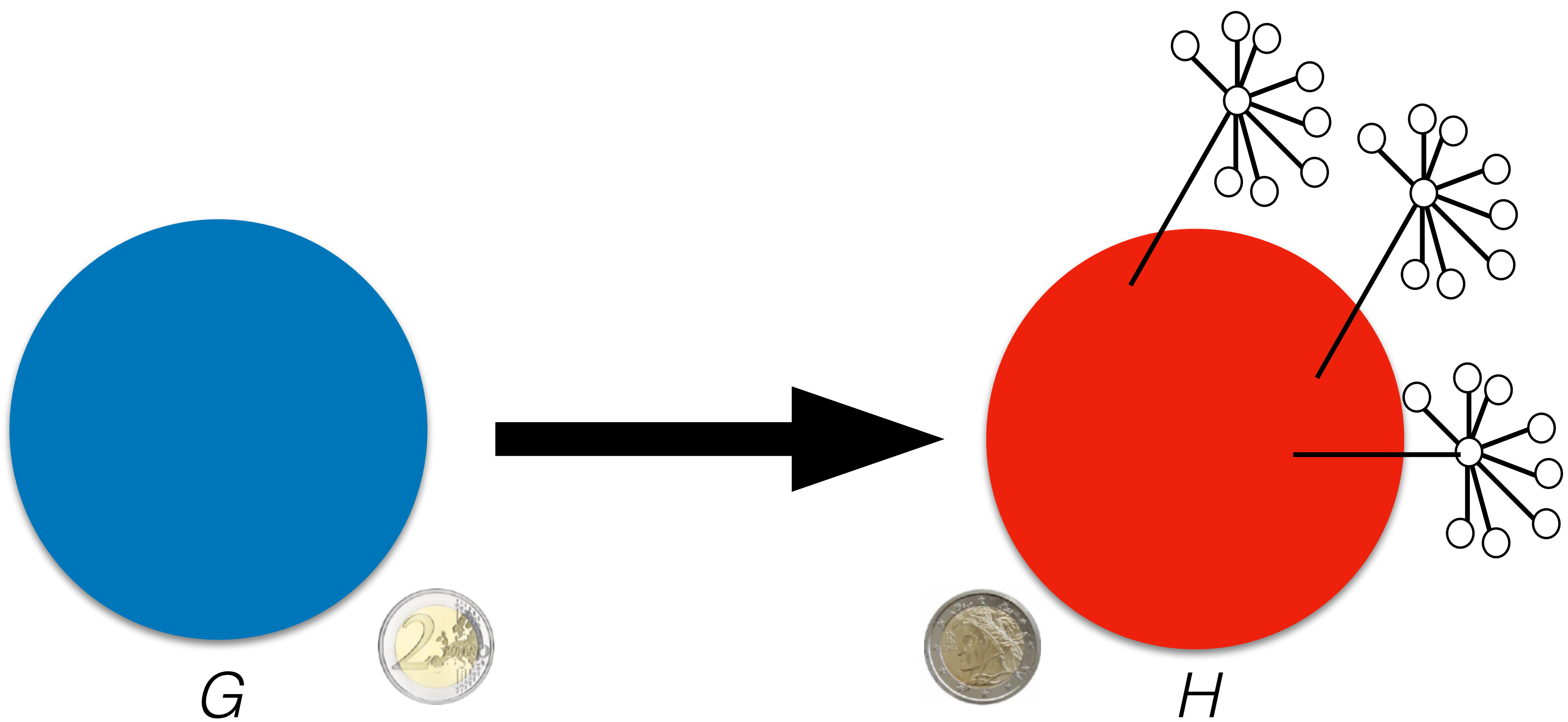

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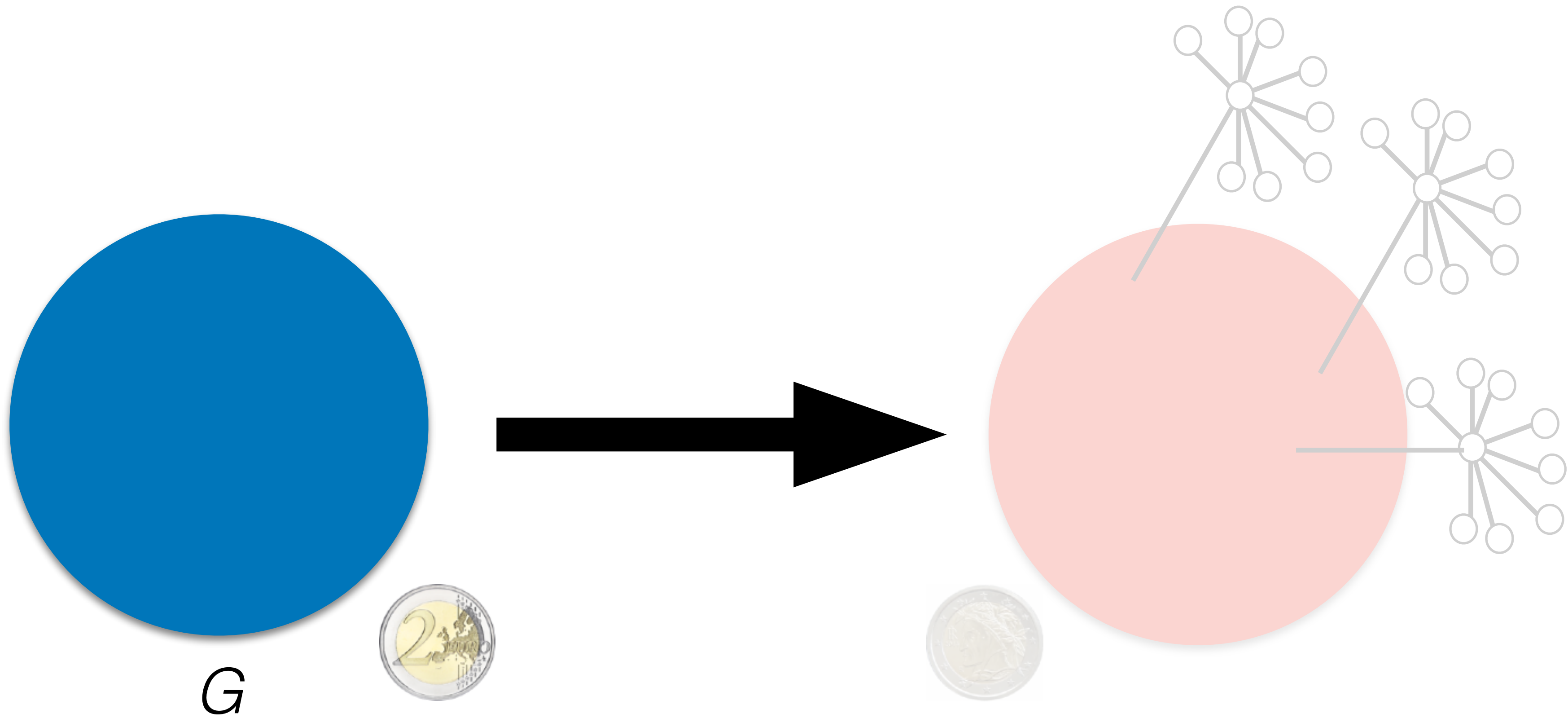

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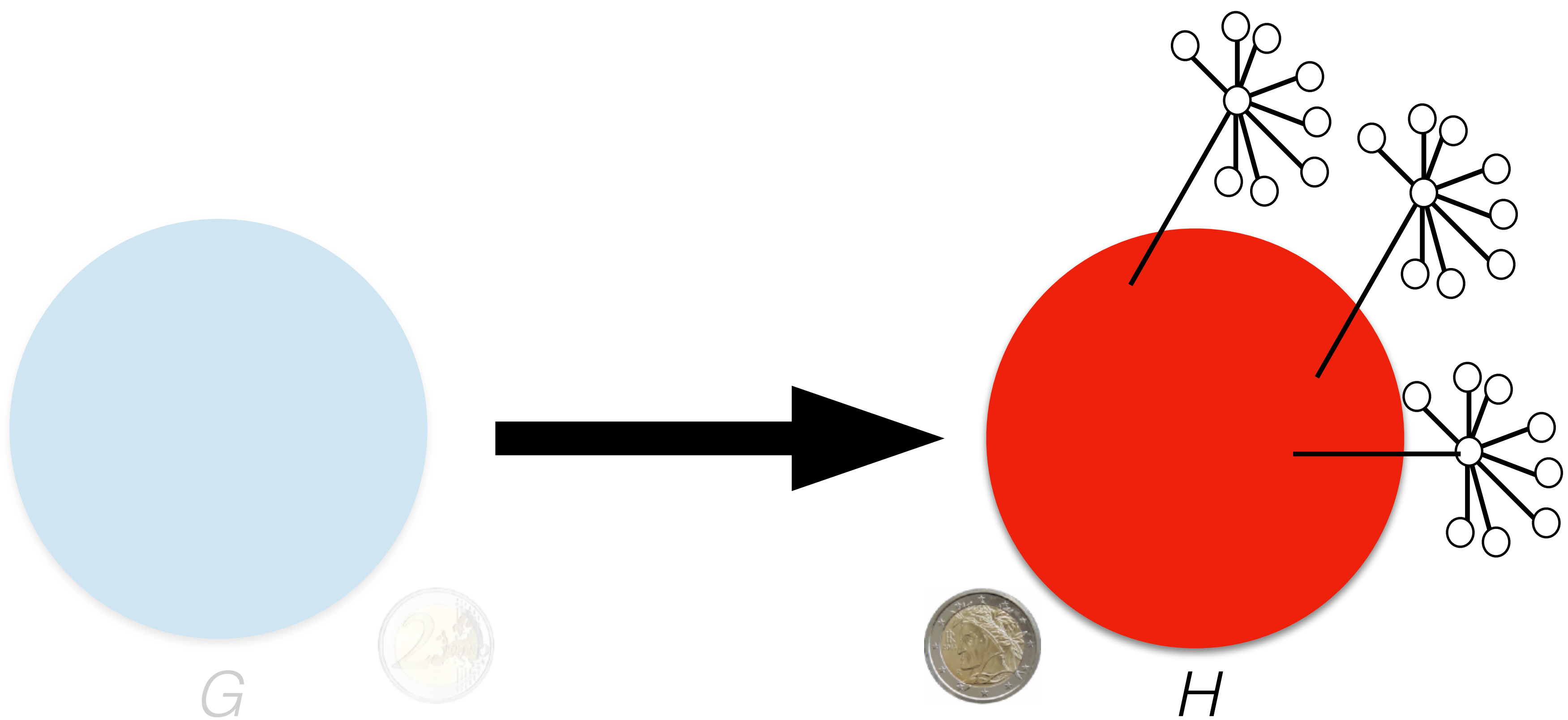

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By showing that the algorithm cannot detect whether it is running on G or H, we prove that the algorithm cannot solve a number of problems.

The Graph G

[C., Haddadan,'18]

- This approach can be made to work with *G* being a *G(n, d/n)* graph, with $d \sim \log n$

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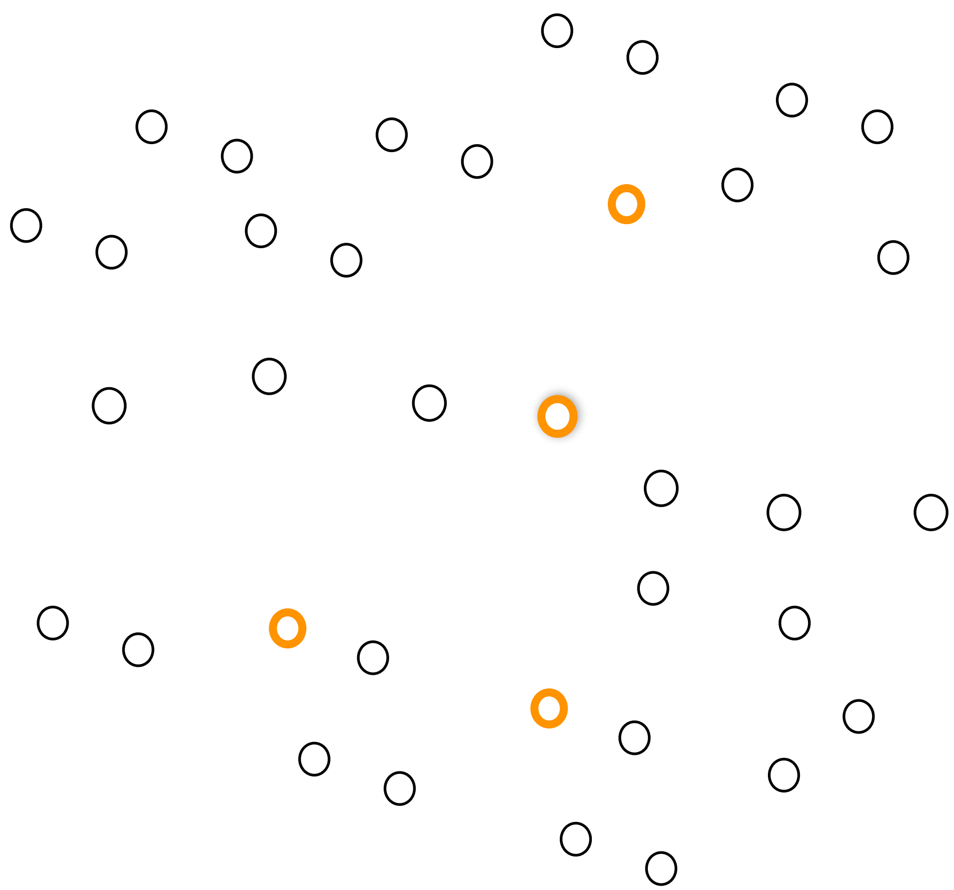

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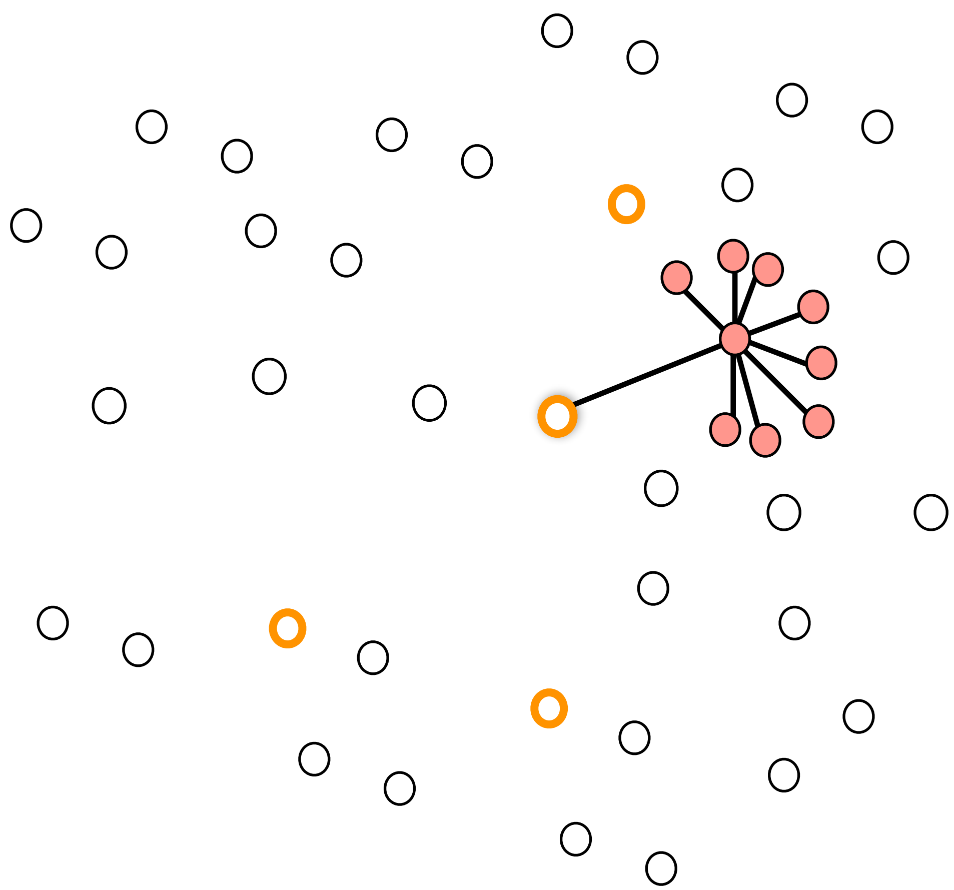

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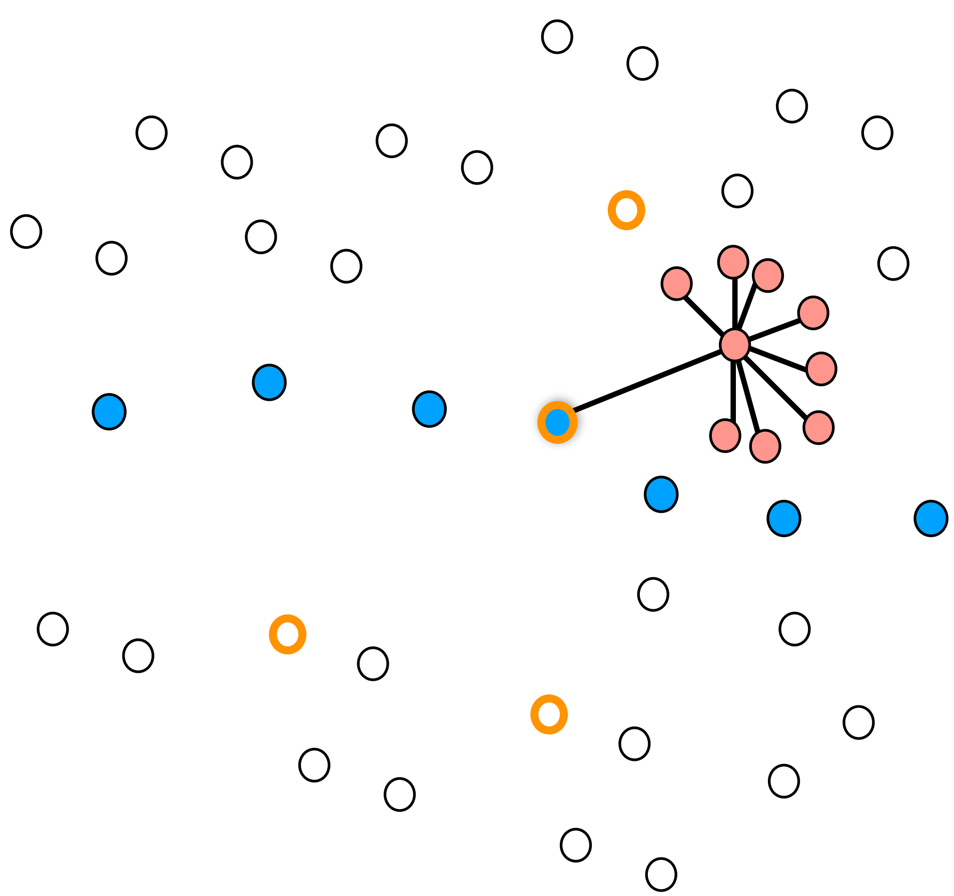

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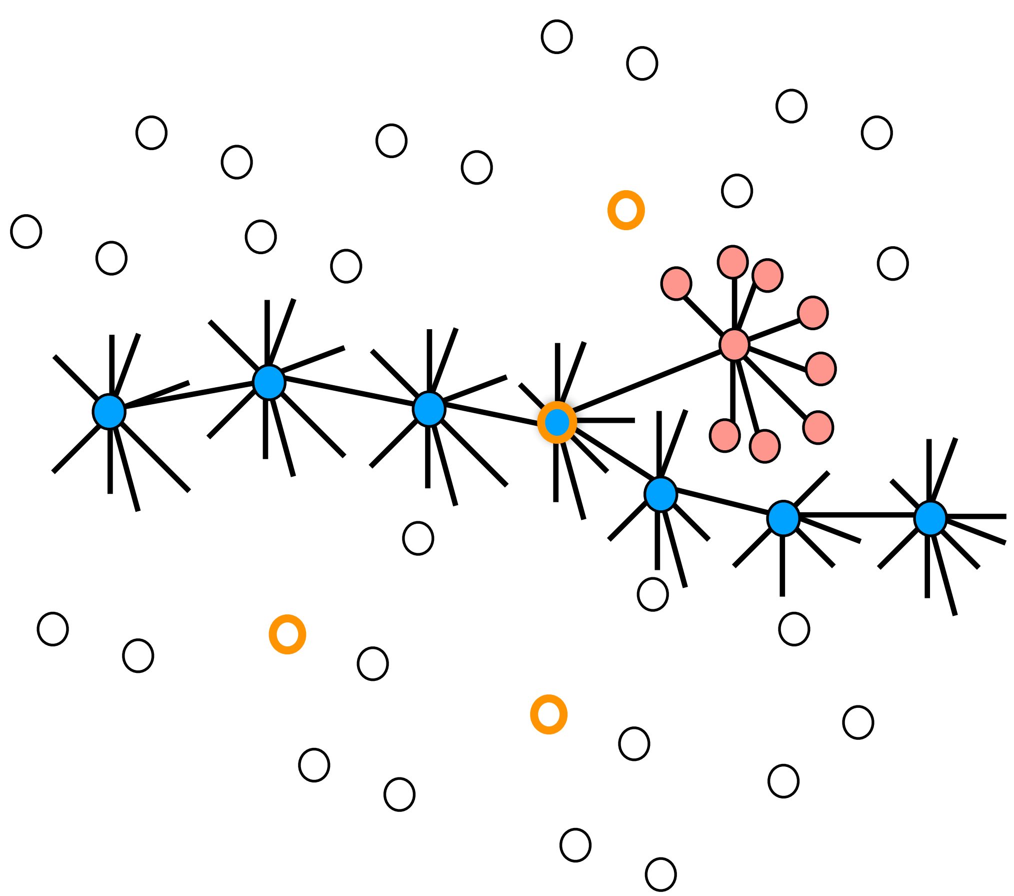

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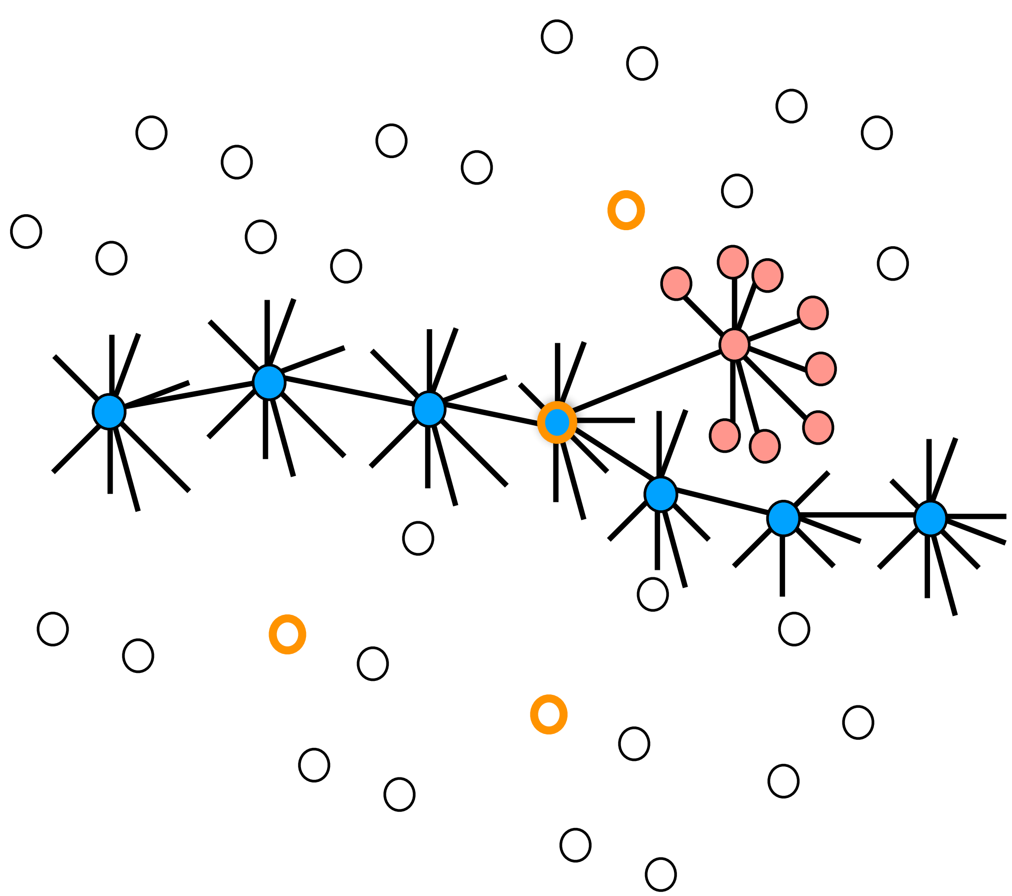

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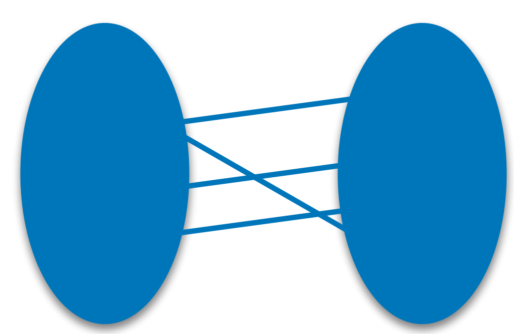

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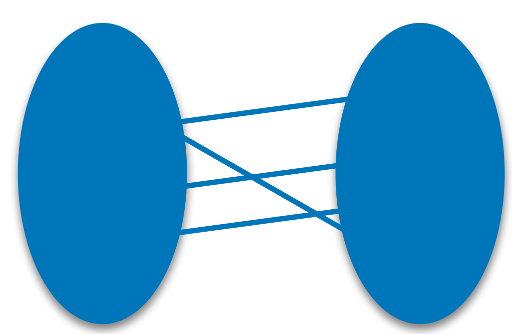
- Therefore, we pick two independent *G(n/2, p)*'s, and join them with a random matching of $< n / 2$ edges,
    - the number of edges allows us to control the mixing time *T* of the resulting *G*.



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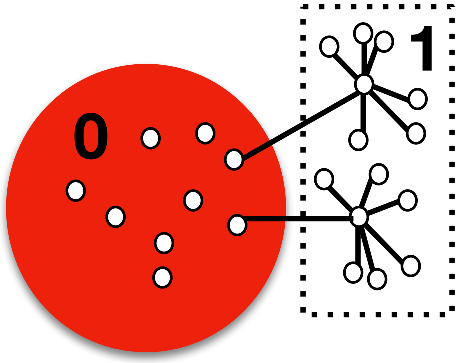

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  - approximate the average value of a bounded function on the nodes to an o(1) error,
  - approximate the number of nodes of G to *any given constant,*
  - approximate the average degree of G to *any given constant.*

Applications

	Upper Bound
Average of a Bounded Function	$O(\mathrm{t_{mix}}\, \mathrm{d_{avg}} \log(\delta^{-1})\epsilon^{-2})$ **Max-Degree**
Uniform Sample	$O(\mathrm{t_{mix}}\, \mathrm{d_{avg}} \log(\epsilon^{-1}))$ **Max-Degree/Rejection-Sampling**

Applications

	Upper Bound	Lower Bound
Average of a Bounded Function	$O(\mathrm{t}_{\mathrm{mix}}\, \mathrm{d}_{\mathrm{avg}} \log(\delta^{-1})\epsilon^{-2})$ **Max-Degree**	$\Omega(\mathrm{t}_{\mathrm{mix}}\, \mathrm{d}_{\mathrm{avg}} \log(\delta^{-1})\epsilon^{-2})$
Uniform Sample	$O(\mathrm{t}_{\mathrm{mix}}\, \mathrm{d}_{\mathrm{avg}} \log(\epsilon^{-1}))$ **Max-Degree/Rejection-Sampling**	$\Omega(\mathrm{t}_{\mathrm{mix}}\, \mathrm{d}_{\mathrm{avg}})$

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Average of a Bounded Function	$O(\mathrm{t_{mix}}\,\mathrm{d_{avg}}\log(\delta^{-1})\epsilon^{-2})$ **Max-Degree**	$\Omega(\mathrm{t_{mix}}\,\mathrm{d_{avg}}\log(\delta^{-1})\epsilon^{-2})$
Uniform Sample	$O(\mathrm{t_{mix}}\,\mathrm{d_{avg}}\log(\epsilon^{-1}))$ **Max-Degree/Rejection-Sampling**	$\Omega(\mathrm{t_{mix}}\,\mathrm{d_{avg}})$
Number of Vertices	$O(\mathrm{t_{mix}}\max\{\mathrm{d_{avg}}, \lvert\Pi^1\rvert_2^{-1}\}\log(\delta^{-1})\log(\epsilon^{-1})\epsilon^{-2})$ **[Katzir et al.]**	$\Omega(\mathrm{t_{mix}}\,\mathrm{d_{avg}})$

Open Questions

- What is the minimum number of *node queries* to approximate the number of nodes of *G?*
- Can the lower bound, and/or the algorithm of *[Katzir et al]*, be improved?

Open Questions

- In *[C., Dasgupta, Kumar, Lattanzi, Sarlós,'16]* we also studied the number of node accesses to return a node with probability proportional to some power of its degree.

- Can one obtain tight lower and upper bounds for this problem?