Fast Approximate Shortest Paths in the Congested Clique

Michal Dory, Technion

Joint work with: Keren Censor-Hillel (Technion), Janne Korhonen (IST Austria), Dean Leitersdorf (Technion)

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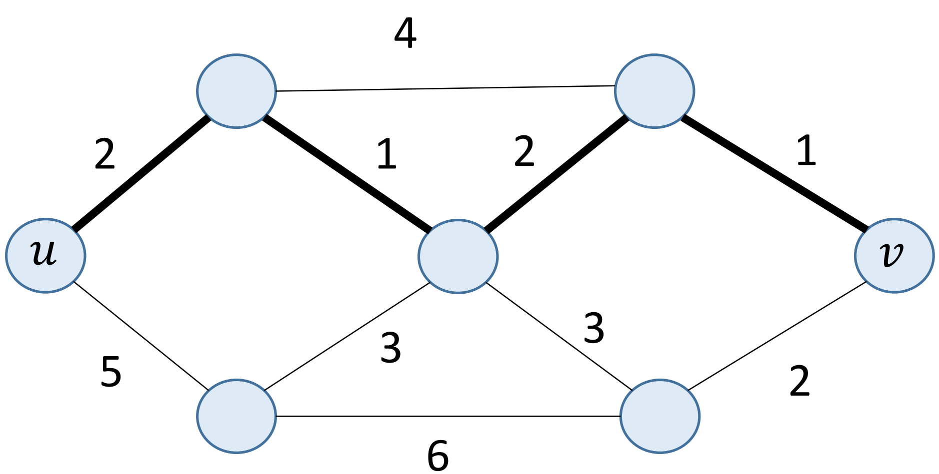
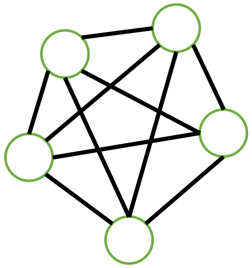
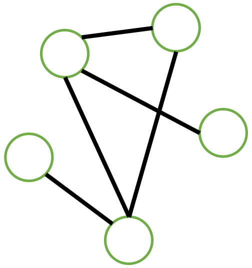
Distance Computation

- All-pairs shortest paths (APSP)
- Single-source shortest paths (SSSP)
- Multi-source shortest paths (MSSP)



The Congested Clique model





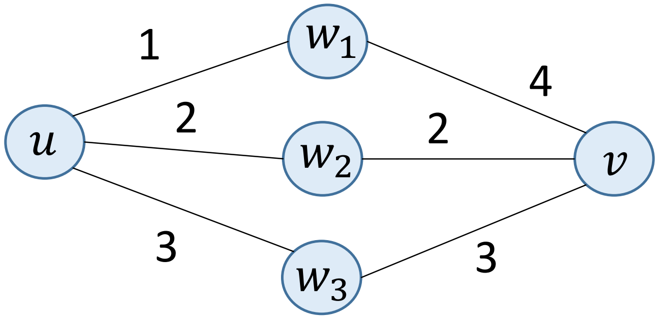
Communication Network **Input Graph**

- n vertices
- Synchronous rounds, $\Theta(\log n)$-bit messages
- All-to-All communication
- Input and output are local

Computing Distances using Matrix Multiplication

- A – weighted adjacency matrix
- Distance product:

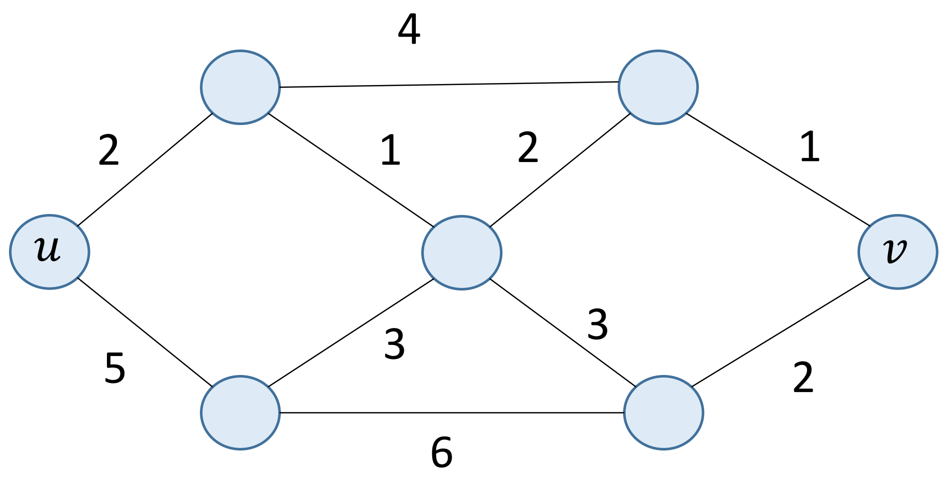
$$A^2[u,v] = \min_{w} A(u,w) + A(w,v)$$



- This is the minimum weight path between u and v of at most 2 edges

Computing Distances using Matrix Multiplication

- Similarly, $A^i[u, v]$ = minimum weight path between u and v of at most i edges (hops).
- <u>Our goal:</u> compute A^n



Computing Distances using Matrix Multiplication

- Our goal: compute A^n
- Requires $O(\log n)$ matrix multiplications:

$$A \rightarrow A^2 \rightarrow A^4 \rightarrow \ldots \rightarrow A^n$$

- How fast can we multiply matrices?

Computing Distances using Matrix Multiplication

$O\left(n^{1-2/\omega}\right) = O(n^{0.158})$	Ring	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
$O\left(n^{1/3}\right)$	Semiring	
	Rectangular, Multiple instances, more	[Le Gall '16]

Computing Distances using Matrix Multiplication

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<tr><td>$O(n^{0.158})$</td><td>• Exact unweighted undirected APSP
• $\left(1+o(1)\right)$-approximation for weighted directed APSP</td><td rowspan="2">[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela ‘15]</td></tr>
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All complexities are polynomial!

What about approximations?

- We can compute a **spanner**: a sparse subgraph that approximates the distances.

$\tilde{O}(n^{1/k})$	$(2k-1)$-approximation for **weighted undirected** APSP

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Still polynomial for any constant k!

Computing Distances in the Congested Clique

Can we get constant approximation for APSP in sub-polynomial time?

Computing Distances in the Congested Clique

Can we get constant approximation for APSP in sub-polynomial time?

- <u>For SSSP:</u>

$O(\epsilon^{-3}\text{polylog}\, n)$-round $(1+\epsilon)$-approximation [Becker, Karrenbauer, Krinninger, Lenzen ‘17]

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Only for a single source!

Our Results: APSP

$O(\log^2 n/\epsilon)$	• $(2+\epsilon)$-approximation for **unweighted undirected** APSP • $(3+\epsilon)$-approximation for **weighted undirected** APSP

First polylog constant-factor approximation!

$(2-\epsilon)$-APSP implies MM	[Dor, Halperin, Zwick ‘00 Korhonen, Suomela ‘18]

Our Results: MSSP and more

$O(\log^2 n/\epsilon)$	$(1+\epsilon)$-approximation weighted undirected **MSSP** with $O(n^{1/2})$ sources
$O(\log^2 n/\epsilon)$	Near $(3/2)$-approximation for **diameter**
$\tilde{O}(n^{1/6})$	Exact weighted undirected **SSSP**

Previous results:

$\tilde{O}(n^{1/3})$	Exact weighted **SSSP**	[Censor-Hillel, Kaski, Korhonen, Lenzen, Paz, Suomela '15]
$O(\epsilon^{-3}\text{polylog}\, n)$	$(1+\epsilon)$-**SSSP**	[Becker, Karrenbauer, Krinninger, Lenzen '17]

Our Techniques

- We can multiply *sparse* matrices faster:

$O\left(1 + \frac{(\rho_S \rho_T)^{1/3}}{n^{1/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Leitersdorf, Turner '18]

- ρ_A= density of A, the average number of non-zero entries on a row
- Example: $O(1)$ rounds for $O(n^{3/2})$ edges.

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  - We want to compute distances in *general* graphs.

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- How can we use this?

  - Even if A is sparse, A^2 can be dense.

  - We want to compute distances in *general* graphs.

Many *building blocks* for distance computation are actually based on computations in *sparse* graphs

Building blocks for distance computation

- k-nearest: for each vertex, compute distances to k nearest vertices

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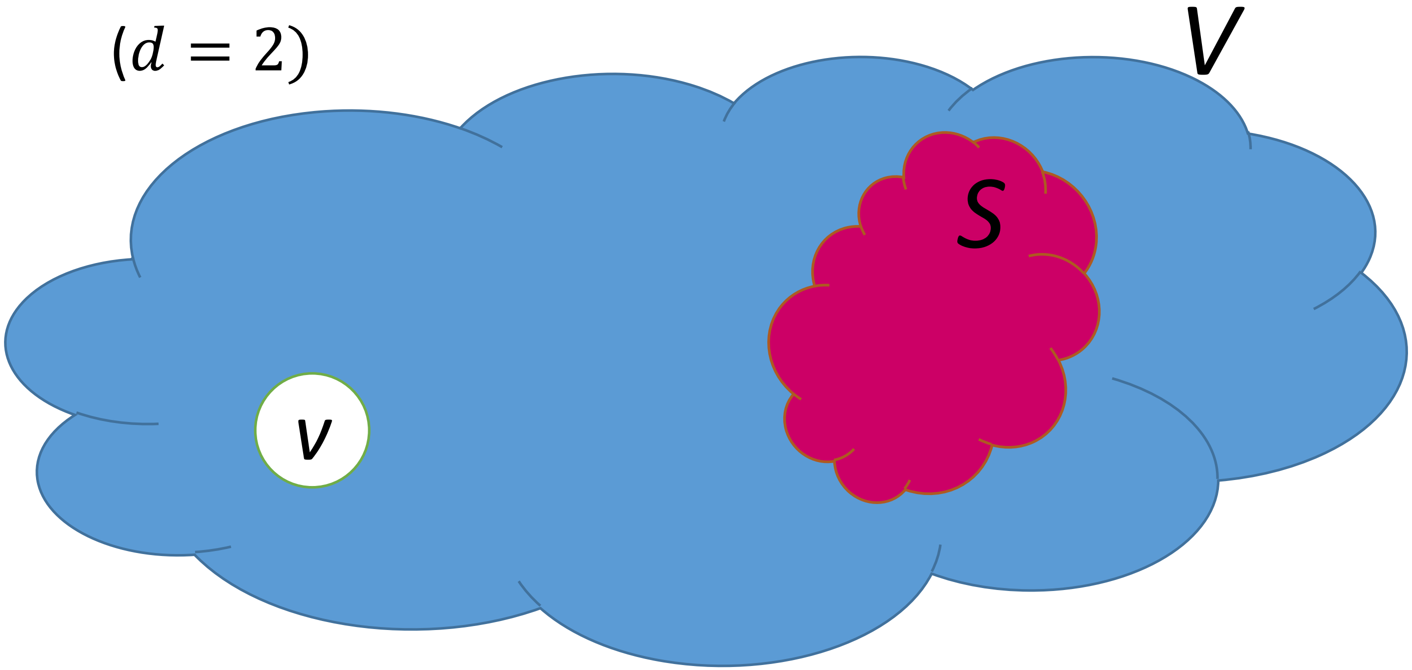

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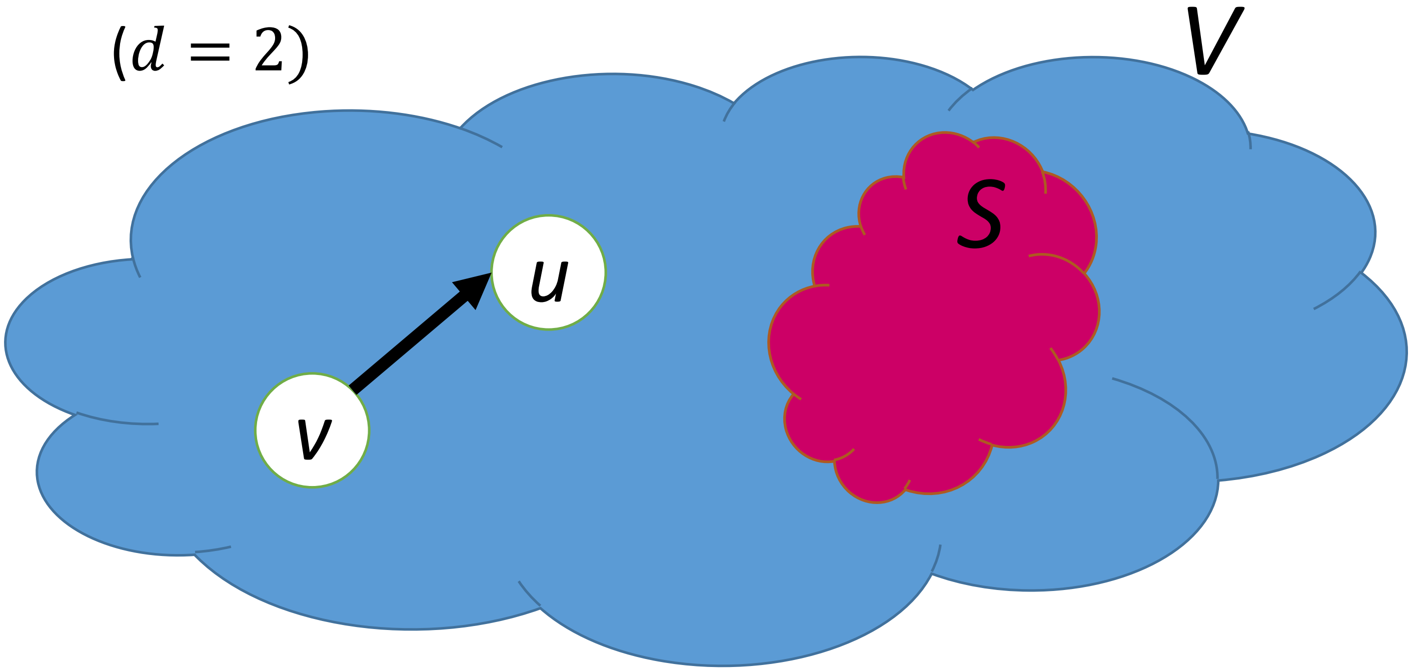

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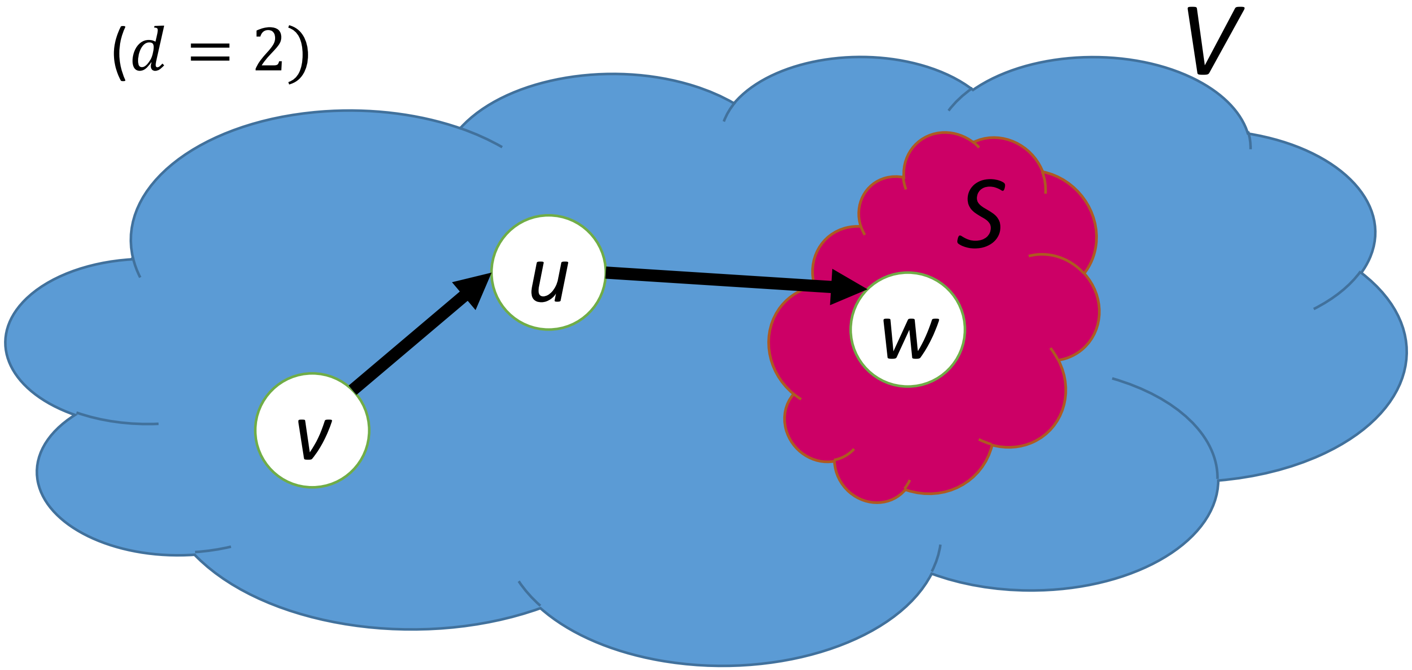
$(d = 2)$



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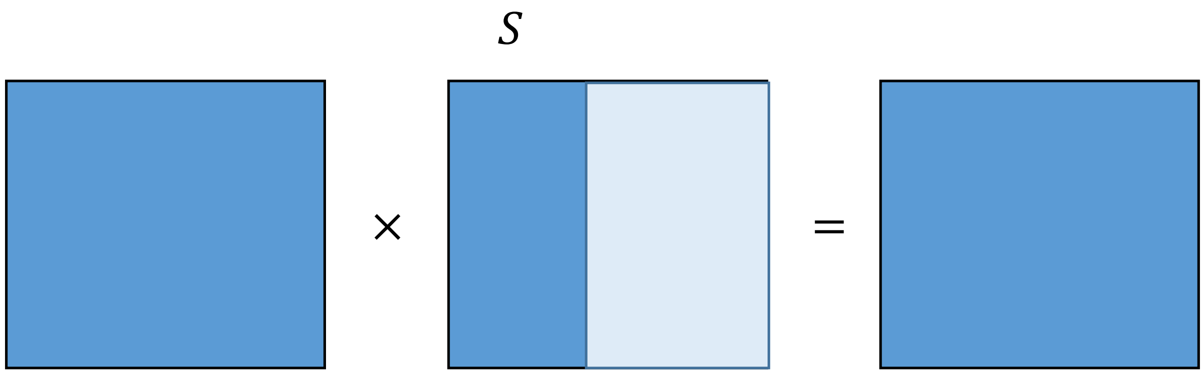

$(d = 2)$

Multiplication of sparse matrix by dense matrix: previous MM algorithm is *still polynomial*

Building blocks for distance computation

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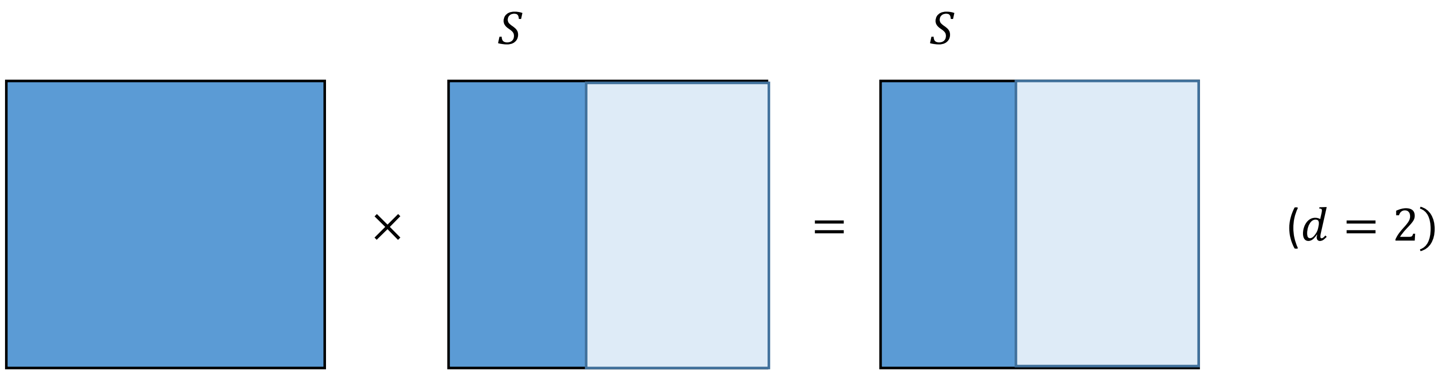

Output matrix is also sparse!

Building blocks for distance computation

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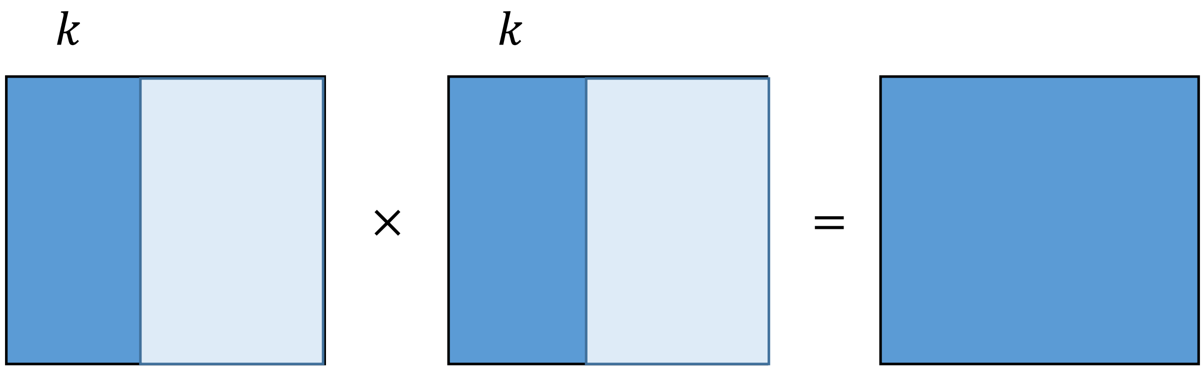
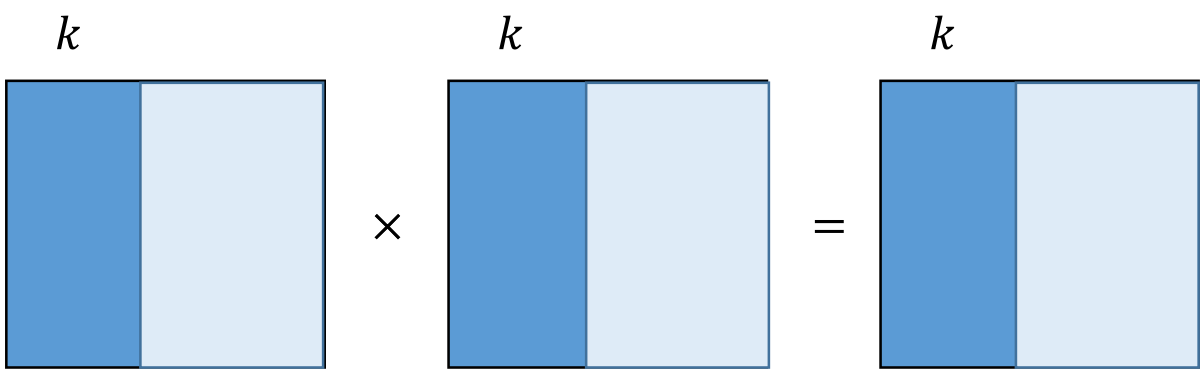


It's enough to look only at the k closest vertices to each vertex

Building blocks for distance computation

- *k-nearest*: for each vertex, compute distances to k nearest vertices



It's enough to look only at the k closest vertices to each vertex: also in the *output*

We don't know the identity of the k closest vertices before the computation

New Matrix Multiplication algorithm

- Previous algorithm:

$O\left(1+\frac{(\rho_S\rho_T)^{1/3}}{n^{1/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Leitersdorf, Turner '18]

- Our algorithm:

$O\left(1+\frac{(\rho_S\rho_T\rho_P)^{1/3}}{n^{2/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Dory, Korhonen, Leitersdorf, '19]

$S \times T = P$

New Matrix Multiplication algorithm

- Our algorithm:

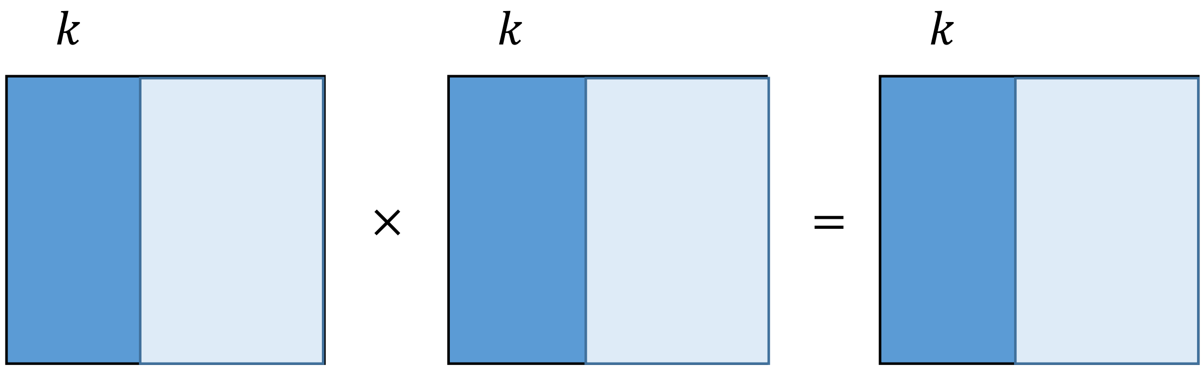
$O\left(1+\frac{(\rho_S \rho_T \rho_P)^{1/3}}{n^{2/3}}\right)$	Semiring, Sparse	[Censor-Hillel, Dory, Korhonen, Leitersdorf, '19]

- Depends also on the sparsity of the *output* matrix
- Even if we don't know the structure of the output matrix, we can *sparsify* the output matrix *on-the-fly*, keeping only ρ_P smallest entries for each row

Application: Distance Tools

k-nearest:

$O\left(\left(\frac{k}{n^{2/3}} + \log n\right)\log k\right)$ rounds ⇨ $O(\log^2 n)$ for $k = n^{2/3}$

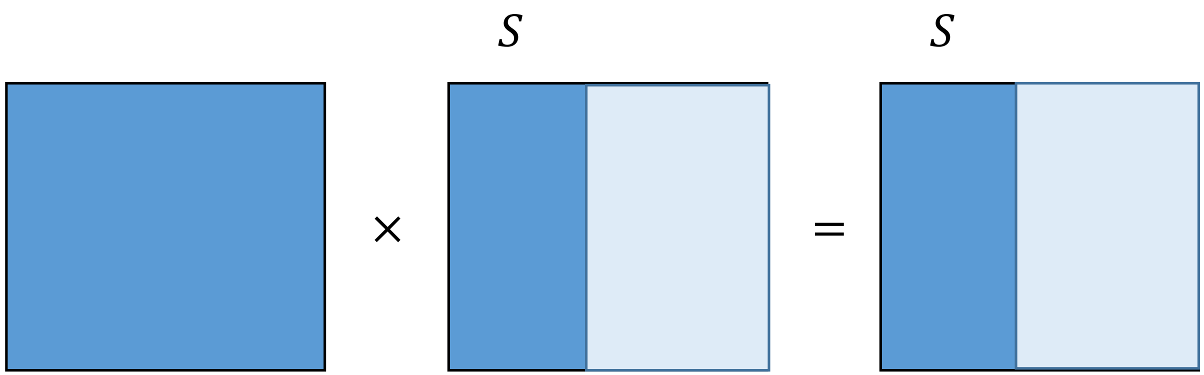


Application: Distance Tools

(S, d, k)-source detection:

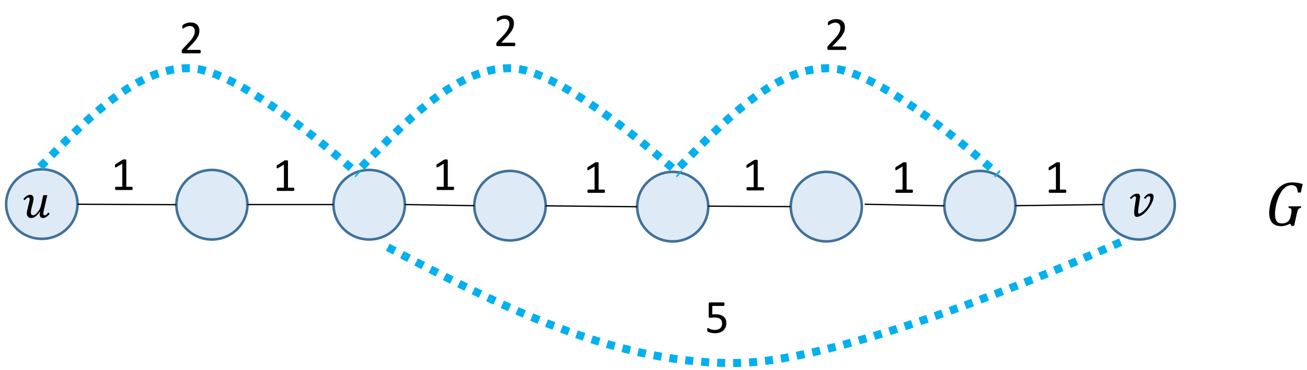
$O\left(\left(\frac{m^{1/3}|S|^{2/3}}{n}+1\right)d\right)$ rounds ($m =$ number of edges)

- To exploit the sparsity we need $\boldsymbol{d = n}$ multiplications - too expensive!



Solution: Hopsets

(β, ϵ)-hopset H:
A graph $H = (V, E')$, such that the β-hop distances in $G \cup H$ give $(1 + \epsilon)$-approximation for the distances in G



Enough to look at β-hop distances!

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Our goal: to have small β and small running time t

What is known?

- We can get $\beta = t = O\left(\frac{\log \log n}{\epsilon}\right)^{\log \log n}$
  [Elkin, Neiman '17]

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Can we get a poly-logarithmic complexity?

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Yes! we can get: $\beta = O\left(\frac{\log n}{\epsilon}\right)$, $t = O\left(\frac{\log^2 n}{\epsilon}\right)$

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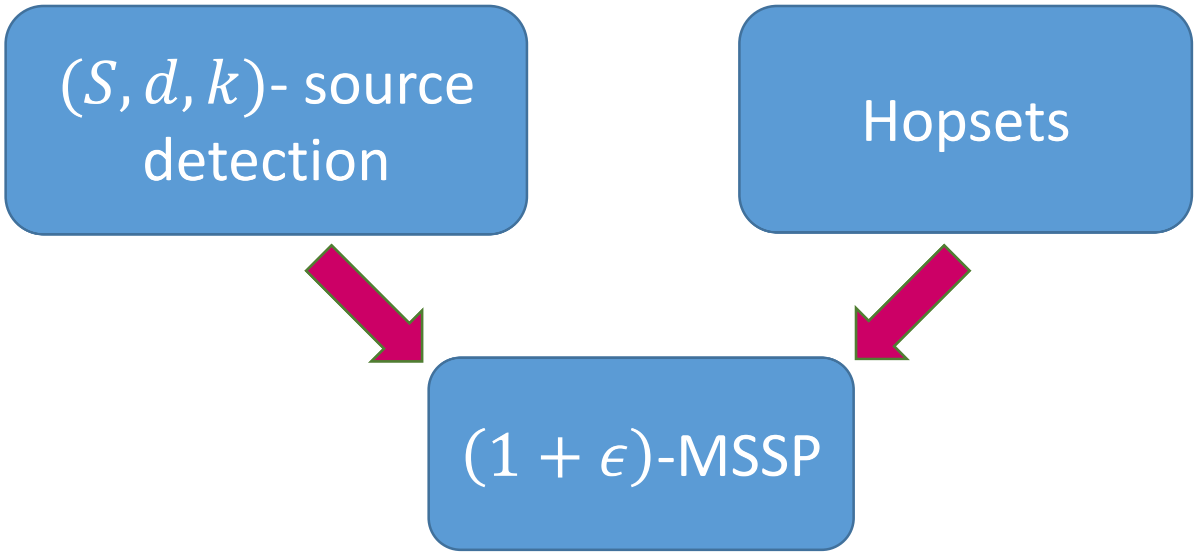
Idea: using our *distance tools* we can implement efficiently the hopset construction of [Elkin, Neiman '17] [Huang, Pettie '19] [Thorup, Zwick '06]

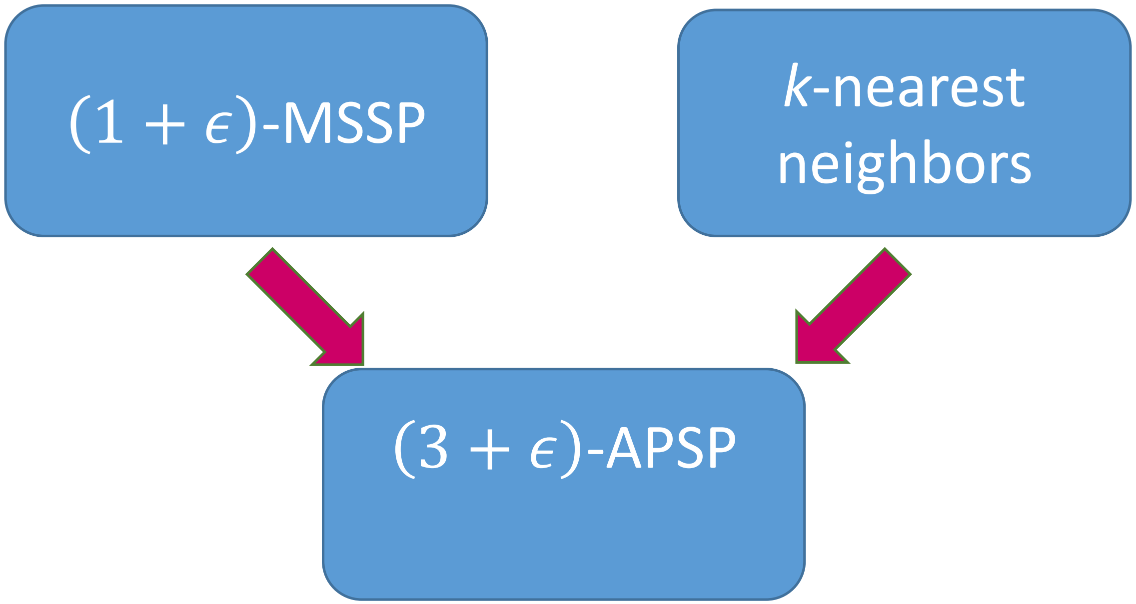
Applications: MSSP



Complexity: $O\left(\left(\frac{|S|^{2/3}}{n^{1/3}} + \log n\right)\frac{\log n}{\epsilon}\right)$

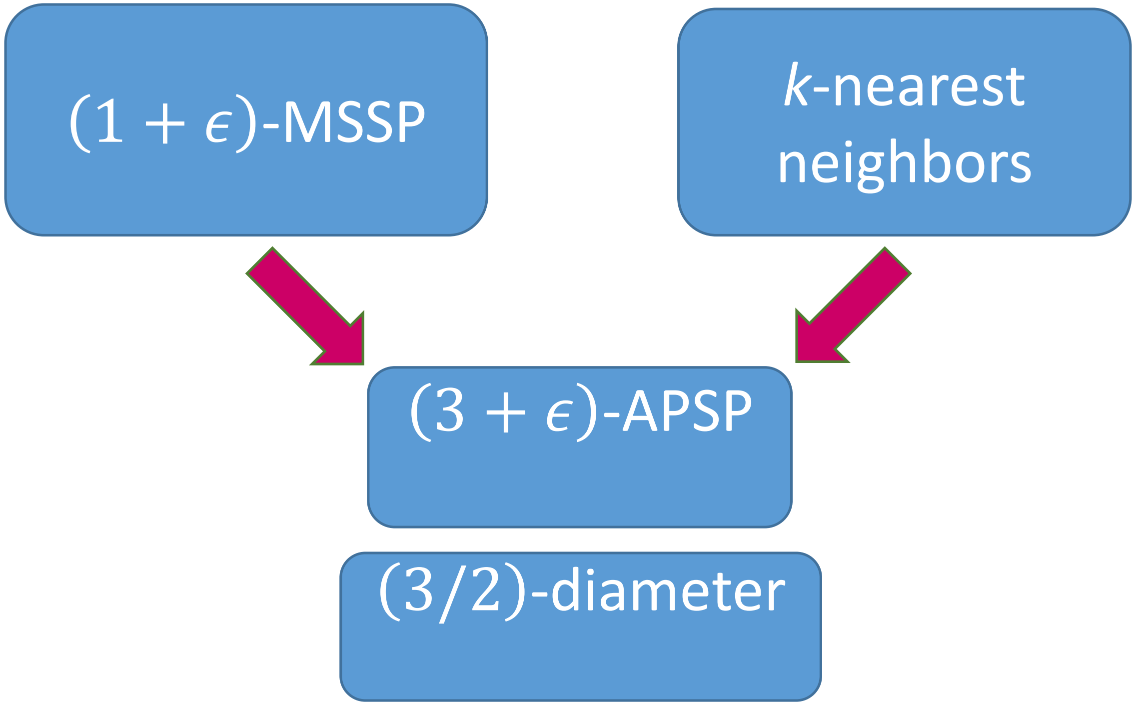
➡ **poly-logarithmic** for $|S| = O(\sqrt{n})$

Applications: weighted APSP



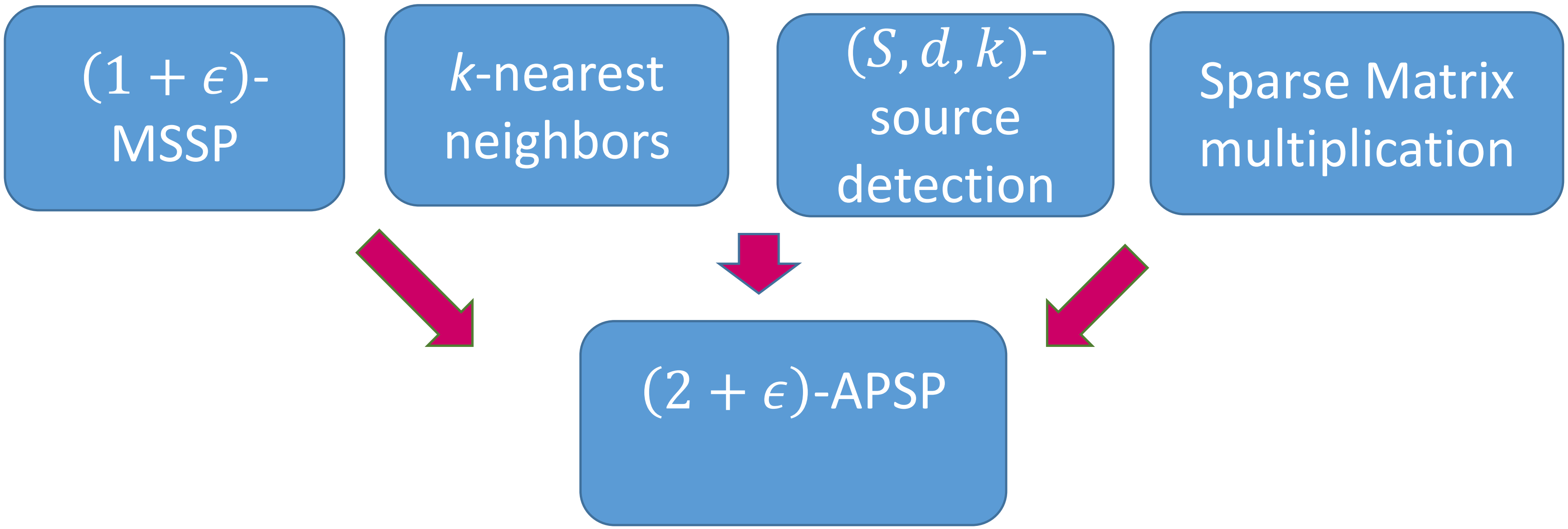
Complexity: $O\left(\frac{\log^2 n}{\epsilon}\right)$

Applications: APSP, diameter



Complexity: $O\left(\frac{\log^2 n}{\epsilon}\right)$

Applications: **unweighted** APSP



<u>Complexity:</u> $O\left(\frac{\log^2 n}{\epsilon}\right)$

Conclusion

- We show a fast algorithm for matrix multiplication that depends on the *sparsity* and is *output*-sensitive.
- Allows to build efficient *distance tools.*
- Together with *hopsets*: polylog algorithms for MSSP, APSP.

Summary

$O(\log^2 n/\epsilon)$	• $(2+\epsilon)$-approximation for **unweighted undirected** APSP • $(3+\epsilon)$-approximation for **weighted undirected** APSP
$O(\log^2 n/\epsilon)$	$(1+\epsilon)$-approximation for **weighted** undirected **MSSP** with $O(n^{1/2})$ sources
$O(\log^2 n/\epsilon)$	Near $(3/2)$-approximation for **diameter**
$\tilde{O}(n^{1/6})$	Exact weighted undirected **SSSP**

Open Questions

- Additional applications for distance tools
- Can we get a $(2+\epsilon)$-approximation for weighted APSP?
- Can we get sub-polynomial algorithm for exact SSSP? Or directed SSSP?