Optimal Distributed Covering Algorithms

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accepted to DISC

Min Weight Hitting Set (Hypergraph Vertex Cover)

Input:

n - # elements

$e_1, \dots, e_m \subseteq [n]$ - sets

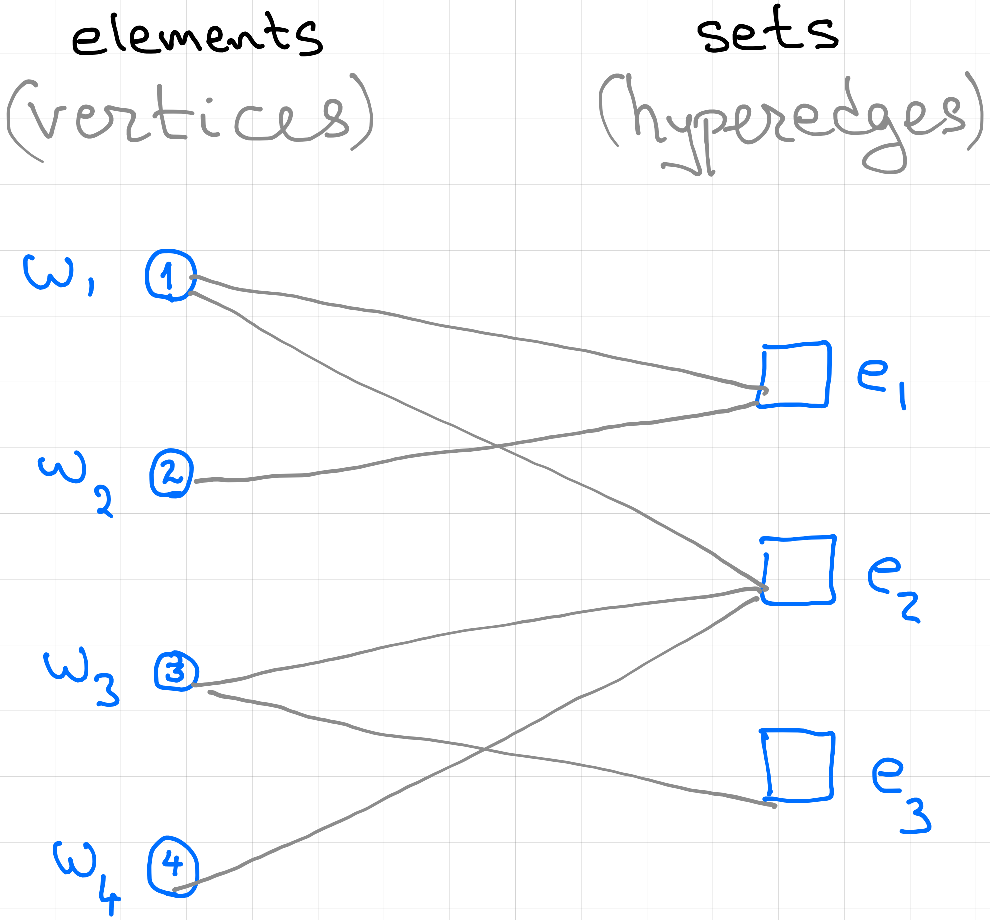
$w_1, \dots, w_n \in \mathbb{N}^+$ - weights

Hitting Set:

$X \subseteq [n]$ s.t.

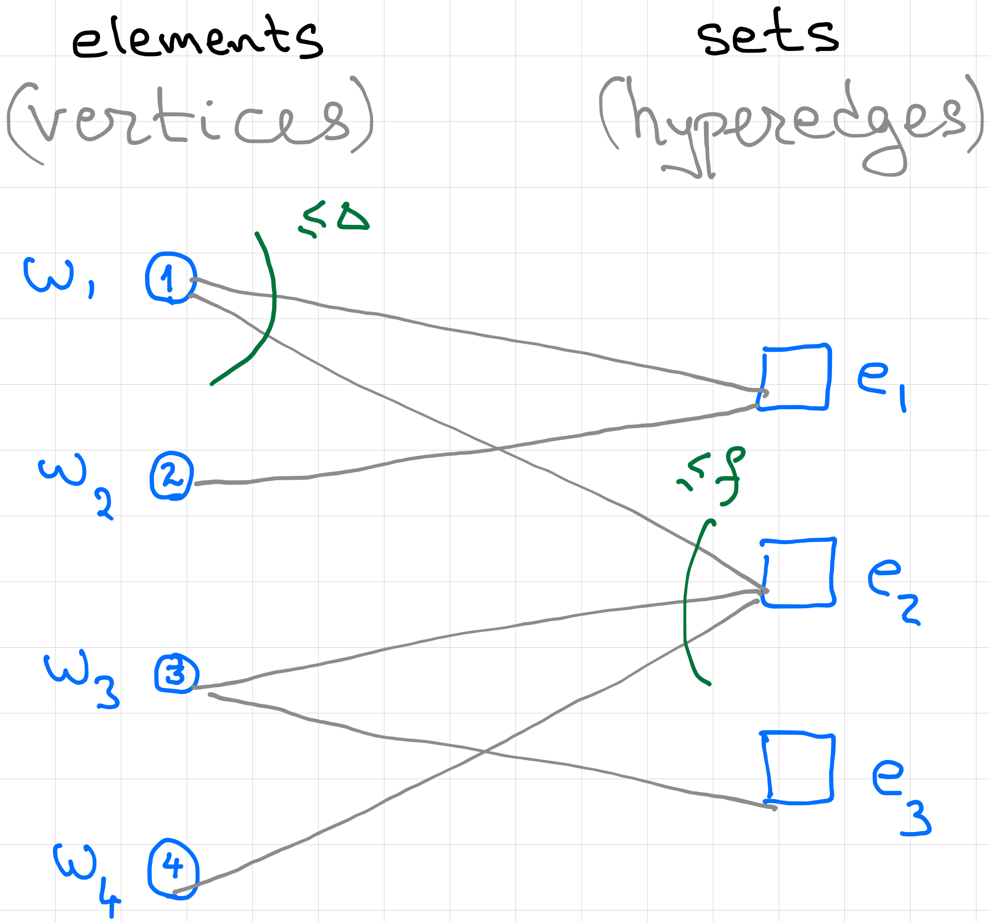
$\forall i \quad e_i \cap X \neq \phi$

Goal: $\min \ w(X)$



Min Weight Hitting Set (Hypergraph Vertex Cover)

distributed setting:



- bipartite network of vertices & hyperedges equipped with "ports"

- local input : w_i

- know : Δ, f, ε

(can do without Δ, f)

- CONGEST

$*\ m = poly(n)$, $w_{max} = poly(n)$

Lower Bound on Number of Communication Rounds

Theorem ([KMW16])

Any distributed constant-factor approximation algorithm requires $\Omega(\log \Delta / \log \log \Delta)$ *rounds to terminate.*

lower bound holds for:

- every constant approximation ratio
- unweighted graphs, and
- even if the message lengths are not bounded.

Weighted Graph Vertex Cover Results

det.	weighted	approximation	time	algorithm
yes	no	3	$O(\Delta)$	[PS09]
yes	no	2	$O(\Delta^2)$	[Ast+09]
yes	yes	2	$O(1)$ for $\Delta \leq 3$	[Ast+09]
yes	yes	2	$O(\Delta + \log^* n)$	[PR01]
yes	yes	2	$O(\Delta + \log^* W)$	[AS10]
yes	yes	2	$O(\log^2 n)$	[KVY94]
yes	yes	2	$O(\log n \log \Delta / \log^2 \log \Delta)$	[Ben+18]
no	yes	2	$O(\log n)$	[GKP08; KY11]
yes	**yes**	**2**	$\boldsymbol{O(\log n)}$	**This work**
yes	yes	$2 + \epsilon$	$O(\epsilon^{-4} \log(W \cdot \Delta))$	[Hoc82; KMW06]
yes	yes	$2 + \epsilon$	$O(\log \epsilon^{-1} \log n)$	[KVY94]
yes	yes	$2 + \epsilon$	$O(\epsilon^{-1} \log \Delta / \log \log \Delta)$	[BCS17; EGM18]
yes	yes	$2 + \epsilon$	$O\left(\frac{\log \Delta}{\log \log \Delta} + \frac{\log \epsilon^{-1} \log \Delta}{\log^2 \log \Delta}\right)$	[Ben+18]
yes	**yes**	$\boldsymbol{2 + \epsilon}$	$\boldsymbol{O\left(\frac{\log \Delta}{\log \log \Delta} + \log \epsilon^{-1} \cdot (\log \Delta)^{0.001}\right)}$	**This work**
yes	yes	$2 + \frac{\log \log \Delta}{c \cdot \log \Delta}$	$O(\log \Delta / \log \log \Delta)$	[BCS17], $\forall c = O(1)$
yes	yes	$2 + (\log \Delta)^{-c}$	$O(\log \Delta / \log \log \Delta)$	[Ben+18], $\forall c = O(1)$
yes	**yes**	$\boldsymbol{2 + 2^{-c \cdot (\log \Delta)^{0.99}}}$	$\boldsymbol{O(\log \Delta / \log \log \Delta)}$	**This work**, $\boldsymbol{\forall c = O(1)}$

Weighted Hypergraph Vertex Cover Results

weighted	approximation	time	algorithm
yes	f	$O\left(f^2\Delta^2 + f\Delta\log^* W\right)$	[AS10]
yes	f	$O\left(f\log^2 n\right)$	[KVY94]
yes	$\boldsymbol{f}$	$\boldsymbol{O\left(f\log n\right)}$	**This work**
no	$f+\epsilon$	$O\left(\epsilon^{-1}\cdot f\cdot\frac{\log(f\Delta)}{\log\log(f\Delta)}\right)$	[EGM18][1]
yes	$f+\epsilon$	$O\left(f\cdot\log(f/\epsilon)\cdot\log n\right)$	[KVY94]
yes	$f+\epsilon$	$O\left(\epsilon^{-4}\cdot f^4\cdot\log f\cdot\log(W\cdot\Delta)\right)$	[KMW06]
yes	$\boldsymbol{f+\epsilon}$	$\boldsymbol{O\left(f\cdot\log(f/\epsilon)\cdot(\log\Delta)^{0.001}+\frac{\log\Delta}{\log\log\Delta}\right)}$	**This work**
no	$f+1/c$	$O\left(\log\Delta/\log\log\Delta\right)$	[EGM18], $\forall f, c=O(1)$
yes	$\boldsymbol{f+2^{-c\cdot(\log\Delta)^{0.99}}}$	$\boldsymbol{O\left(\log\Delta/\log\log\Delta\right)}$	**This work**, $\boldsymbol{\forall f, c=O(1)}$

No depend. on n, W

depend. on ε: $\lg\frac{1}{\varepsilon}$

Primal Dual Scheme

LP relaxation:

$$\min \sum_{v} w_v \cdot x_v$$

$$\text{s.t.} \quad \sum_{v \in e} x_v \geq 1 \quad \forall e$$

covering constraint

$$(1 \geq) \quad x_v \geq 0 \quad \forall v$$

Dual LP:

$$\max \sum_{e} \delta_e$$

$$\text{s.t} \quad \sum_{e \ni v} \delta_e \leq w_v \quad \forall v$$

packing constr.

$$\delta_e \geq 0 \quad \forall e$$

Primal Dual Scheme

$$\min \sum_v w_v \cdot x_v$$
$$\text{s.t.} \quad \sum_{v \in e} x_v \geq 1 \quad \forall e$$
$$x_v \geq 0 \quad \forall v$$

DEF: v is **β-tight** if

$$\beta \cdot w(v) \geq w(v) - \sum_{e \ni v} \delta(e)$$

slack(v)

$$\max \sum_e \delta_e$$
$$\text{s.t} \quad \sum_{e \ni v} \delta_e \leq w_v \quad \forall v$$
$$\delta_e \geq 0 \quad \forall e$$

set $\beta \triangleq \frac{\varepsilon}{f+\varepsilon}$

THM: if $\{\delta_e\}_{e \in E}$ feas

then $w(\beta\text{-Tight}) \leq (f+\varepsilon) \cdot OPT_{frac}$

ALG: find $\{\delta_e\}_e$ feas s.t. β-Tight is a VC

apx ratio (ALG) $\leq f + \varepsilon$

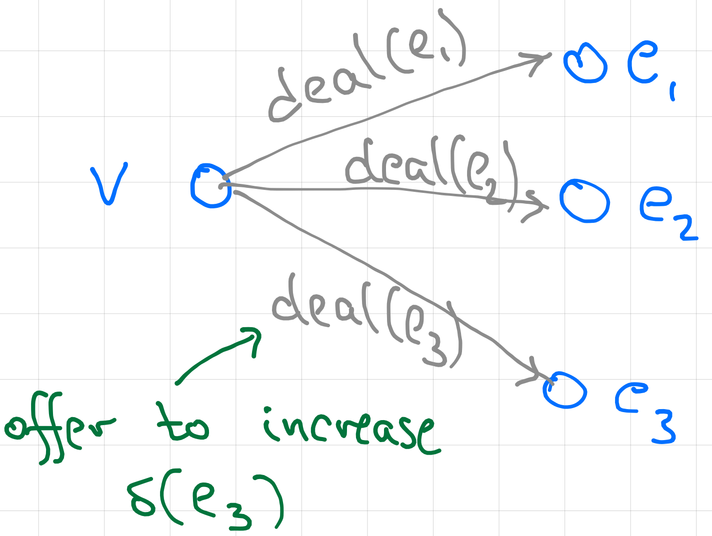
ALG Termination Rules

* v joins cover & terminates if becomes tight
* e terminates if $\exists v \in e \cap$ cover
* v terminates (& does not join cover) if $\forall e \ni v$ terminated

ALG terminates if $\forall v$ & $\forall e$ terminated

Dual Solution Dynamics

$\sum_{e \ni v} \delta_e \leq w_v \quad \forall v$



v sends deal(e_i) to e_i.

update $\delta(e_i) \leftarrow \delta(e_i) + \text{deal}(e_i)$

is v-safe if

$$\sum_{e_i \ni v} \delta(e_i) + \sum_i \text{deal}(e_i) \leq w(v)$$

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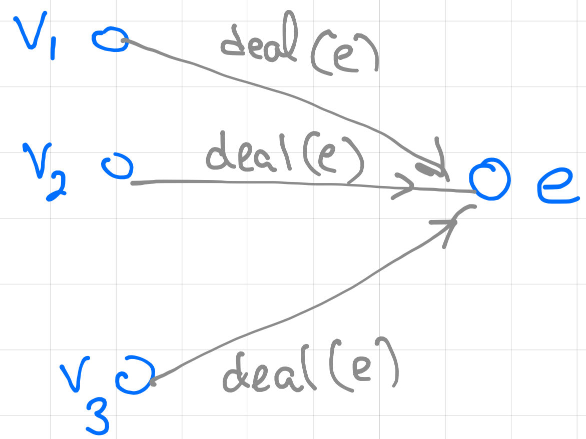
e recv's deal(e)

$$\delta(e) \leftarrow \delta(e) + \text{deal}(e)$$



* e recv's same deal $\forall v \in e$

deal(e) Dynamics

$$\left.\begin{array}{ll} \underset{\text{deal}(e)}{\text{Init}}: & \beta \cdot \min\limits_{v \in e} \dfrac{w(v)}{\deg(v)} \\[2em] \underset{\text{deal}(e)}{\max}: & \beta \cdot w(v) \end{array}\right\} \quad \frac{\max}{\text{init}} \leq \Delta$$

$$\text{deal}_{i+1}(e) = \begin{cases} \alpha \cdot \text{deal}_i(e) & \text{if } \forall v \in e \setminus \text{Tight} \quad \sum\limits_{\substack{e' \ni v \\ e' \text{ not covered}}} \text{deal}_i(e') \leq \frac{\beta}{\alpha} \cdot w(v) \\[2em] \text{deal}_i(e) & \text{Otherwise} \end{cases}$$

RAISE:

$$\mathrm{deal}_{i+1}(e) = \alpha \cdot \mathrm{deal}_i(e)$$

STUCK:

$$\mathrm{deal}_{i+1}(e) = \mathrm{deal}_i(e)$$

$$\mathrm{deal}_{i+1}(e) = \begin{cases} \alpha \cdot \mathrm{deal}_i(e) & \text{if } \forall v \in e \setminus \mathrm{Tight} \quad \sum_{\substack{e' \ni v \\ e' \text{ not covered}}} \mathrm{deal}_i(e') \leq \frac{\beta}{\alpha} \cdot w(v) \\ \mathrm{deal}_i(e) & \text{Otherwise} \end{cases}$$

$$\#\text{Raises} \leq \log_\alpha \frac{\mathrm{MAX}}{\mathrm{INIT}} \leq \log_\alpha \Delta$$

$$\#\text{STUCKS} \leq f \cdot \frac{\alpha}{\beta} \quad \left(\forall v \text{ becomes tight after } \frac{\alpha}{\beta} \right)$$

$$\Longrightarrow \quad \#\text{RNDS} \leq \log_\alpha \Delta + f \cdot \frac{\alpha}{\beta}$$

$$\alpha = \frac{\lg \Delta}{\lg \lg \Delta} \quad \Longrightarrow \quad \#\text{RNDS} \leq O\left(\frac{f^2}{\varepsilon} \cdot \frac{\lg \Delta}{\lg \lg \Delta} \right)$$

Reduced Depend. $\frac{1}{\varepsilon} \longrightarrow \lg \frac{1}{\varepsilon}$

* $\text{slack}(v) \in [\beta \cdot w(v), w(v)]$

* $\lg \frac{1}{\beta}$ "levels" per v accord. to $\log \frac{w(v)}{\text{slack}(v)}$

* within each level $\#\text{stuck}(v) \leq \alpha$

$$\Rightarrow \quad \#\text{stuck} \leq \#\text{levels} \times |e| \times \alpha$$

$$\leq \left(\lg \frac{1}{\beta}\right) \cdot f \cdot \alpha$$

LIE!

$$\Rightarrow \#\text{RNDS} \leq \lg_\alpha \Delta + \left(\lg \frac{f}{\varepsilon}\right) \cdot f \cdot \alpha$$

$$\Rightarrow \alpha \simeq \frac{\lg \Delta}{\lg \lg \Delta} : \quad \#\text{RNDS} \leq O\left(f \cdot \lg\left(\frac{f}{\varepsilon}\right) \cdot \frac{\lg \Delta}{\lg \lg \Delta}\right)$$

f - apx

If $\varepsilon \leq \frac{1}{n \cdot w_{max}} = \frac{1}{poly(n)}$

then $(f+\varepsilon)$-apx is f-apx.

rnds : $O(f \cdot \lg n)$

(Req. fine tuning of α)

More

* Consider ILP $\min w^T x$ s.t $Ax \geq b$

CONGEST alg #RNDS $\sim \frac{\lg \Delta}{\lg\lg \Delta} + f^{1.01} \cdot \lg \frac{1}{\varepsilon} \cdot \lg^{0.01} \Delta \cdot \lg^{1.01} M$

OPEN :

mixed pack & cover ILP