# Breaking the Linear-Memory Barrier in Massively Parallel Computing

## MIS on Trees with Strongly Sublinear Memory

Sebastian Brandt, **Manuela Fischer**, Jara Uitto
ETH Zurich

## Model:

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*Karloff, Suri, Vassilvitskii* [SODA'10]

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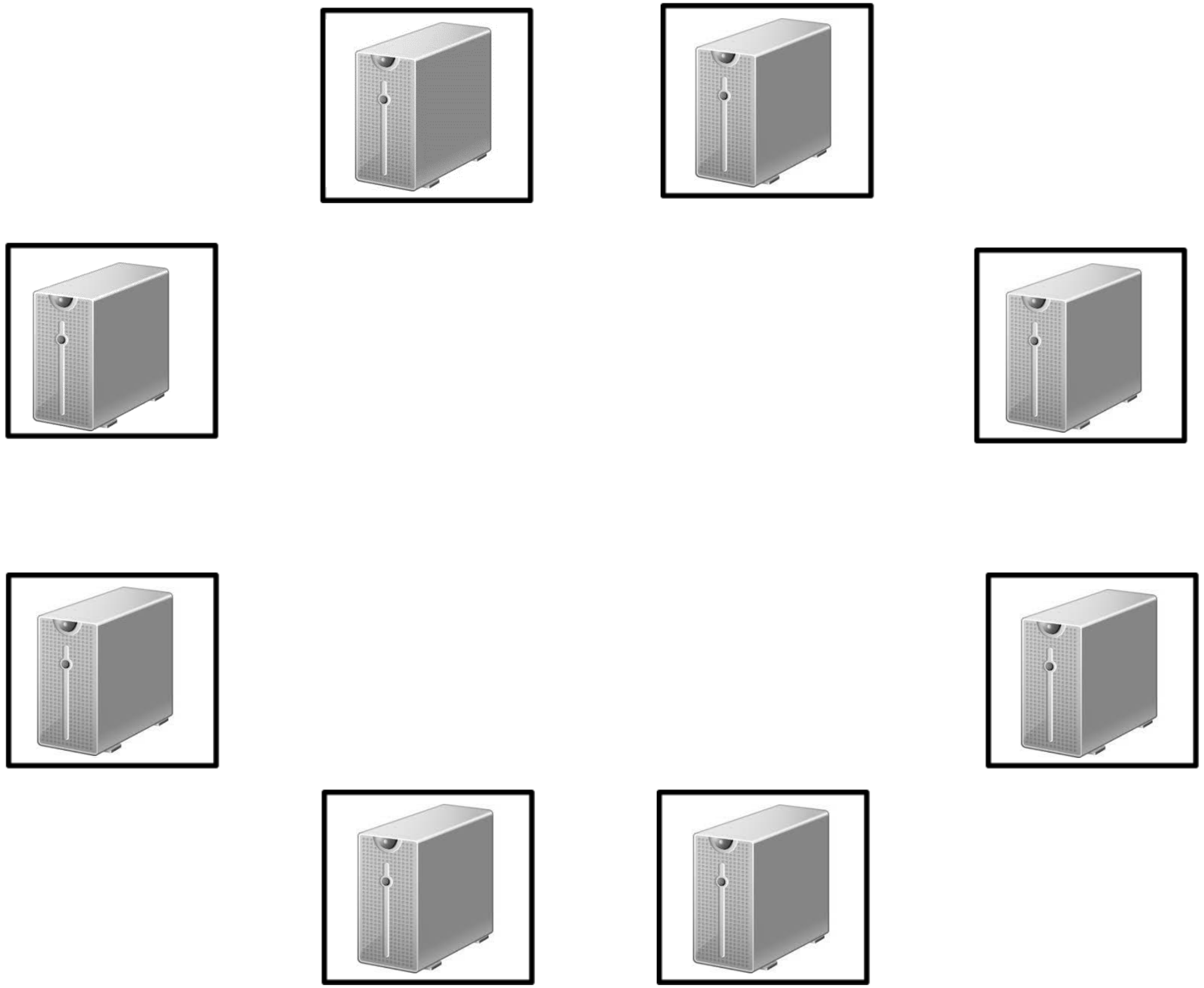

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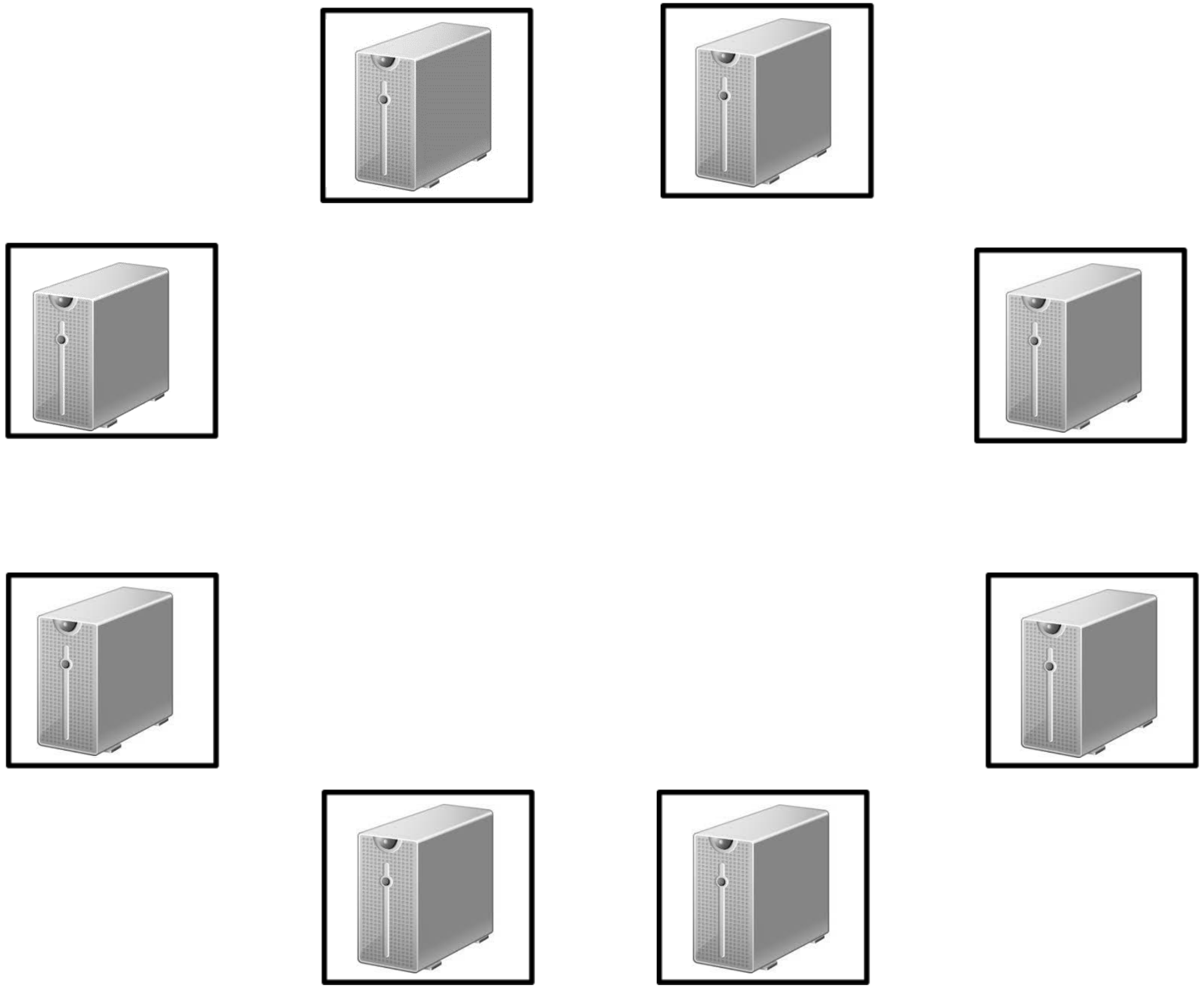

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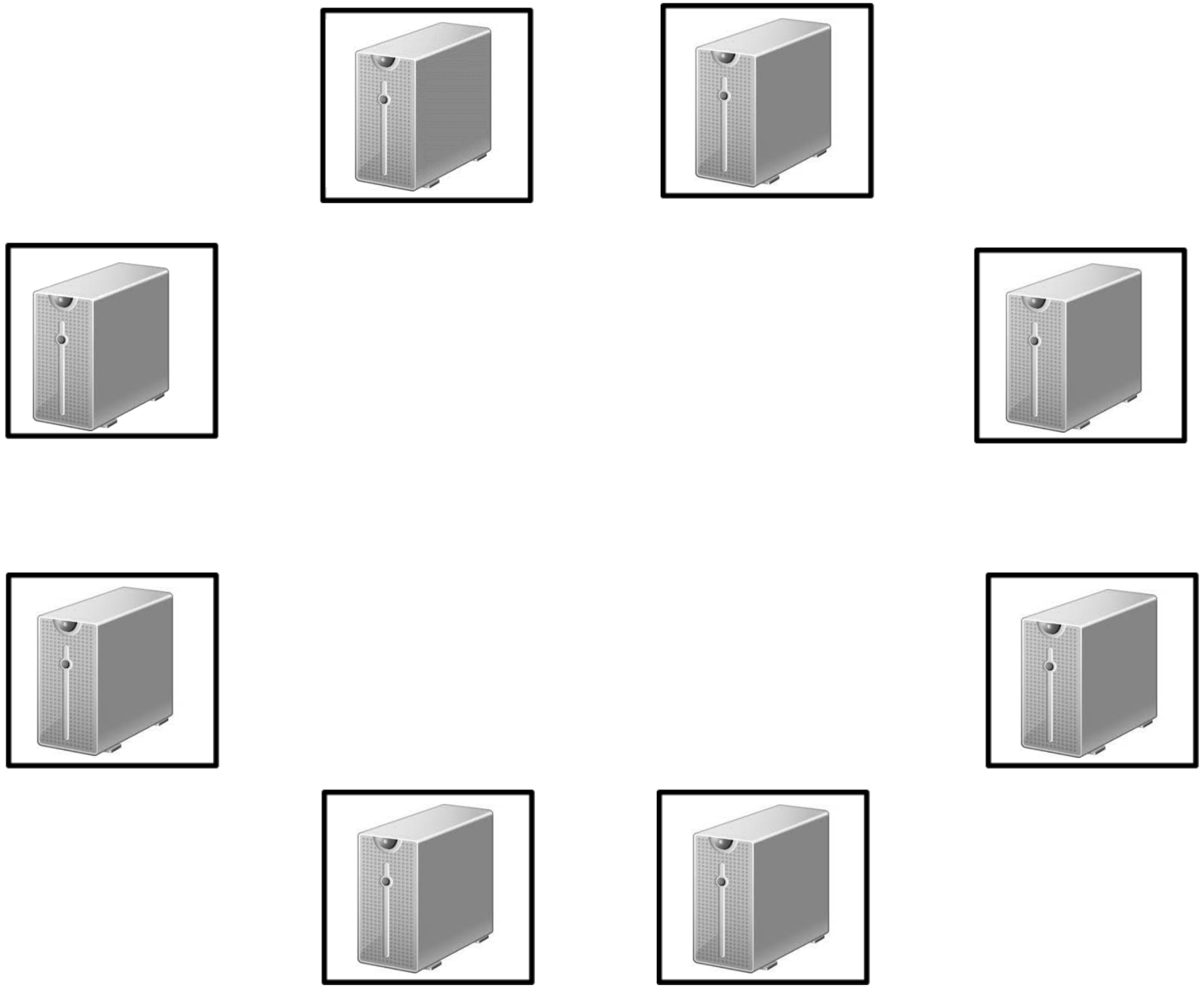

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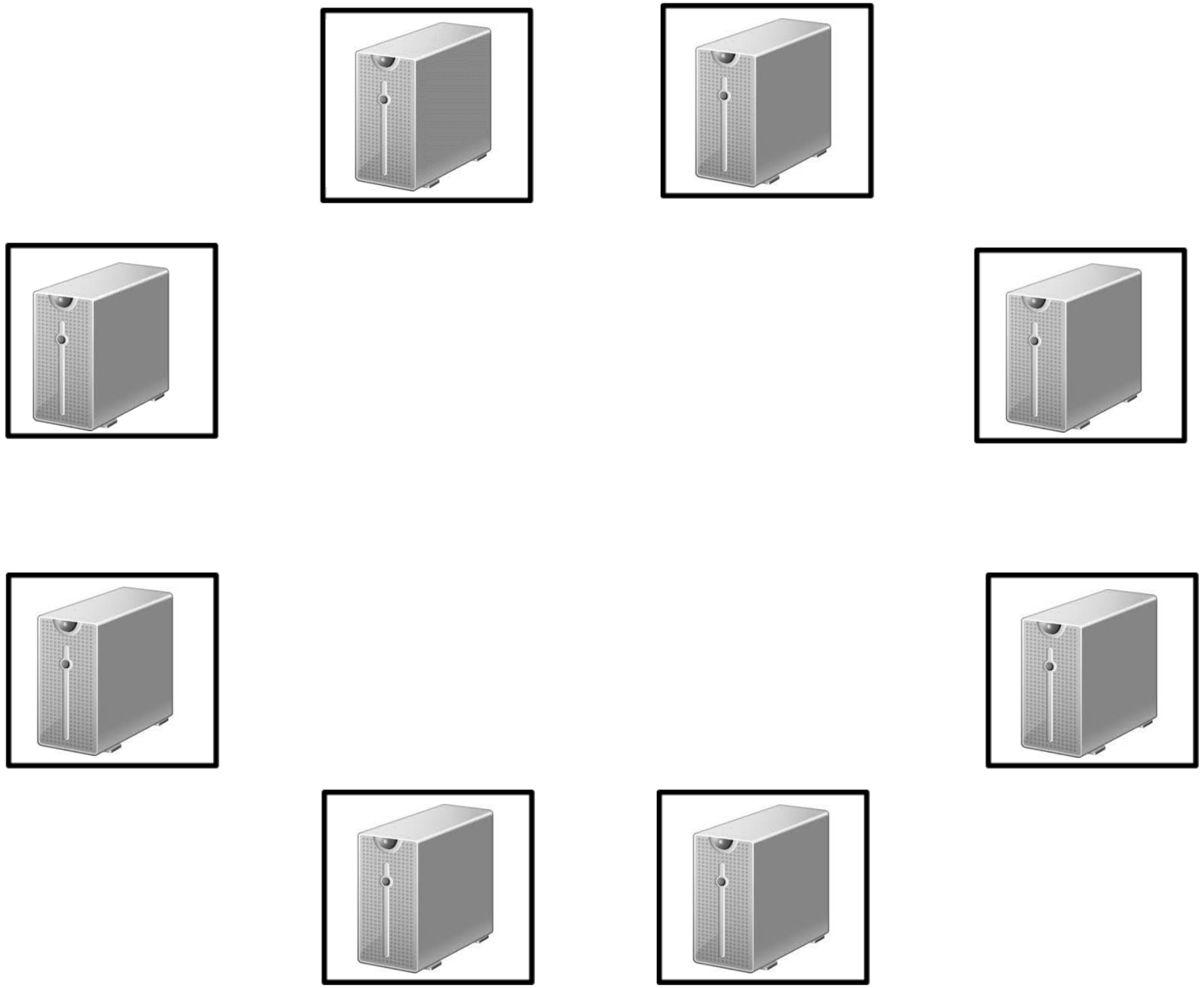

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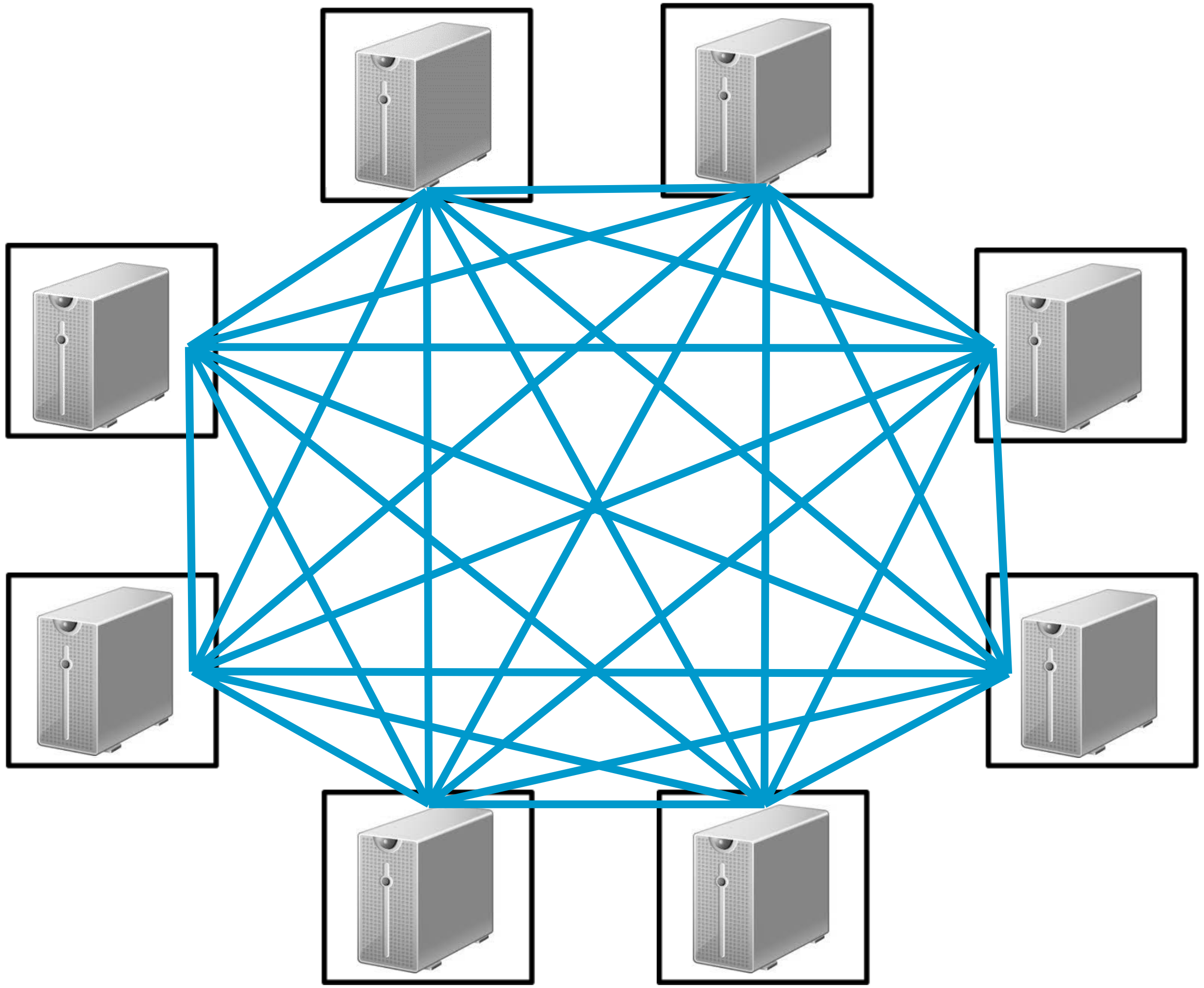

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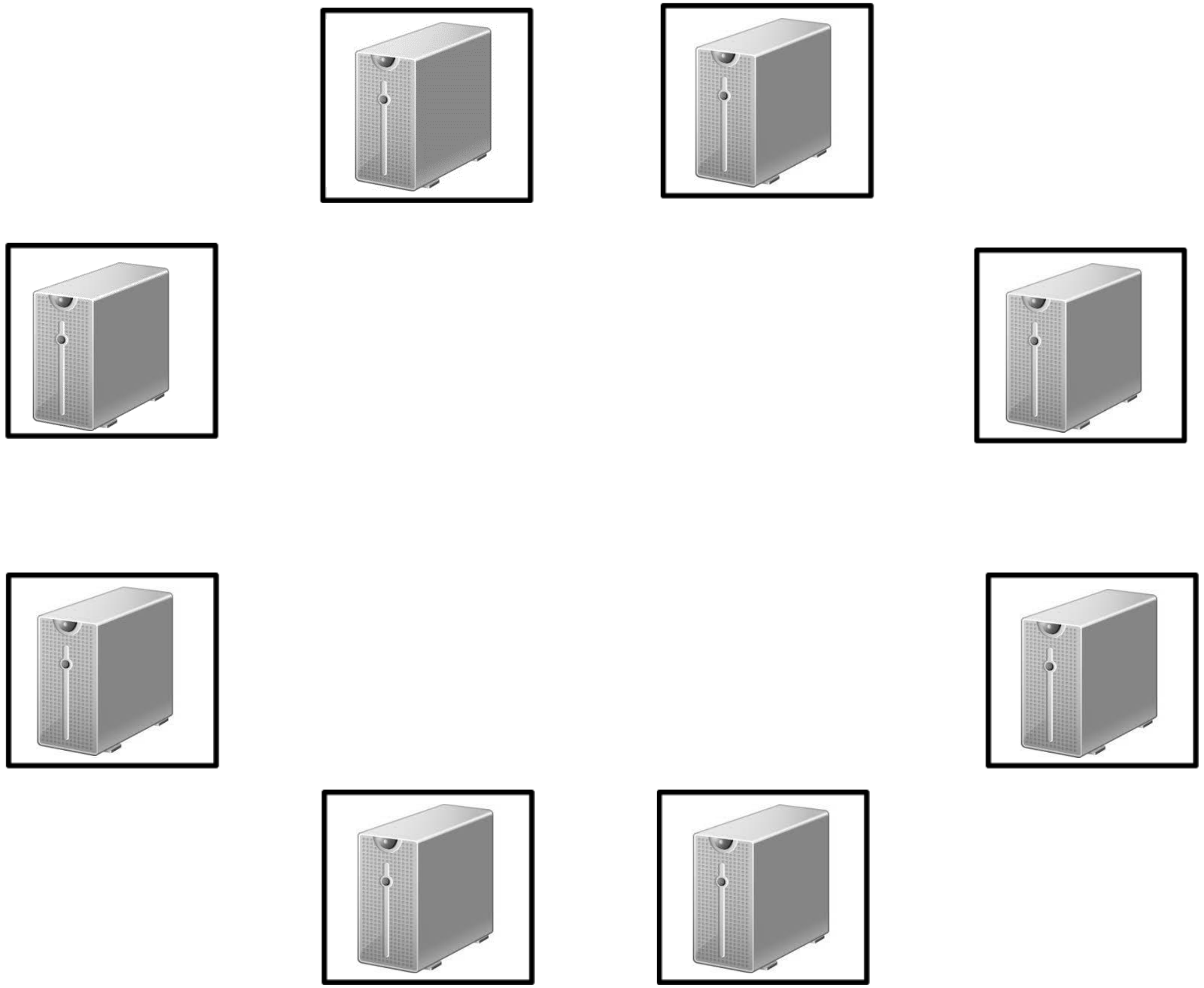

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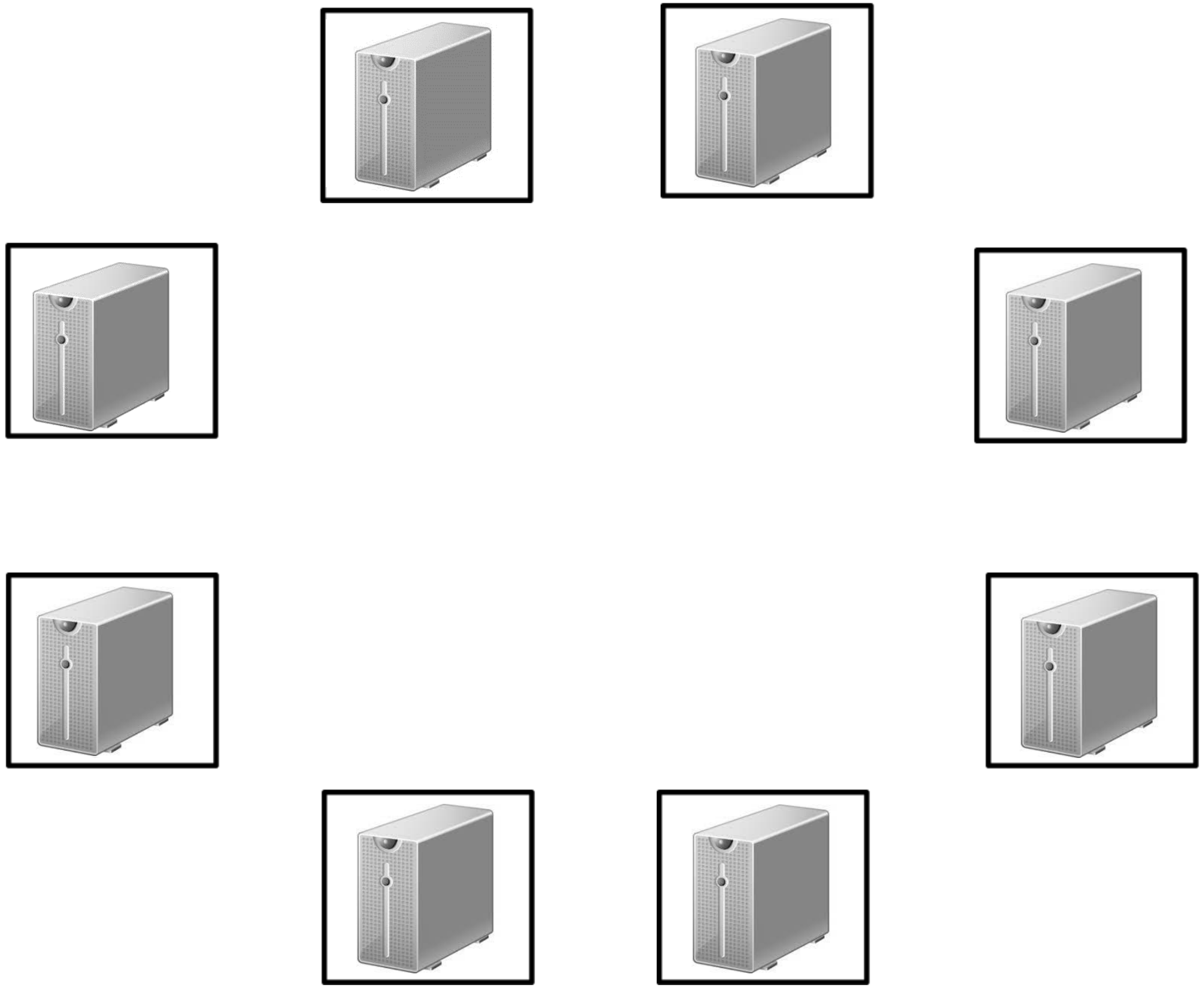

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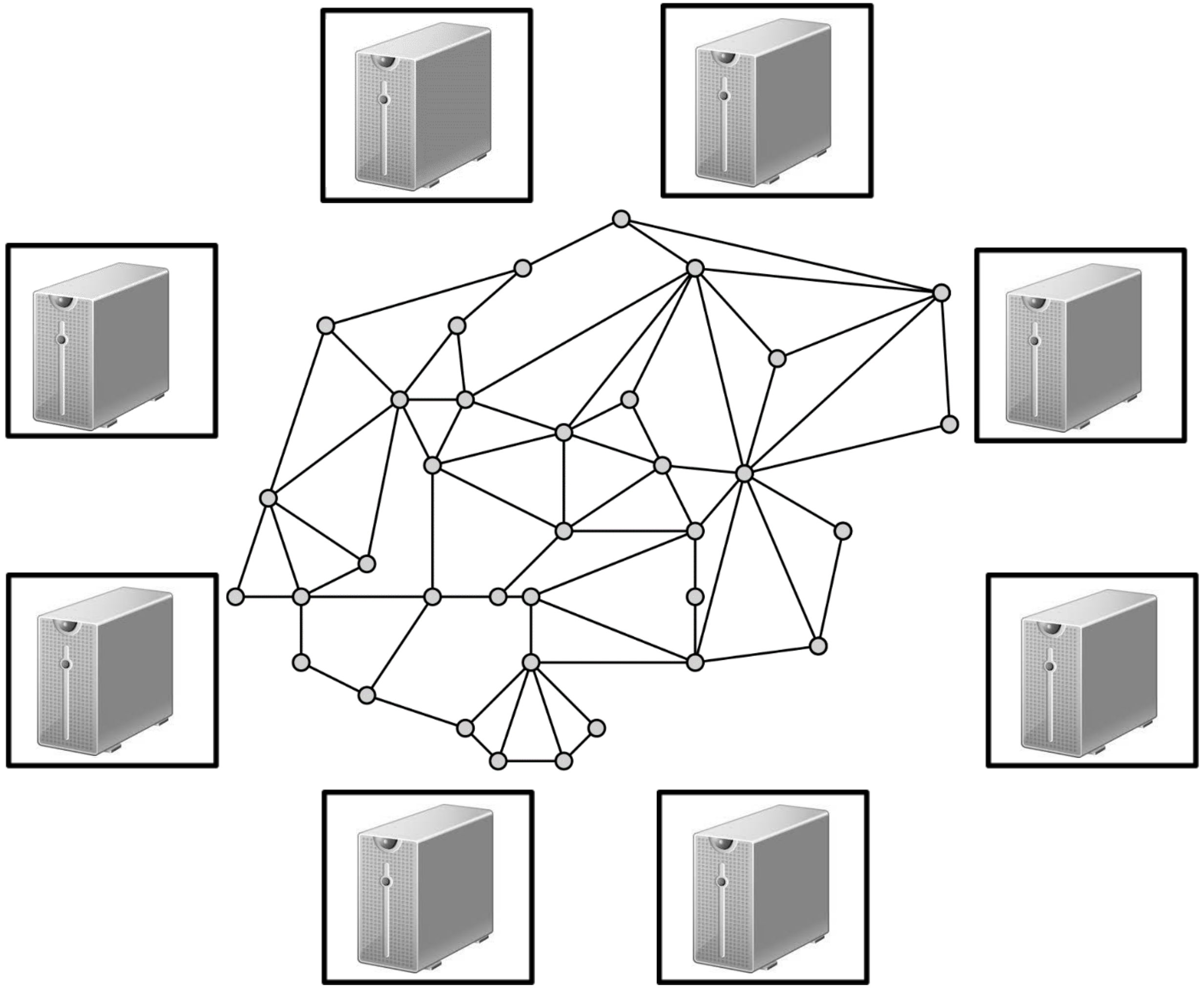

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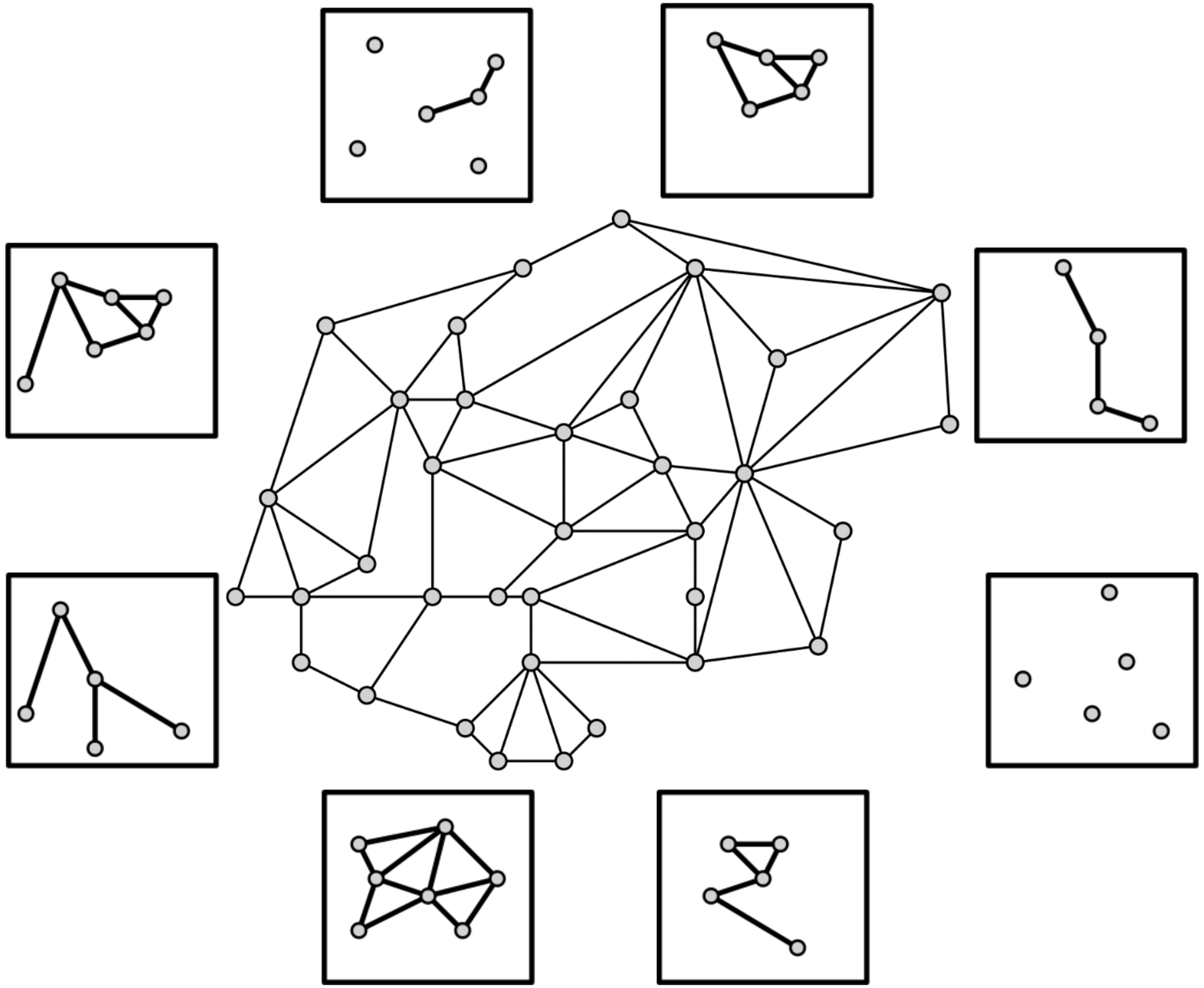

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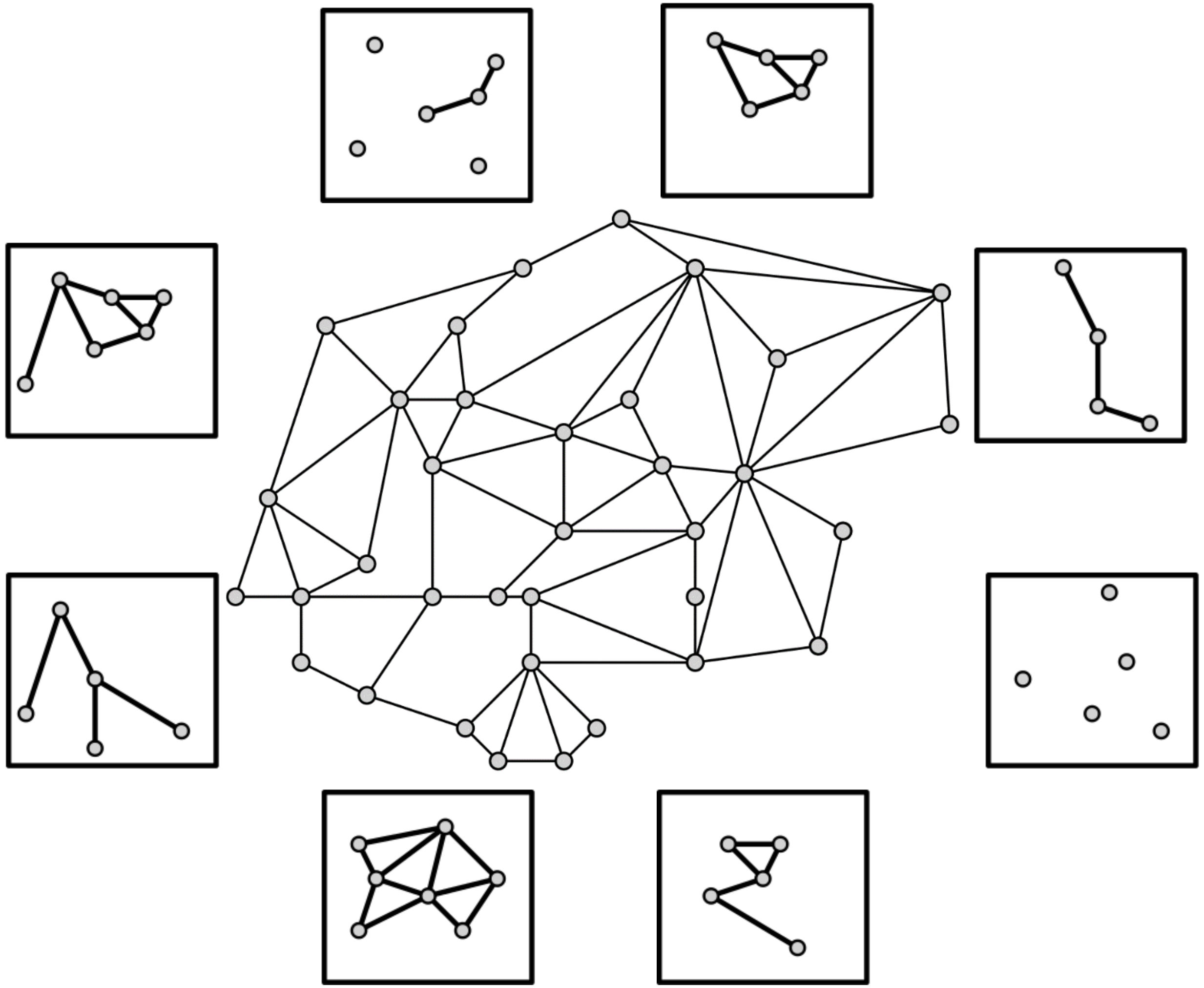

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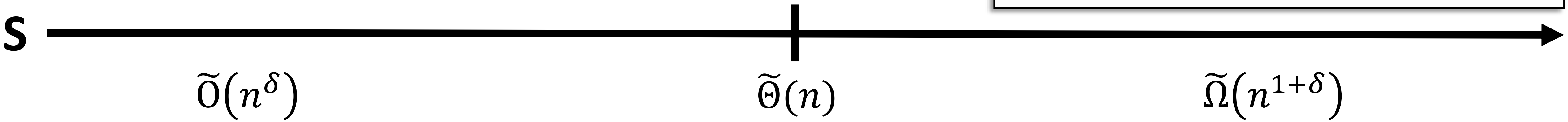

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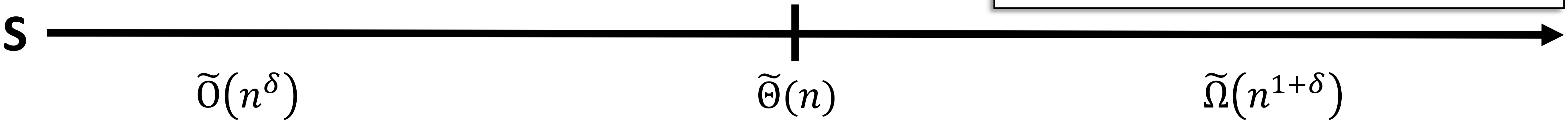

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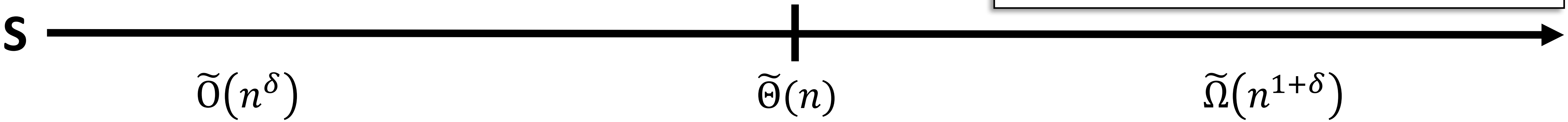

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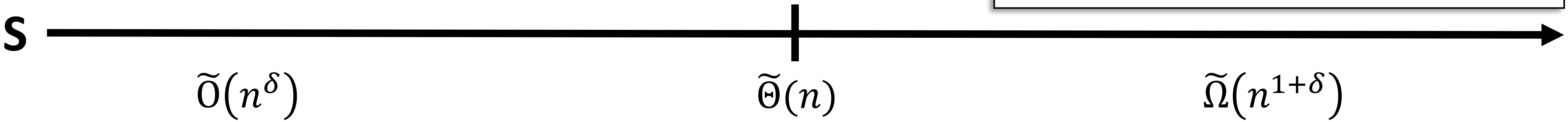

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*Lattanzi et al.* [SPAA'11]

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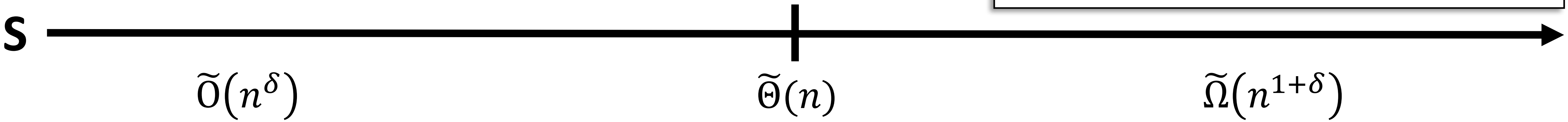

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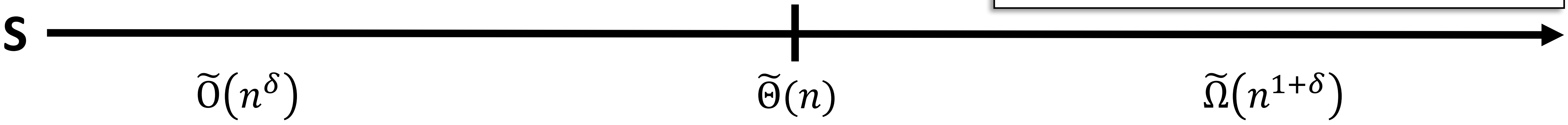

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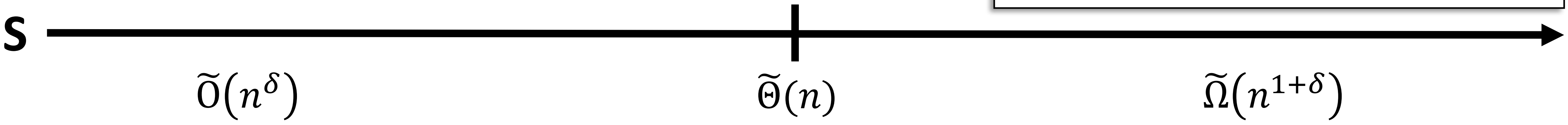

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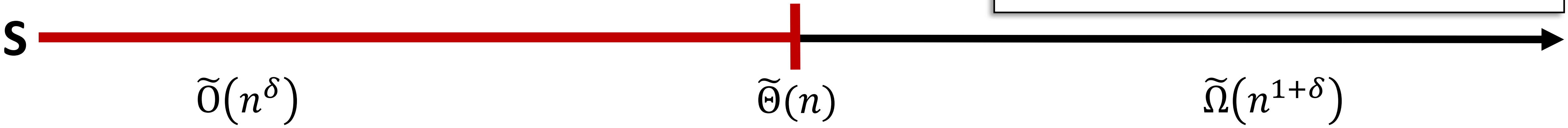

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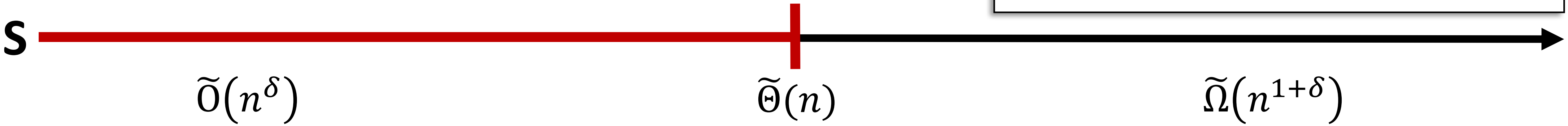

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**Fundamentally?**

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*Ghaffari, Kuhn, Uitto* [FOCS'19]

**Conditional Lower Bound**

$\Omega(\log\log n)$ rounds

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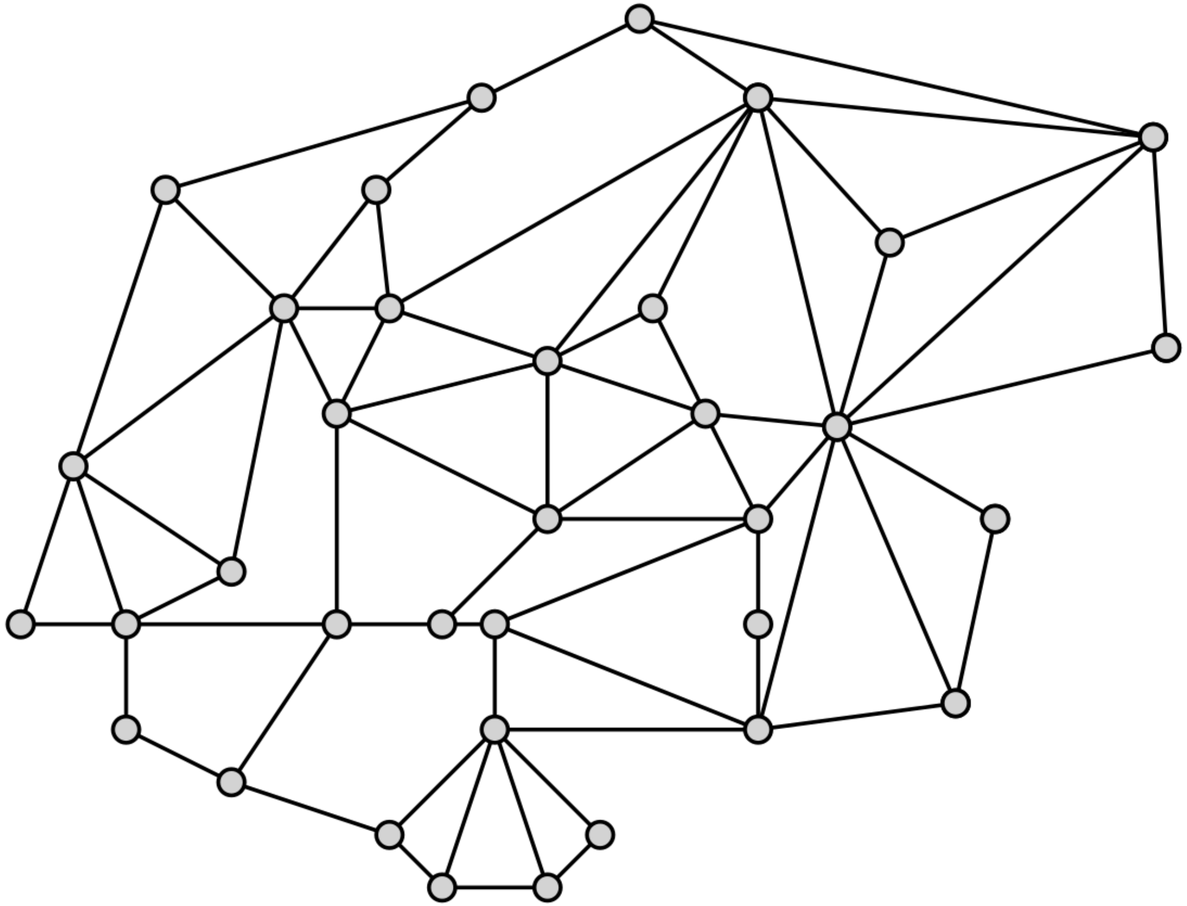

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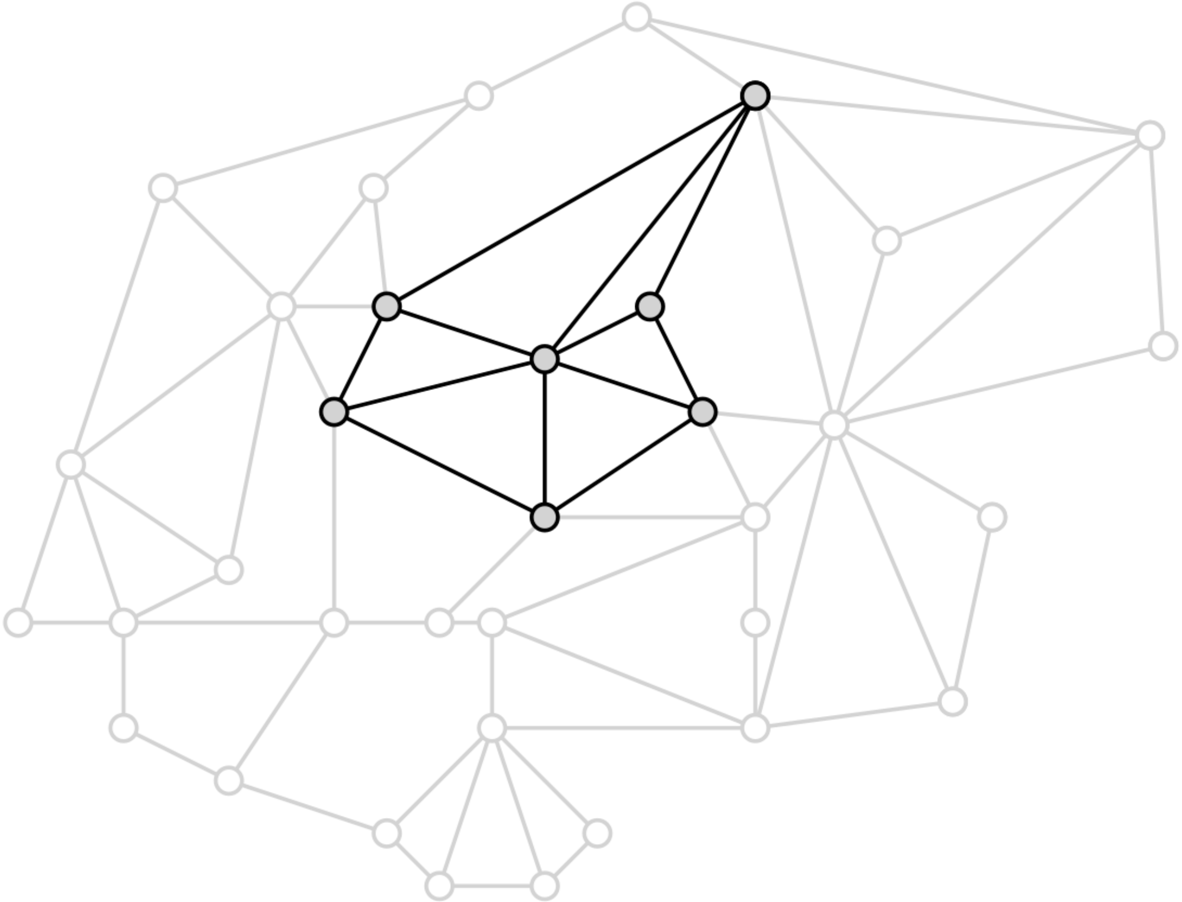

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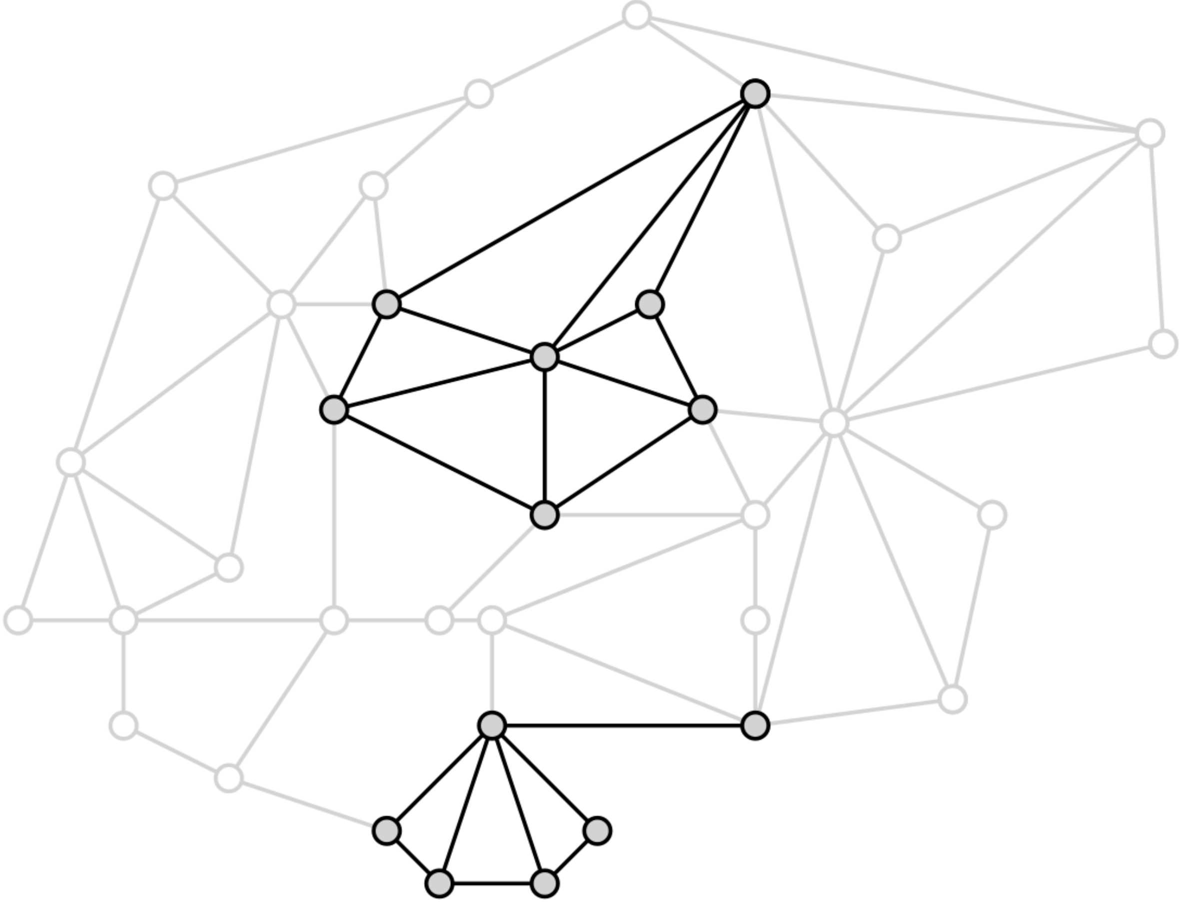

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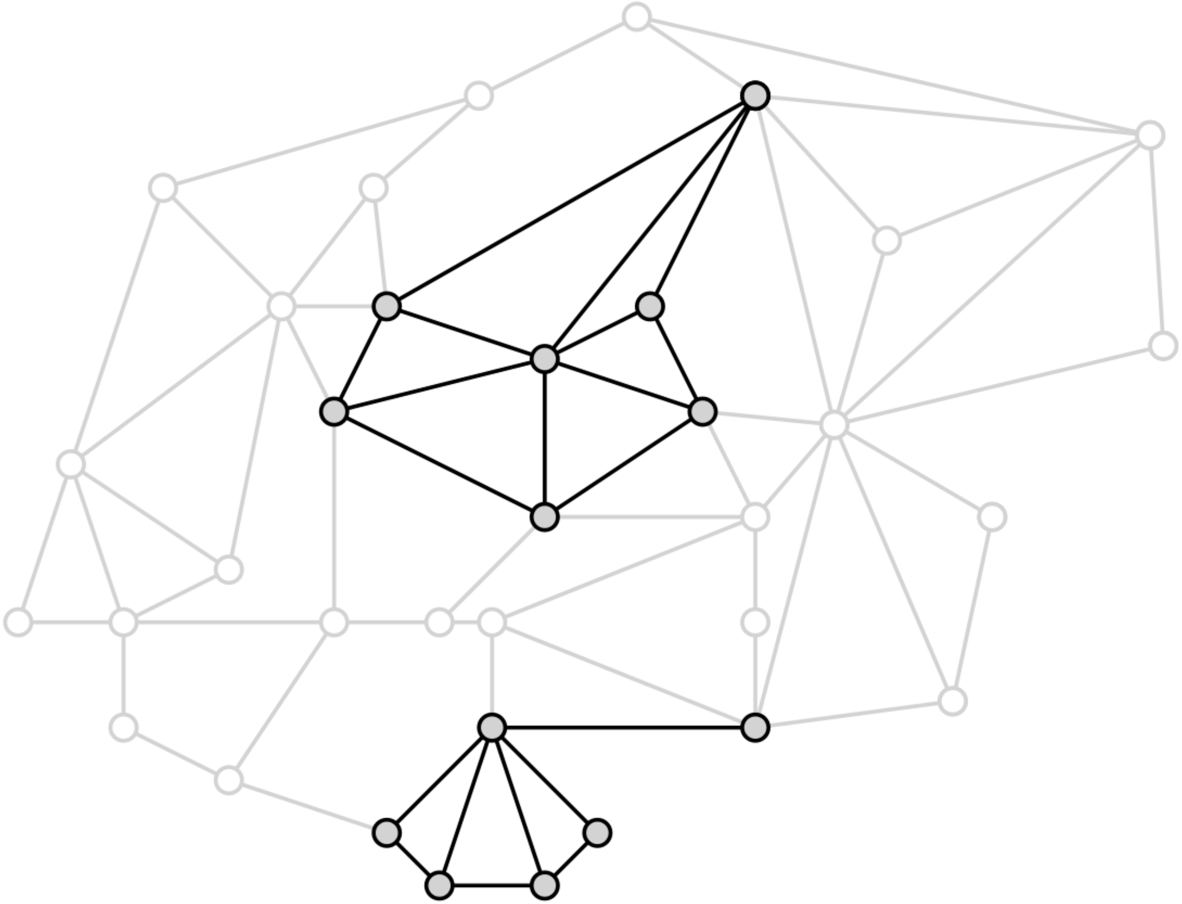

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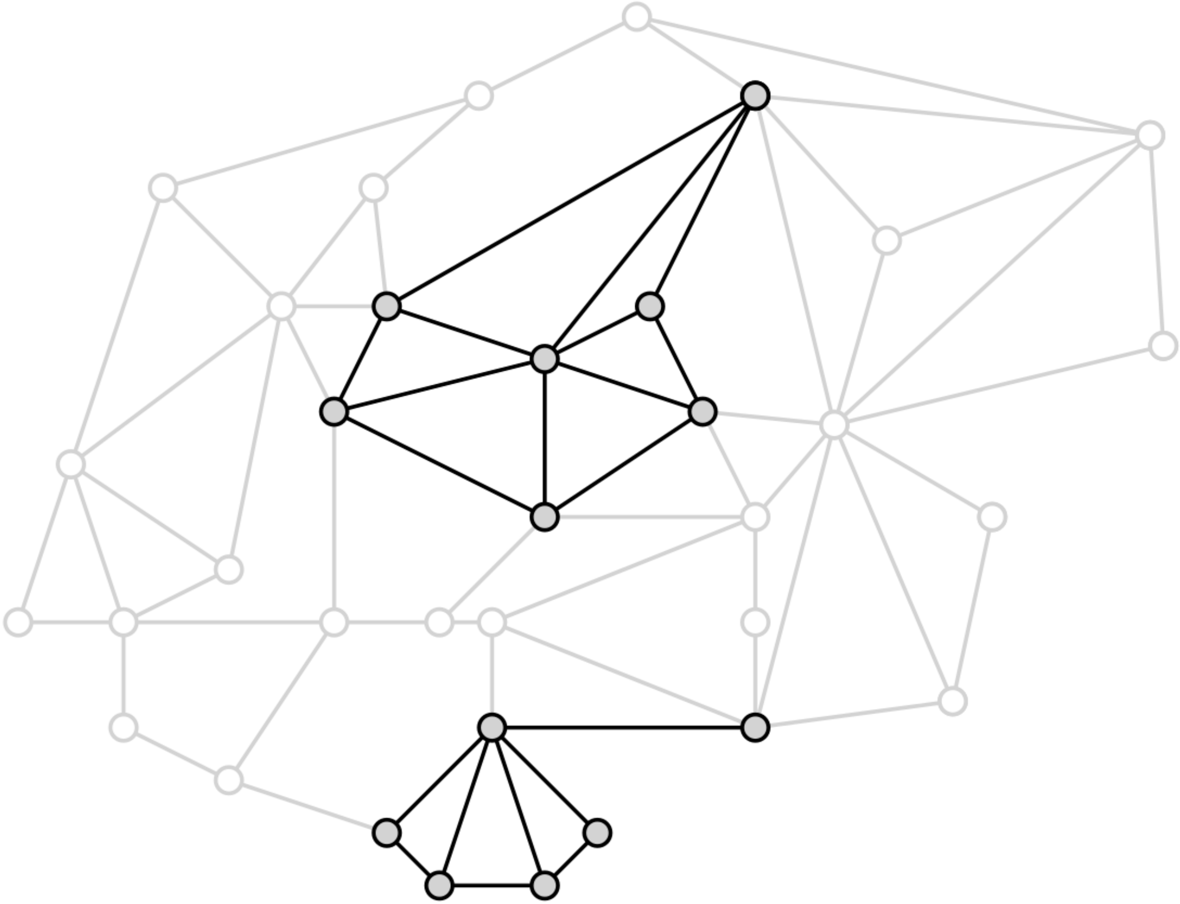

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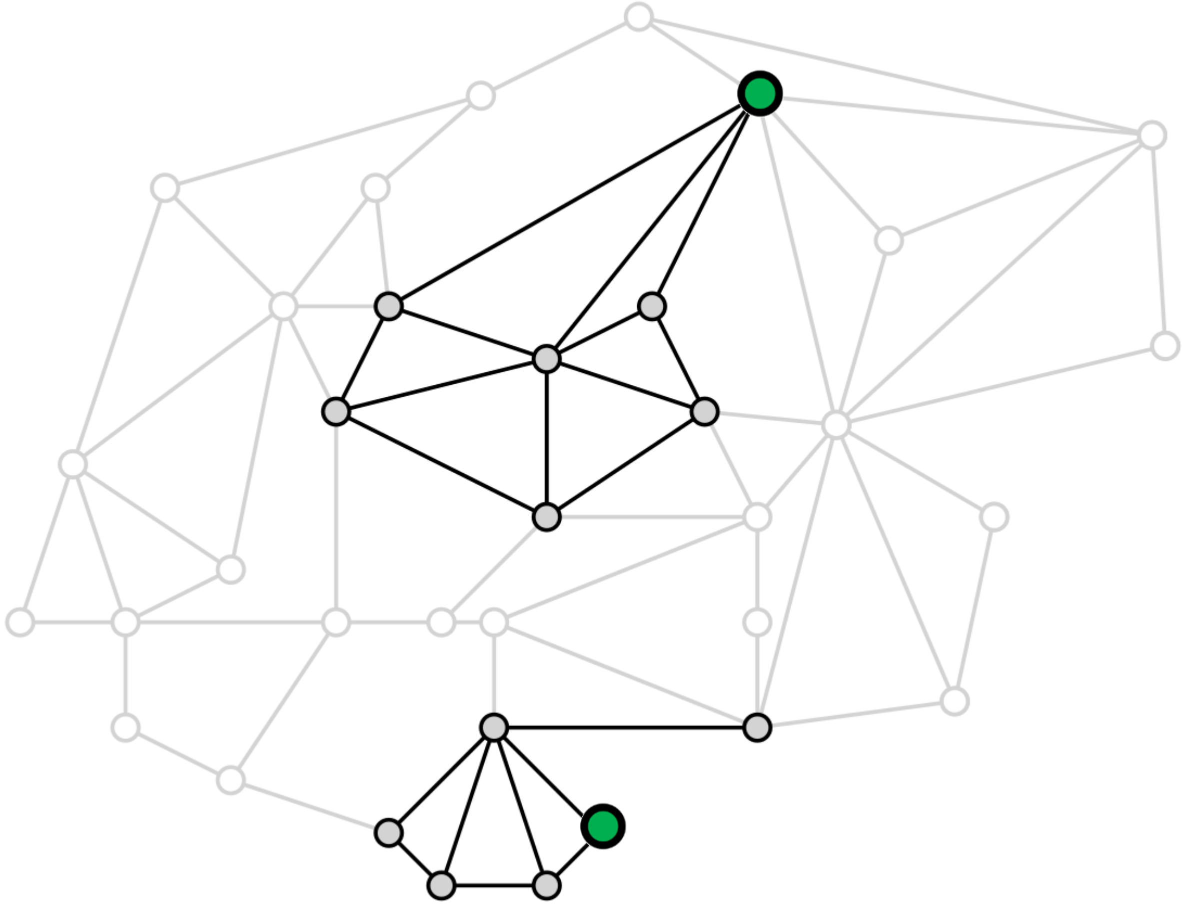

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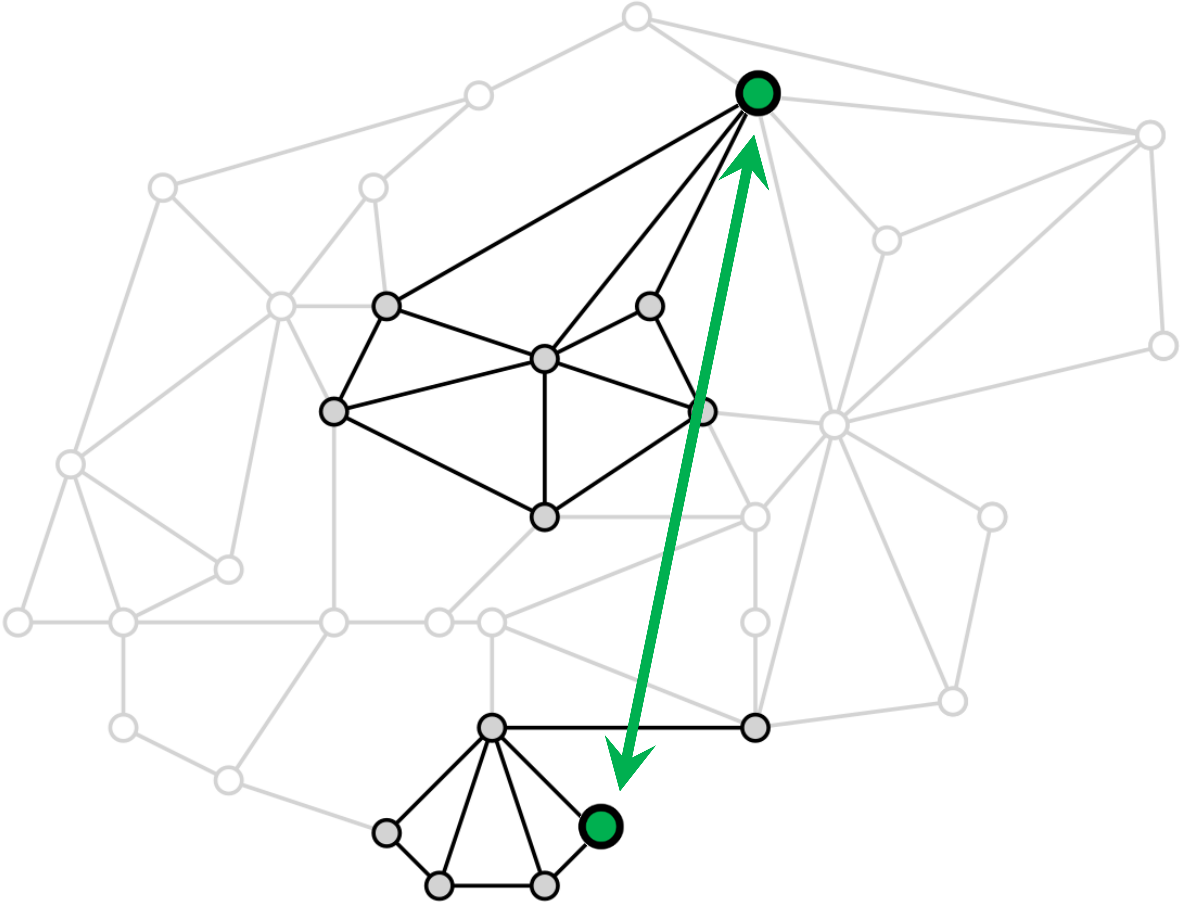

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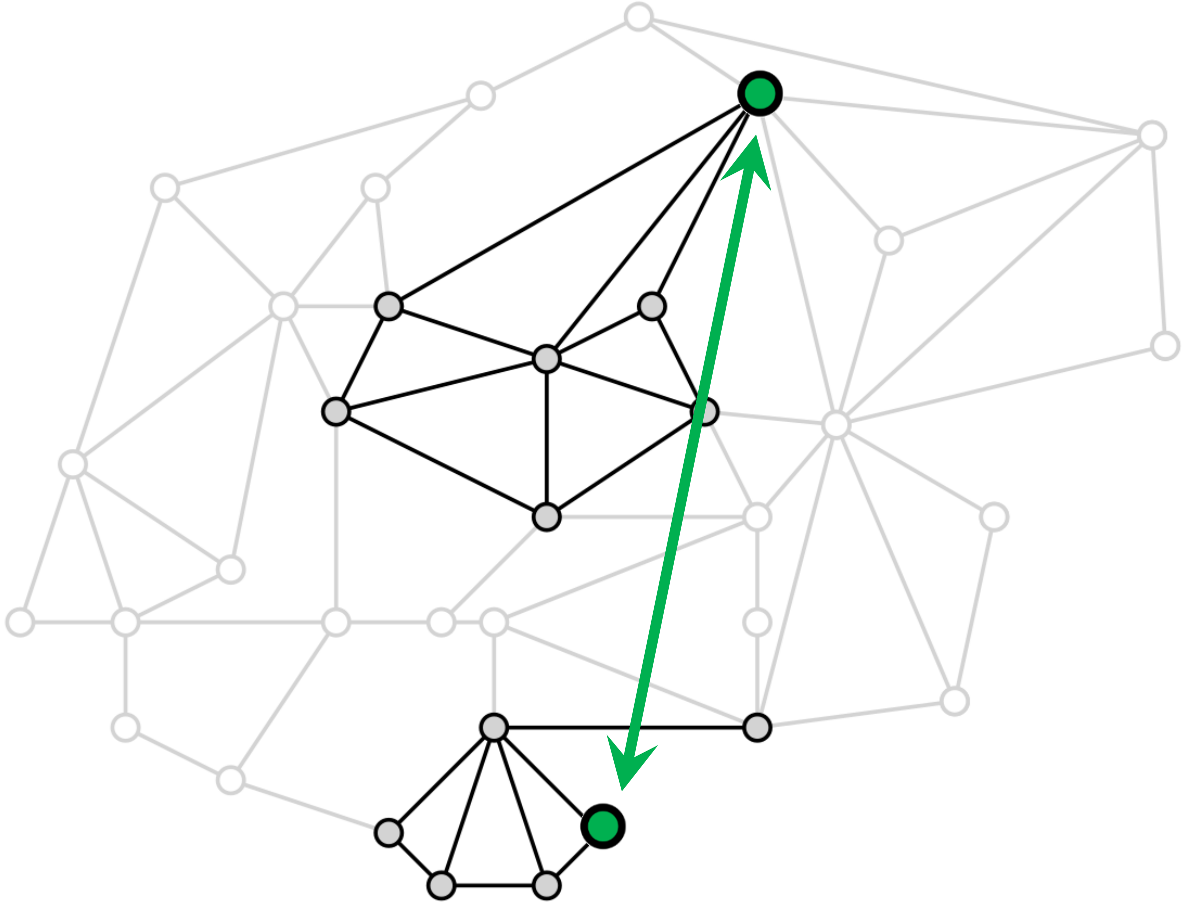

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- exponentially faster than LOCAL algorithms due to shortcuts

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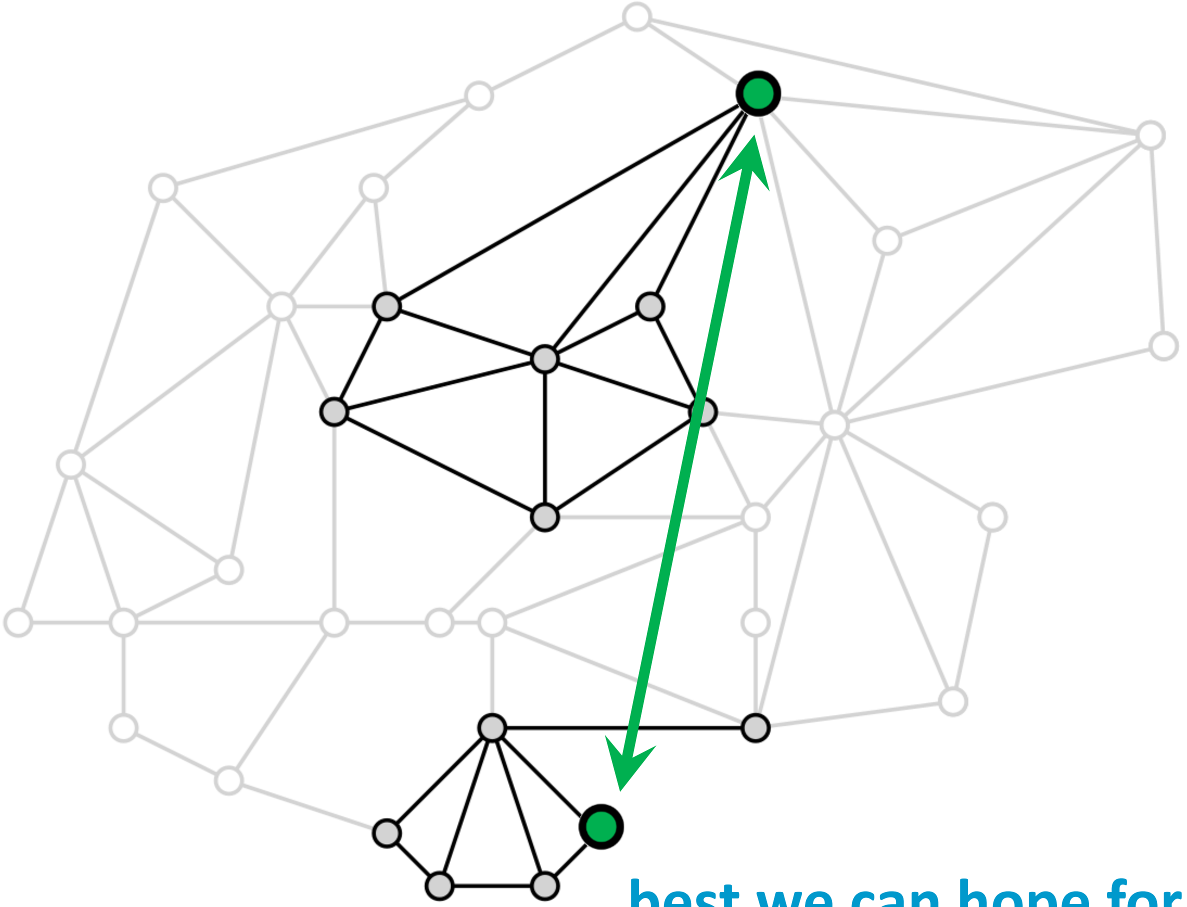

**best we can hope for**

*GKU* [FOCS'19]

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- exponentially faster than LOCAL algorithms due to shortcuts

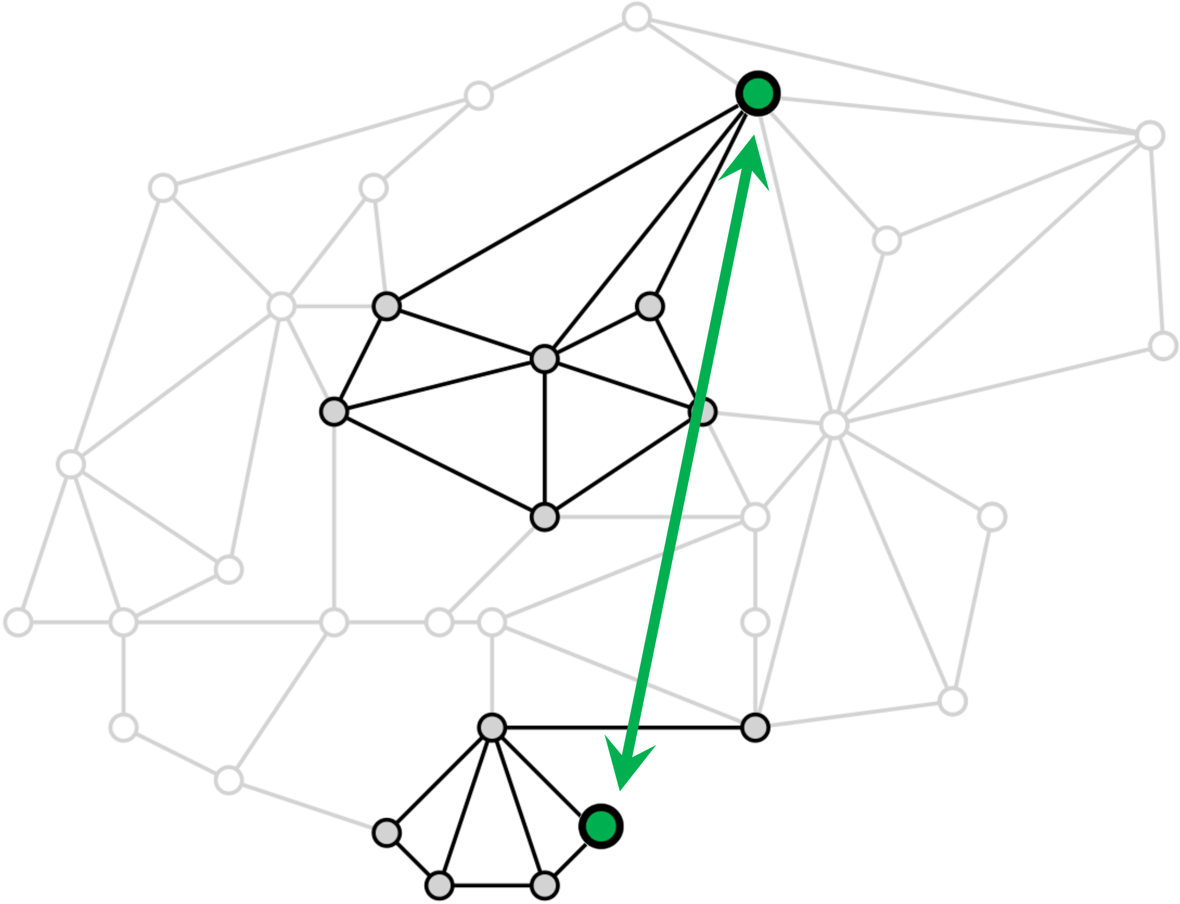
Breaking the Linear-Memory Barrier:

# Efficient MPC Graph Algorithms with Strongly Sublinear Memory

$\boldsymbol{S = O(n^{\delta})}$ local memory

$\boldsymbol{M = O(m/n^{\delta})}$ machines

$\mathbf{poly\,log\,log}\,\boldsymbol{n}$ rounds



**Imposed Locality:**
machines see only subset of nodes, regardless of sparsity of graph

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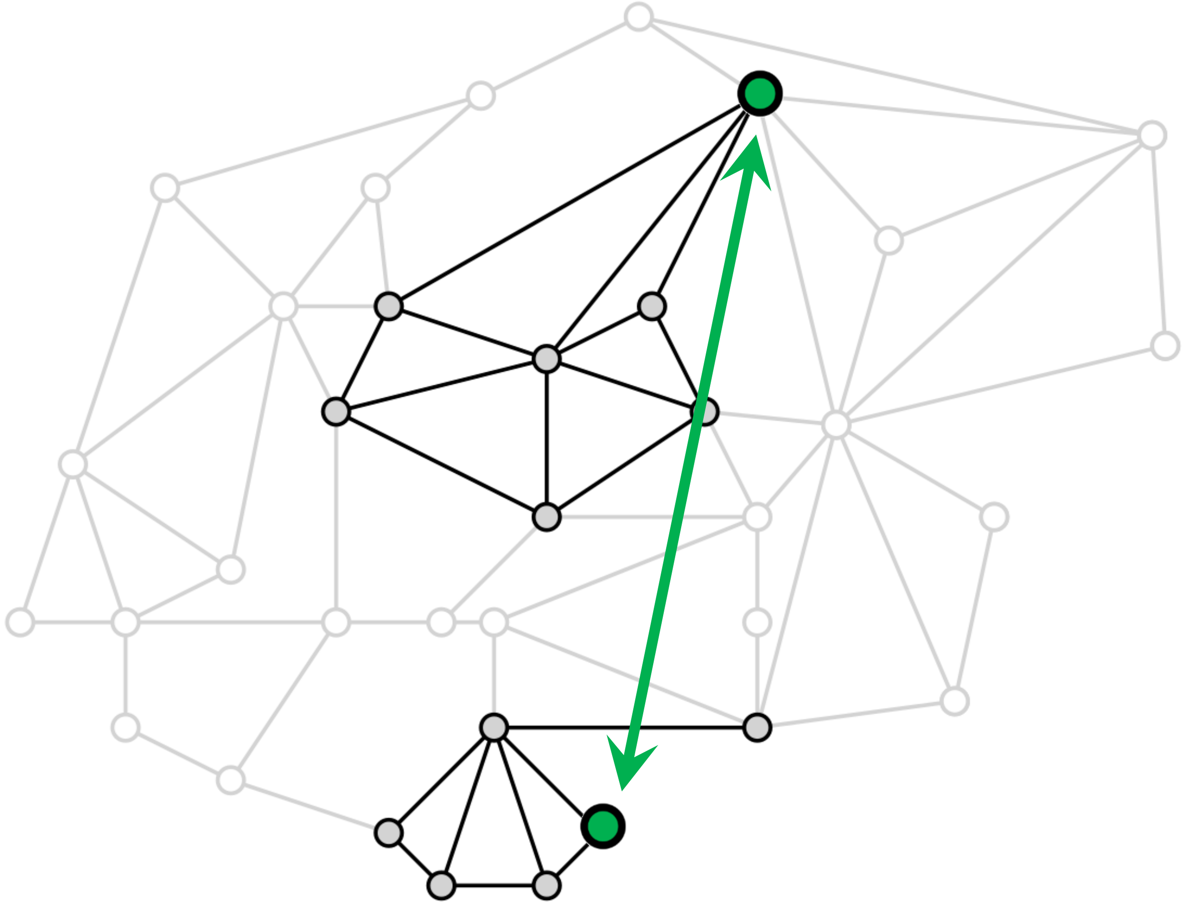

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- polynomially less memory than most MPC algorithms

Problem:

# Maximal Independent Set (MIS)

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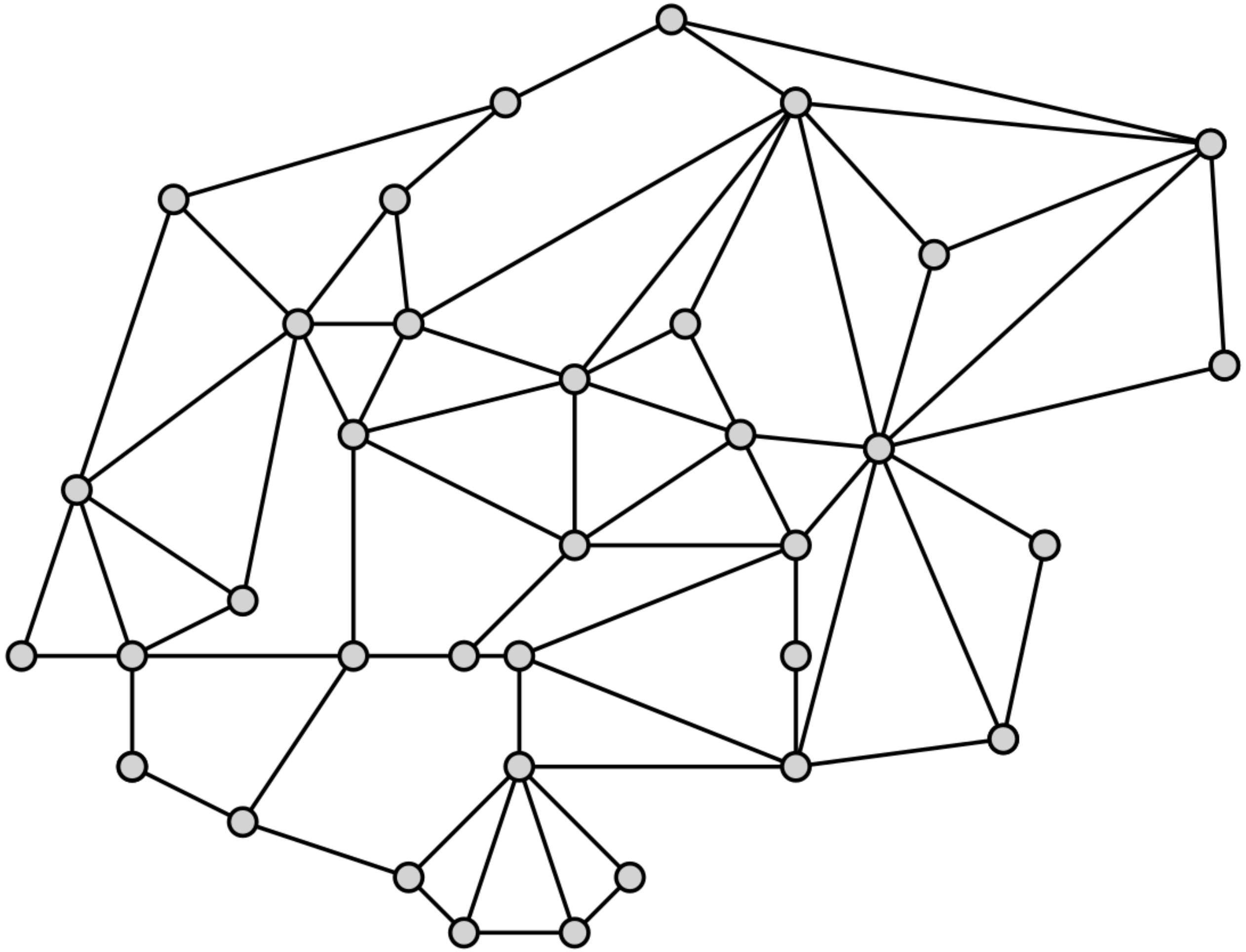
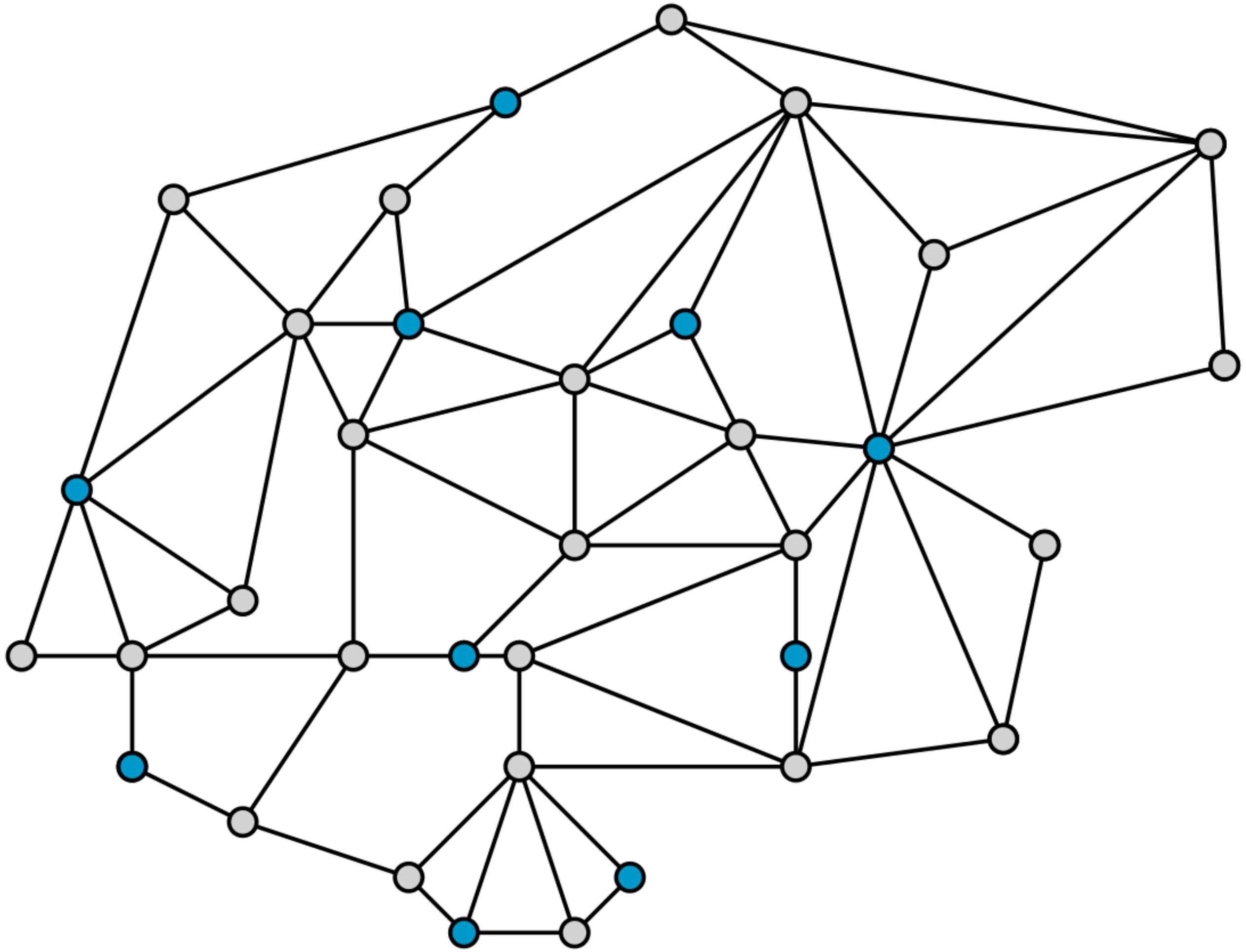


# Maximal Independent Set (MIS)



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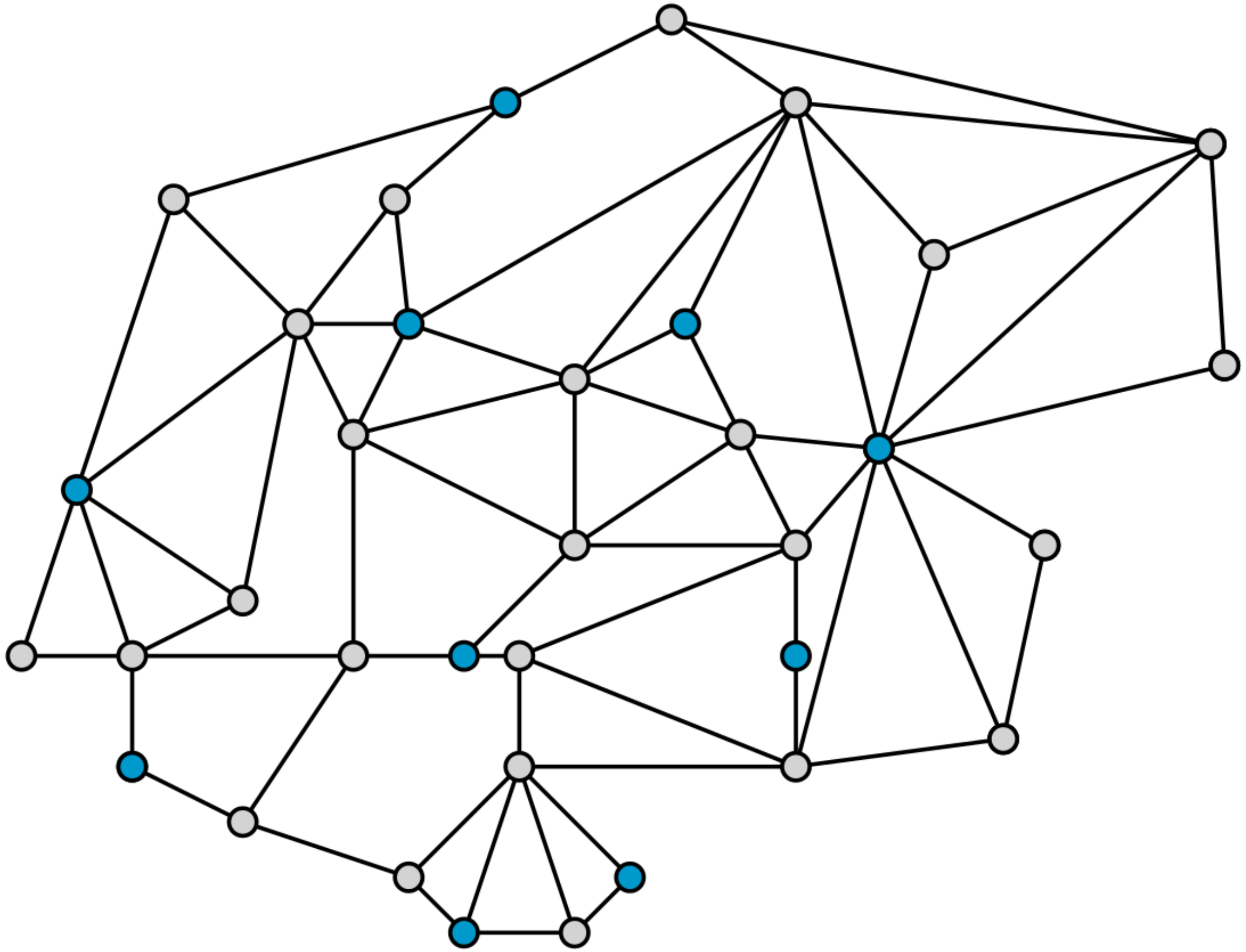
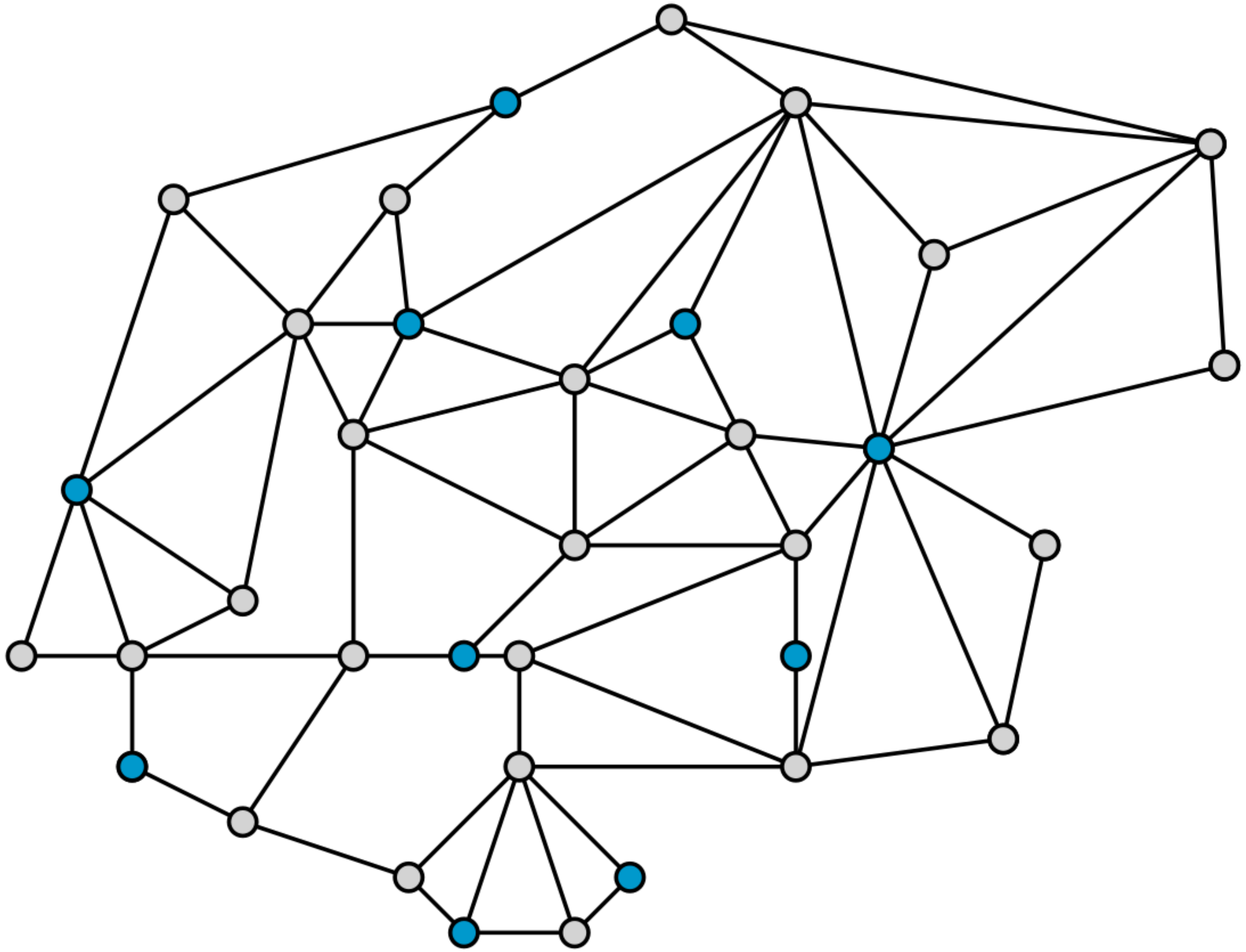


**Independent Set:**
set of non-adjacent nodes

# Maximal Independent Set (MIS)



**Independent Set:**
set of non-adjacent nodes

**Maximal:**
no node can be added
without violating independence

# MIS: State of the Art

$\boldsymbol{M}$ **machines**

$\boldsymbol{S}$ **memory per machine**

$\boldsymbol{M \cdot S = \tilde{O}\,(m+n)}$

**S**

$\widetilde{\mathrm{O}}(n^{\delta})$

$\widetilde{\Theta}(n)$

$\widetilde{\Omega}(n^{1+\delta})$

**Strongly Sublinear Memory:**
$S = \tilde{O}(n^{\delta}), 0 \leq \delta < 1$
No machine sees all nodes.

**Linear Memory:**
$S = \tilde{O}(n)$
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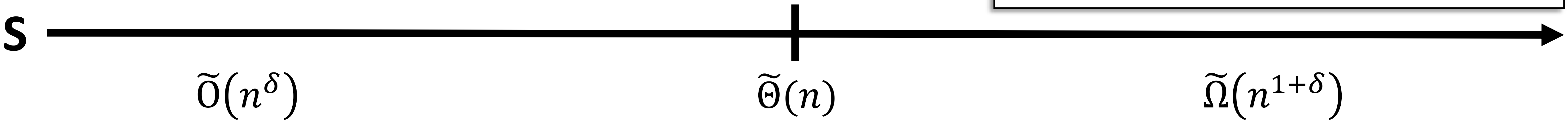

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*Lattanzi et al.* [SPAA'11]
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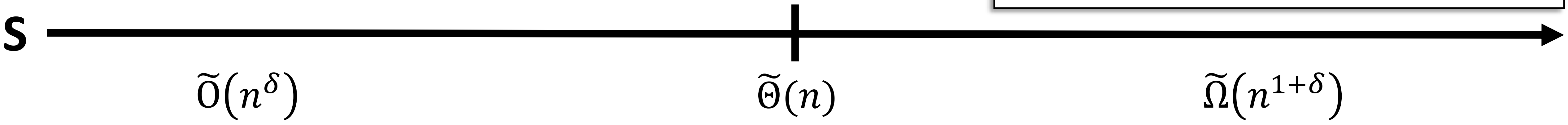

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# MIS: State of the Art on Trees

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*Trivial solution*
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# Our Result

$\boldsymbol{O(\log^3 \log n)}$-round MPC algorithm

with $\boldsymbol{S = \widetilde{O}(n^{\delta})}$ memory that w.h.p. computes MIS on trees.

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$\widetilde{\boldsymbol{O}}(\sqrt{\log \boldsymbol{n}})$ rounds

$\boldsymbol{S} = \widetilde{\boldsymbol{O}}(\boldsymbol{n}^{\delta})$ memory

*Ghaffari and Uitto* [SODA’19]

$\boldsymbol{O}(\log^3 \log \boldsymbol{n})$-round MPC algorithm

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Conditional $\boldsymbol{\Omega}(\log\ \log \boldsymbol{n})$-round lower bound for $\boldsymbol{S} = \widetilde{\boldsymbol{O}}\left(\boldsymbol{n}^{\delta}\right)$

*Ghaffari, Kuhn, and Uitto* [FOCS’19]

# Algorithm

# Algorithm Outline

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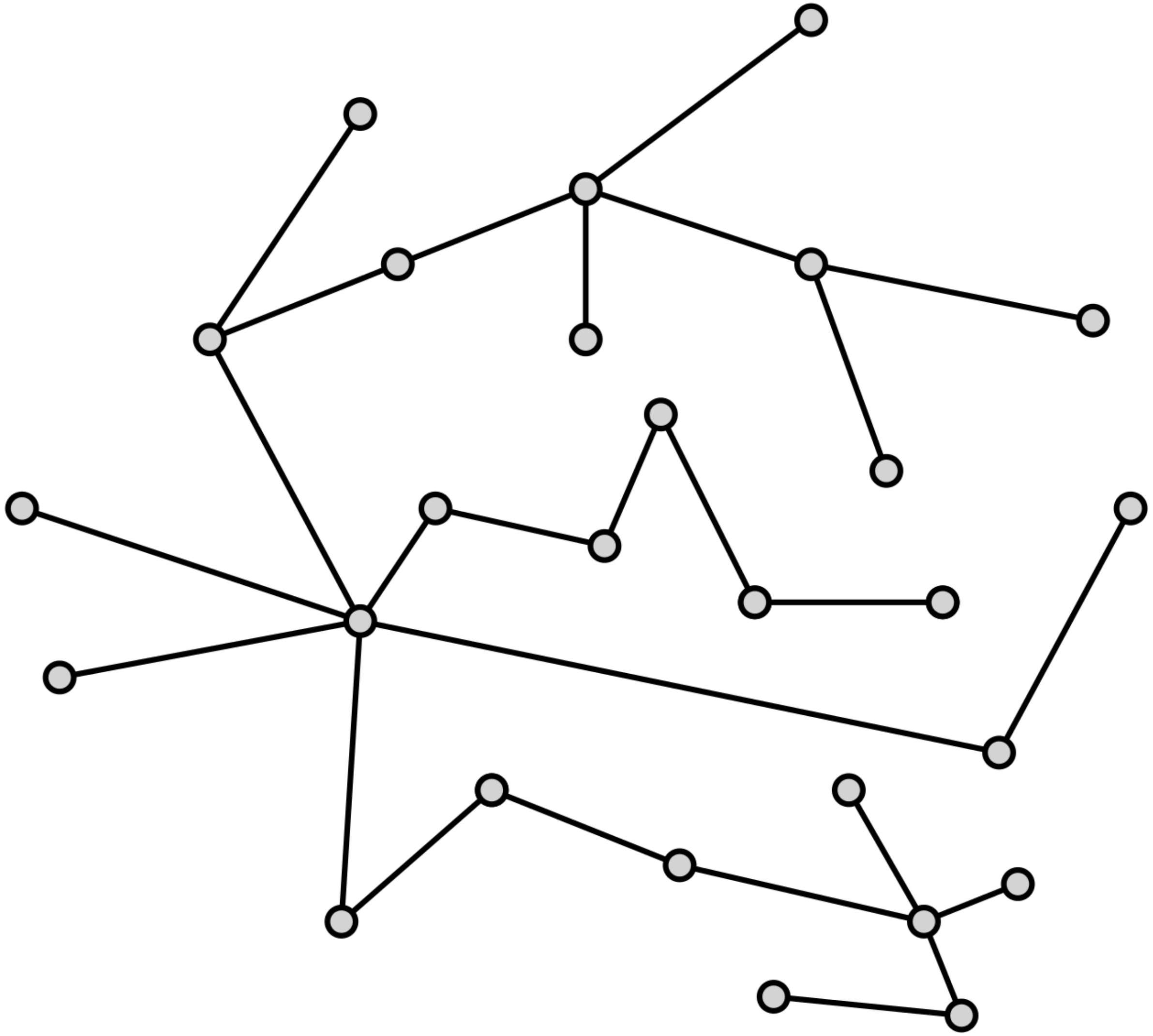
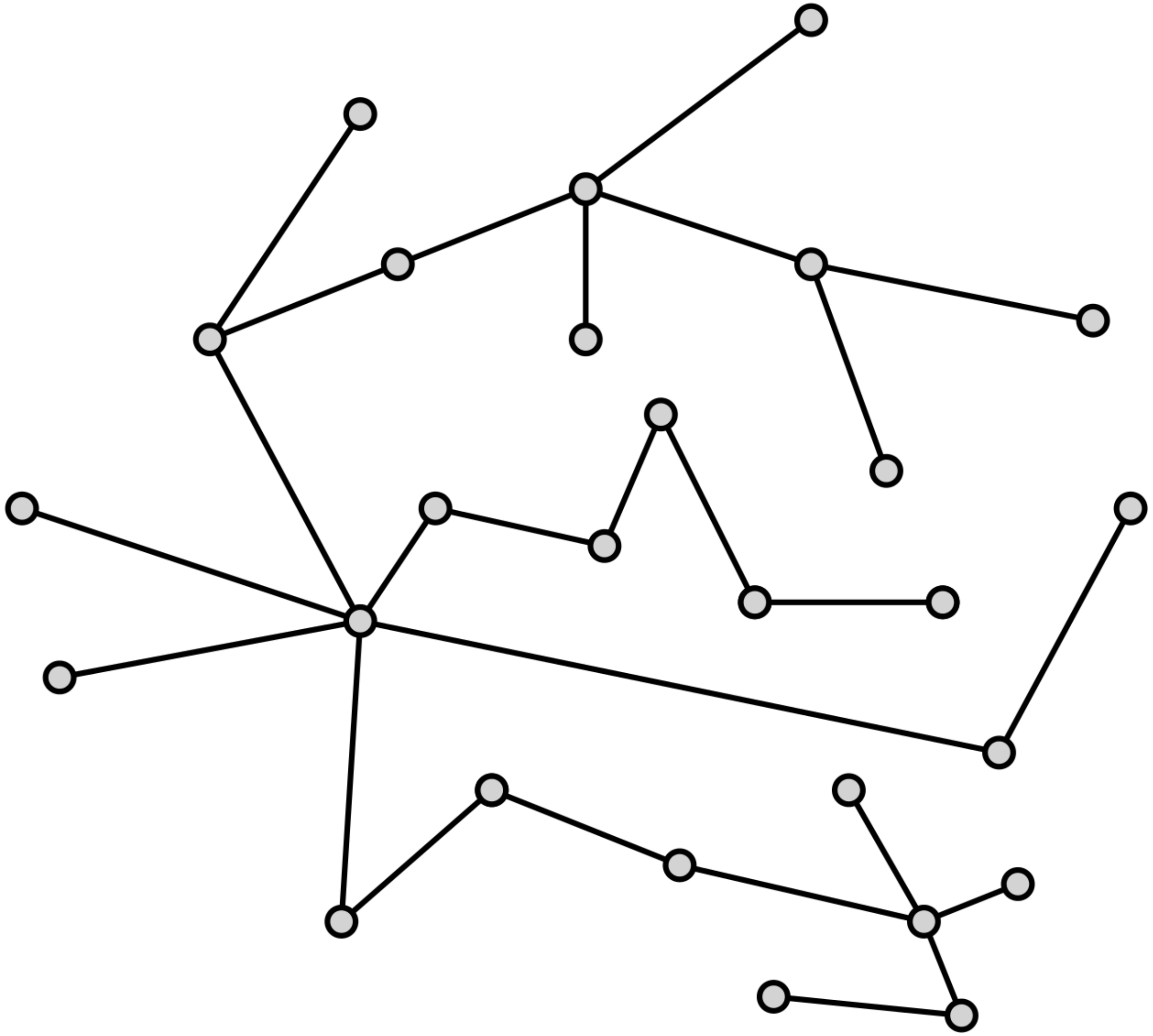


# Algorithm Outline

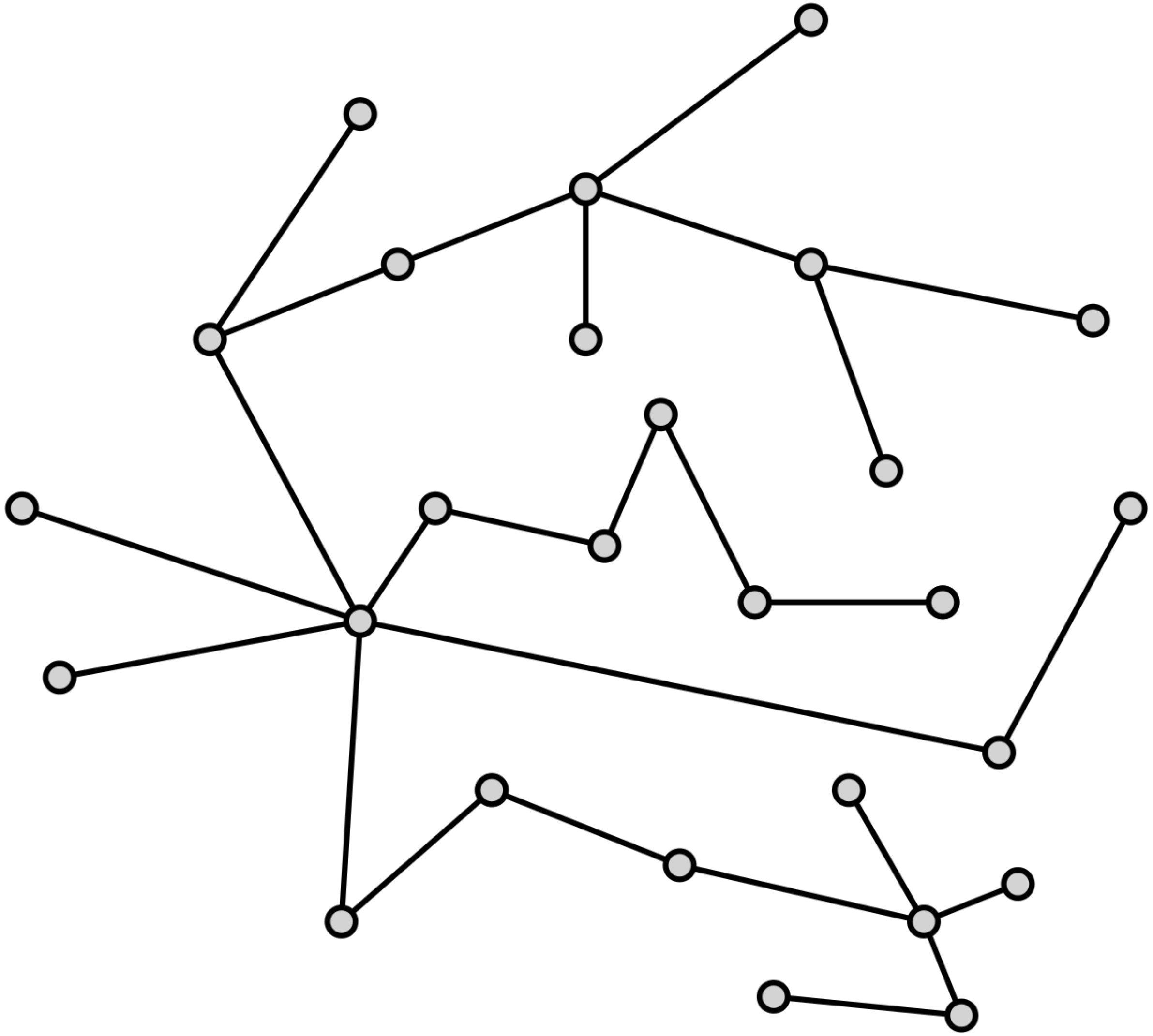
## 1) Shattering



# Algorithm Outline

## 1) Shattering

main LOCAL technique
*Beck* [RSA'91]



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break graph into small components

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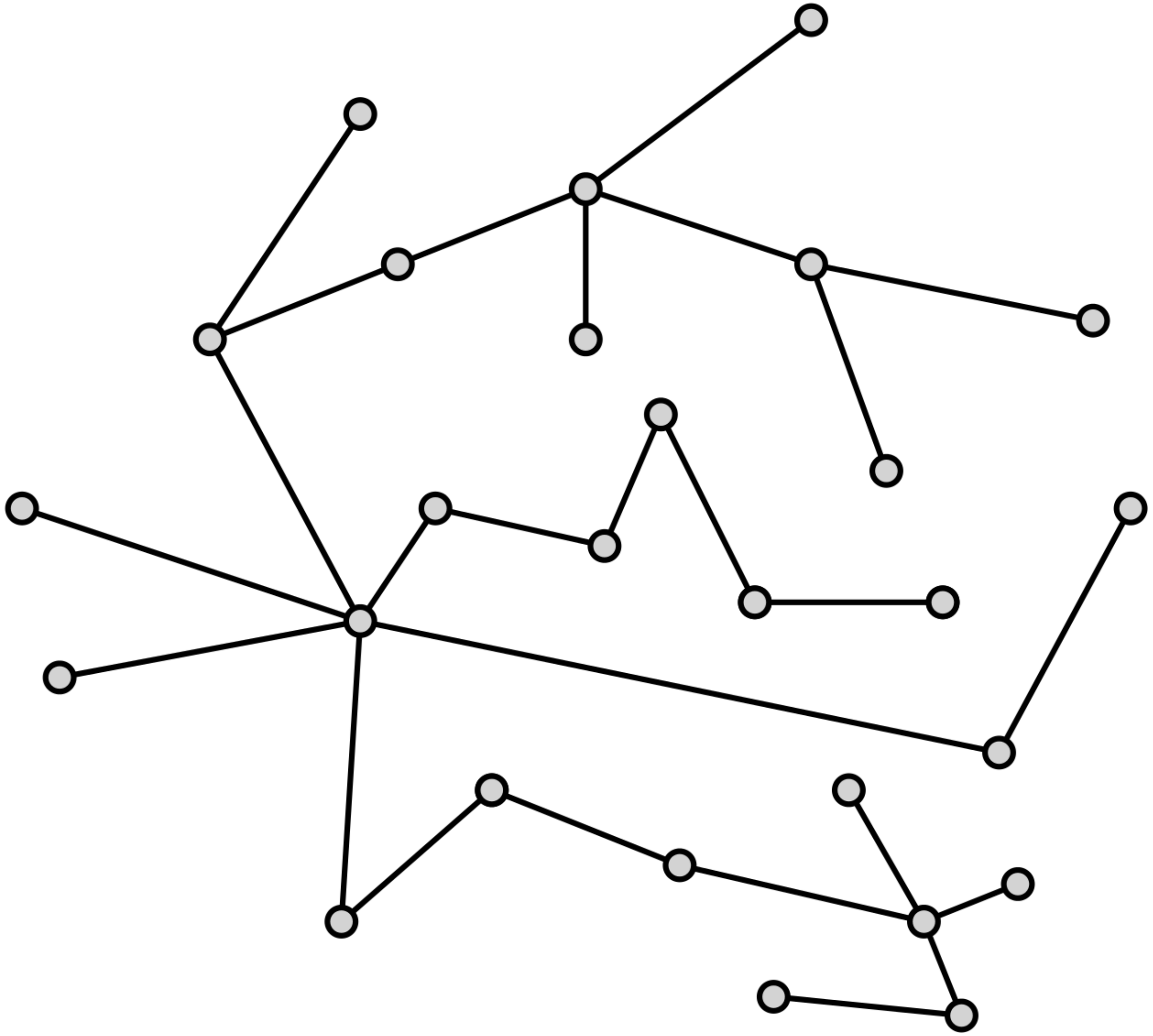

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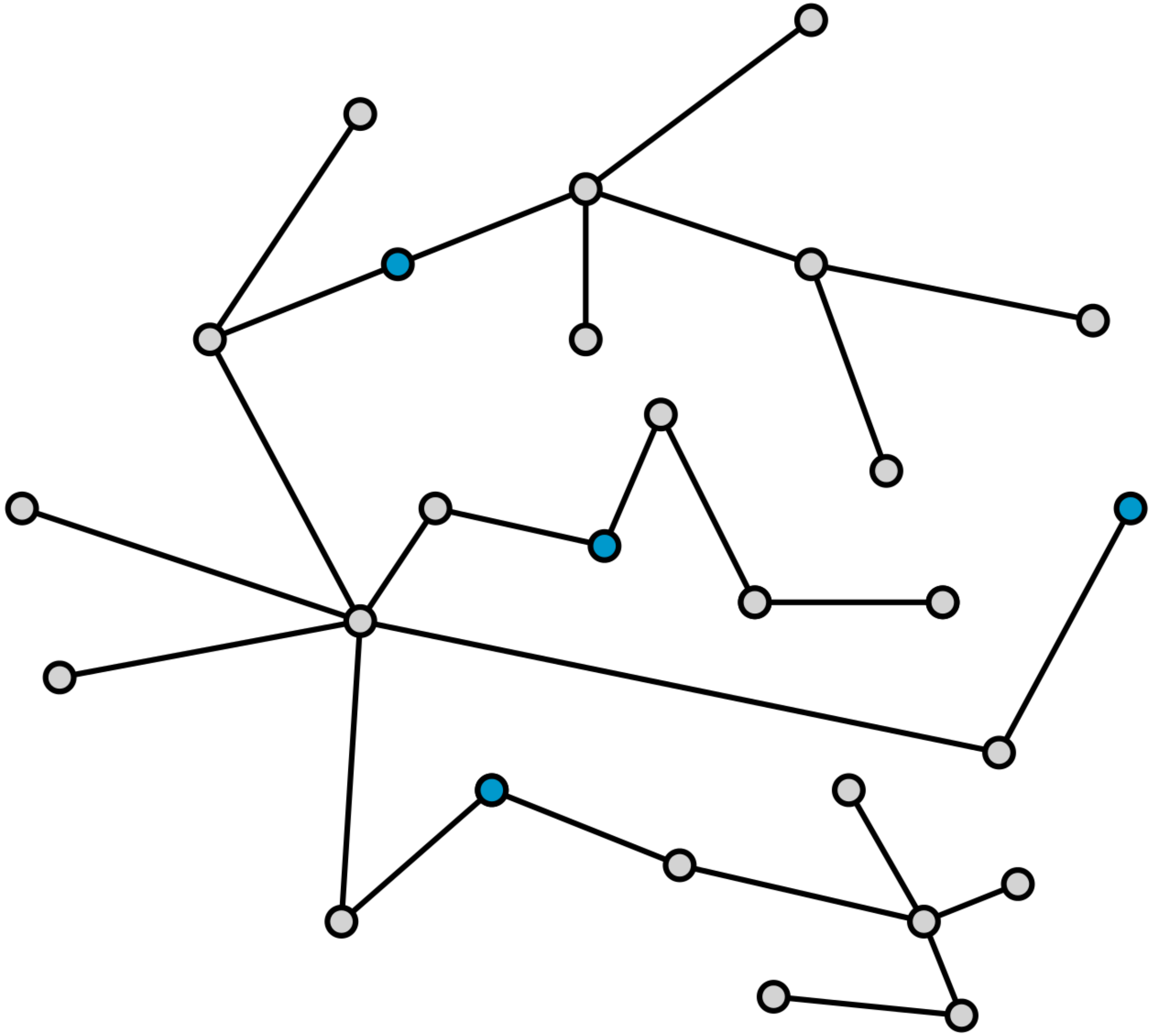

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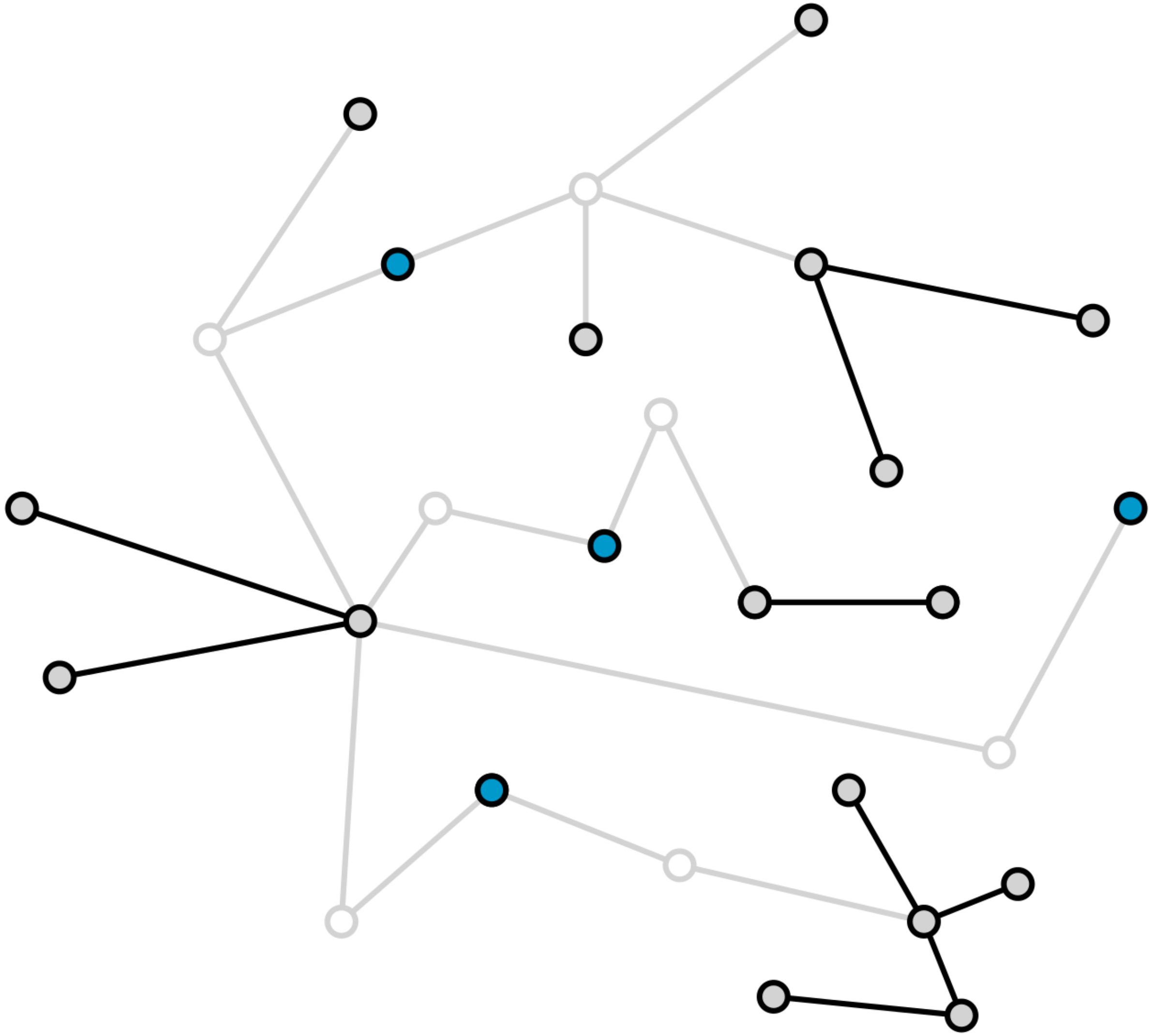

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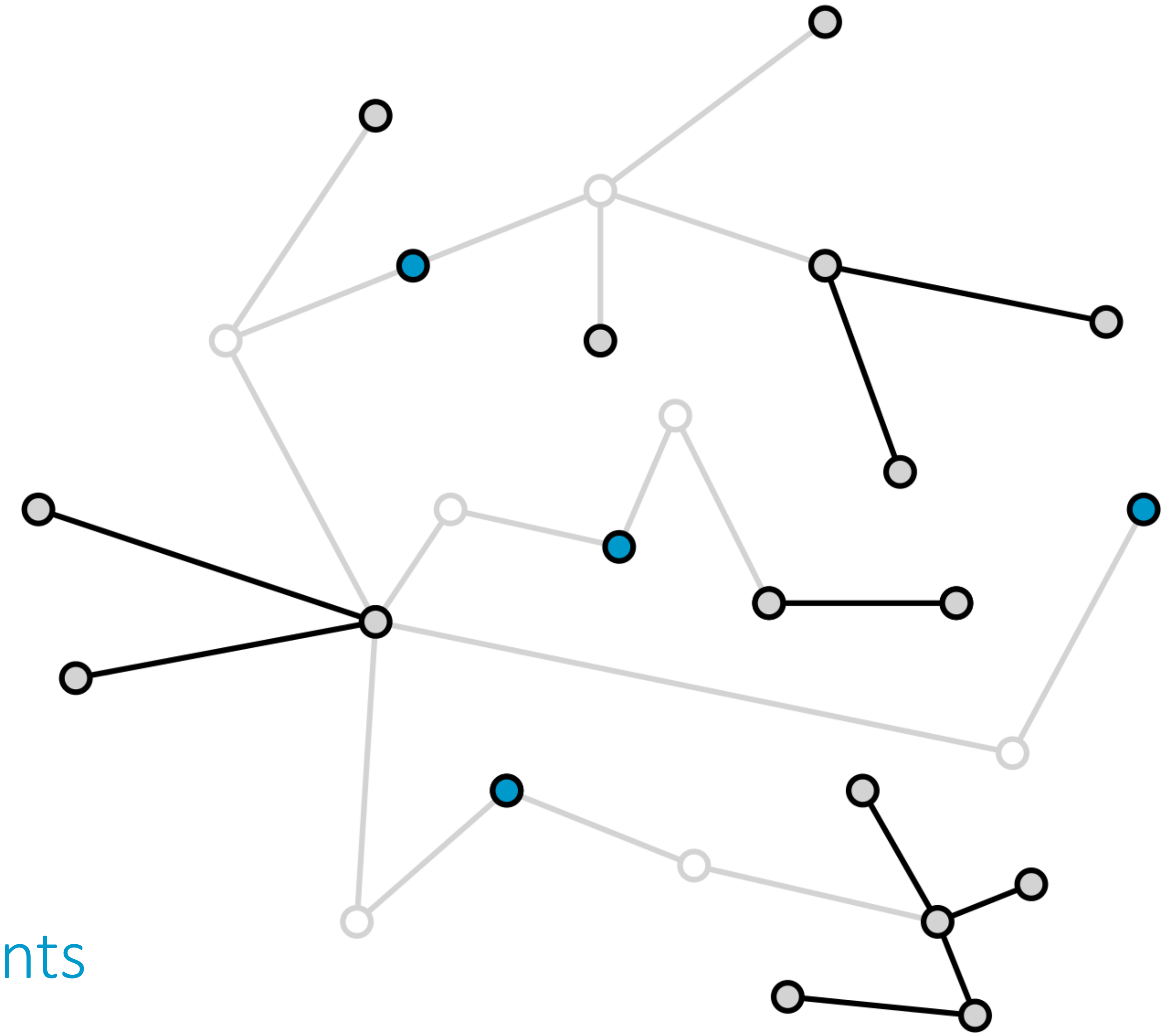

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solve problem on remaining components

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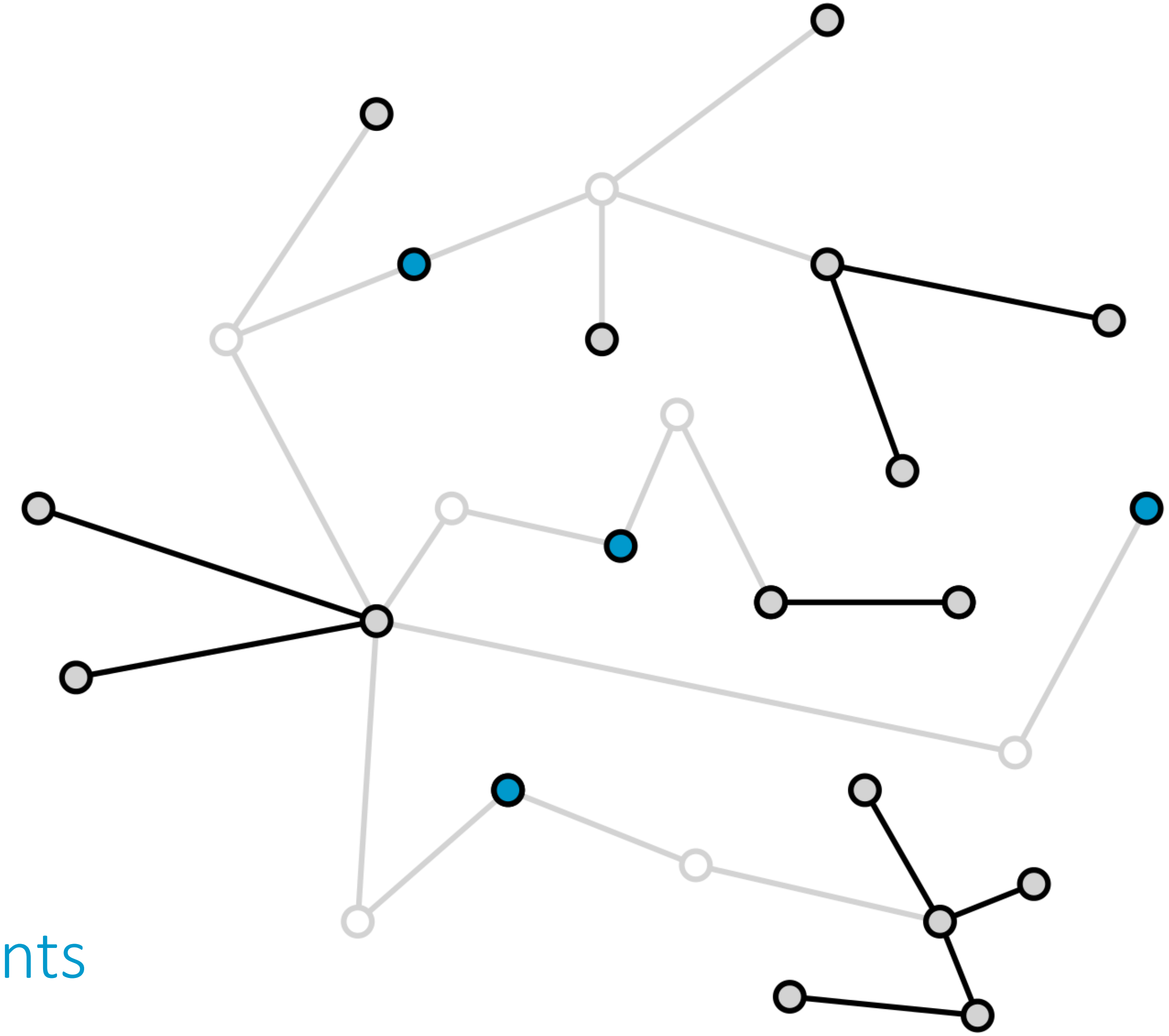

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**i) Gathering of Components**

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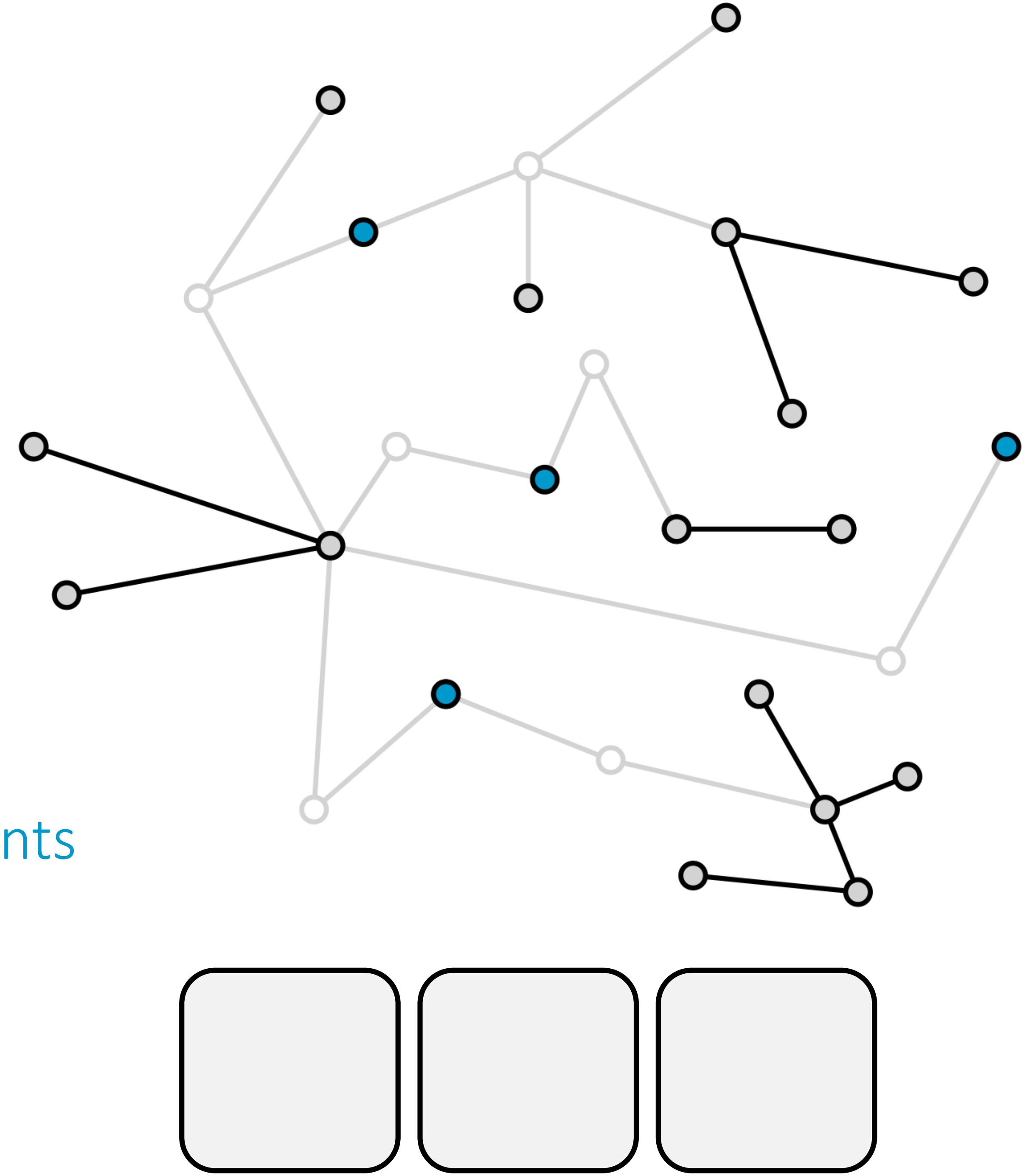

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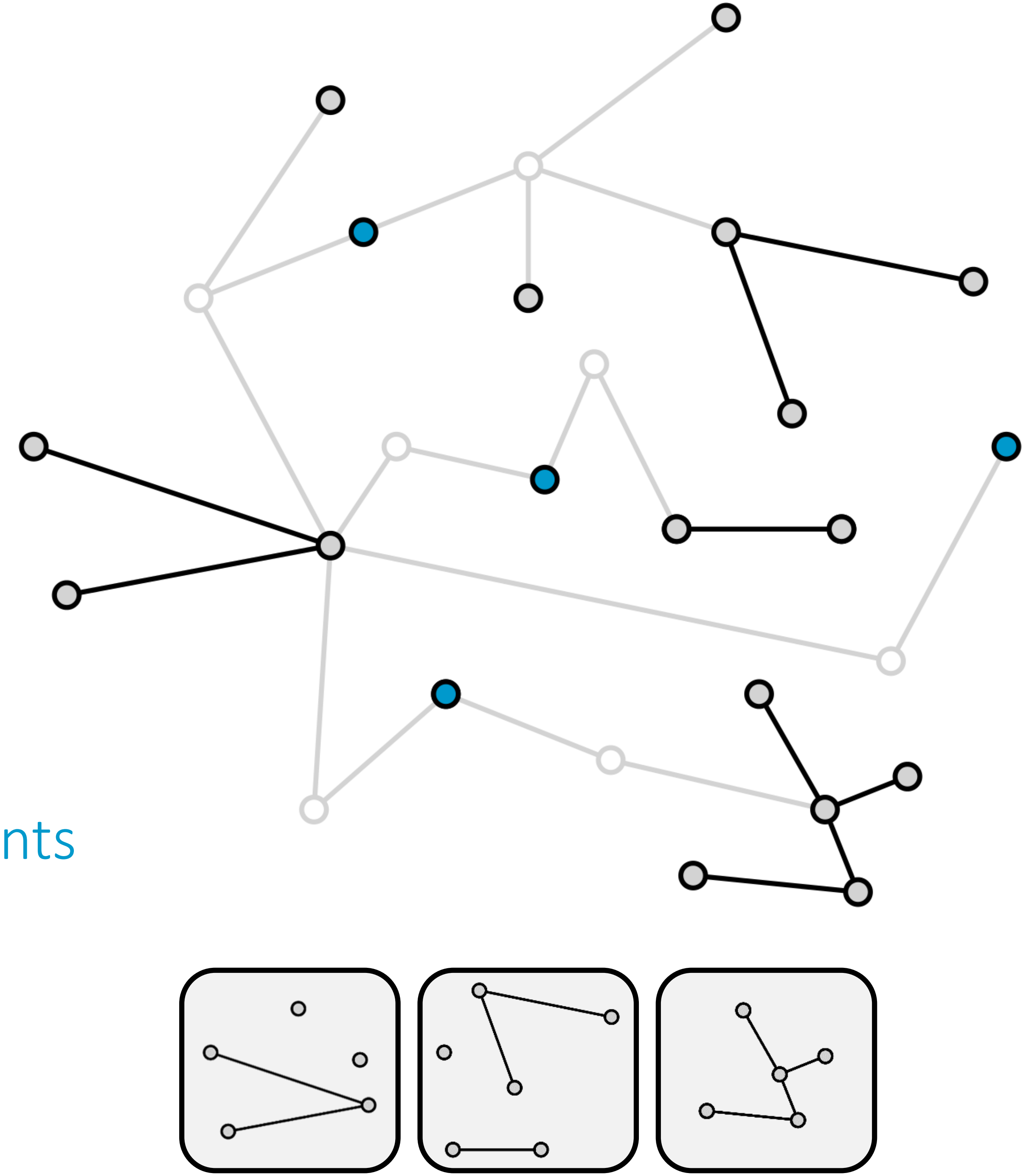

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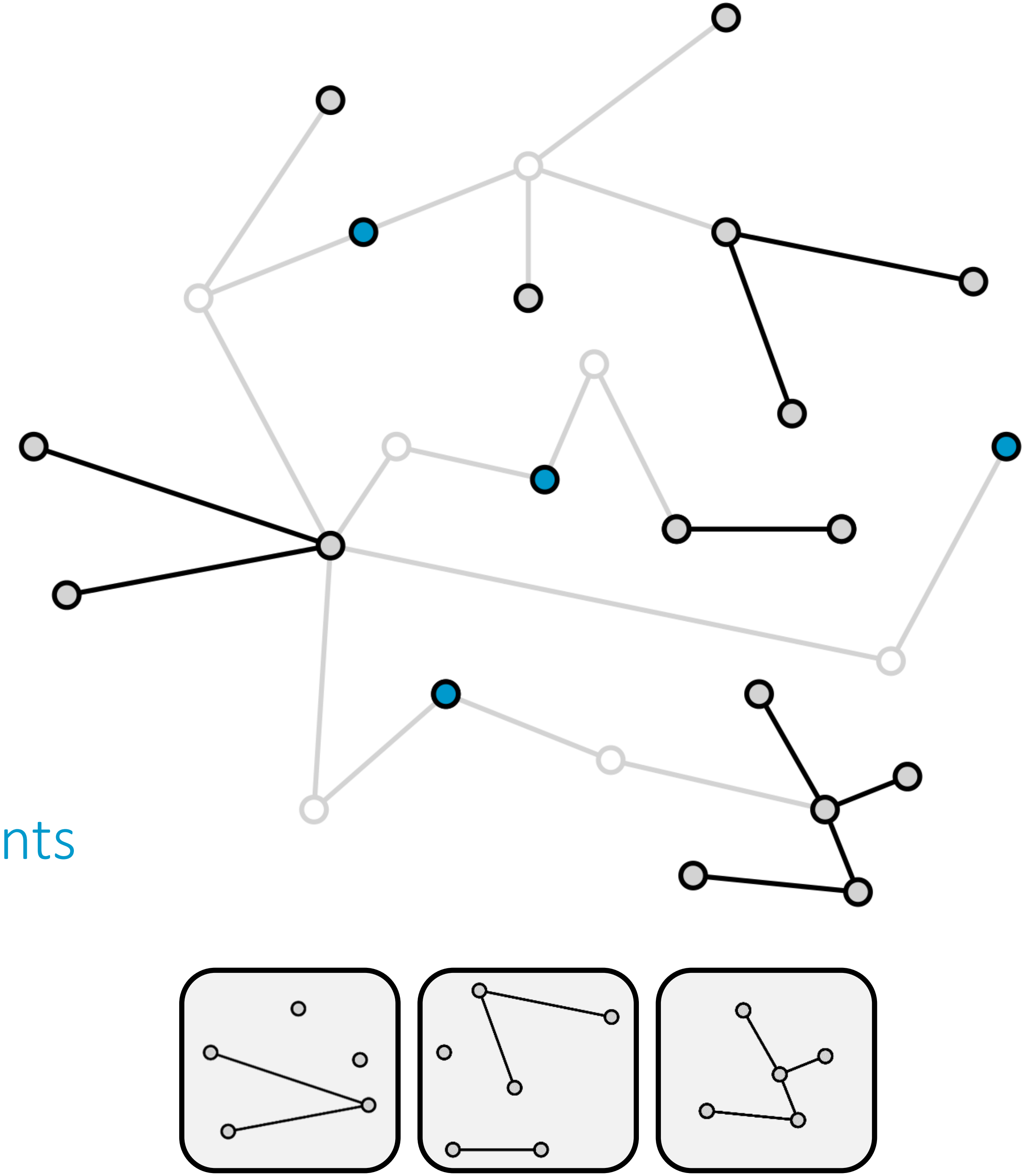

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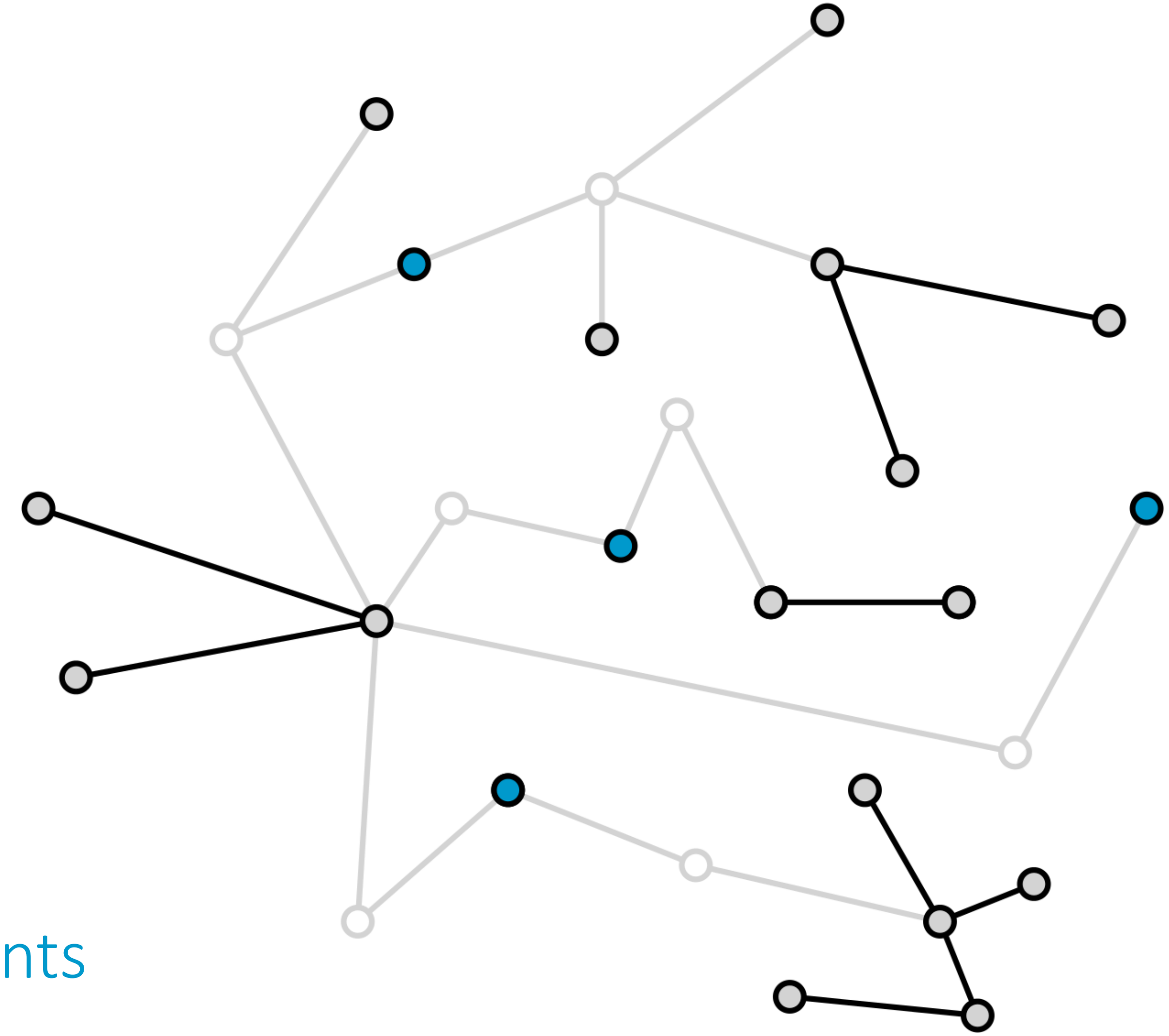
**i) Gathering of Components**

**ii) Local Computation**

# Algorithm Outline



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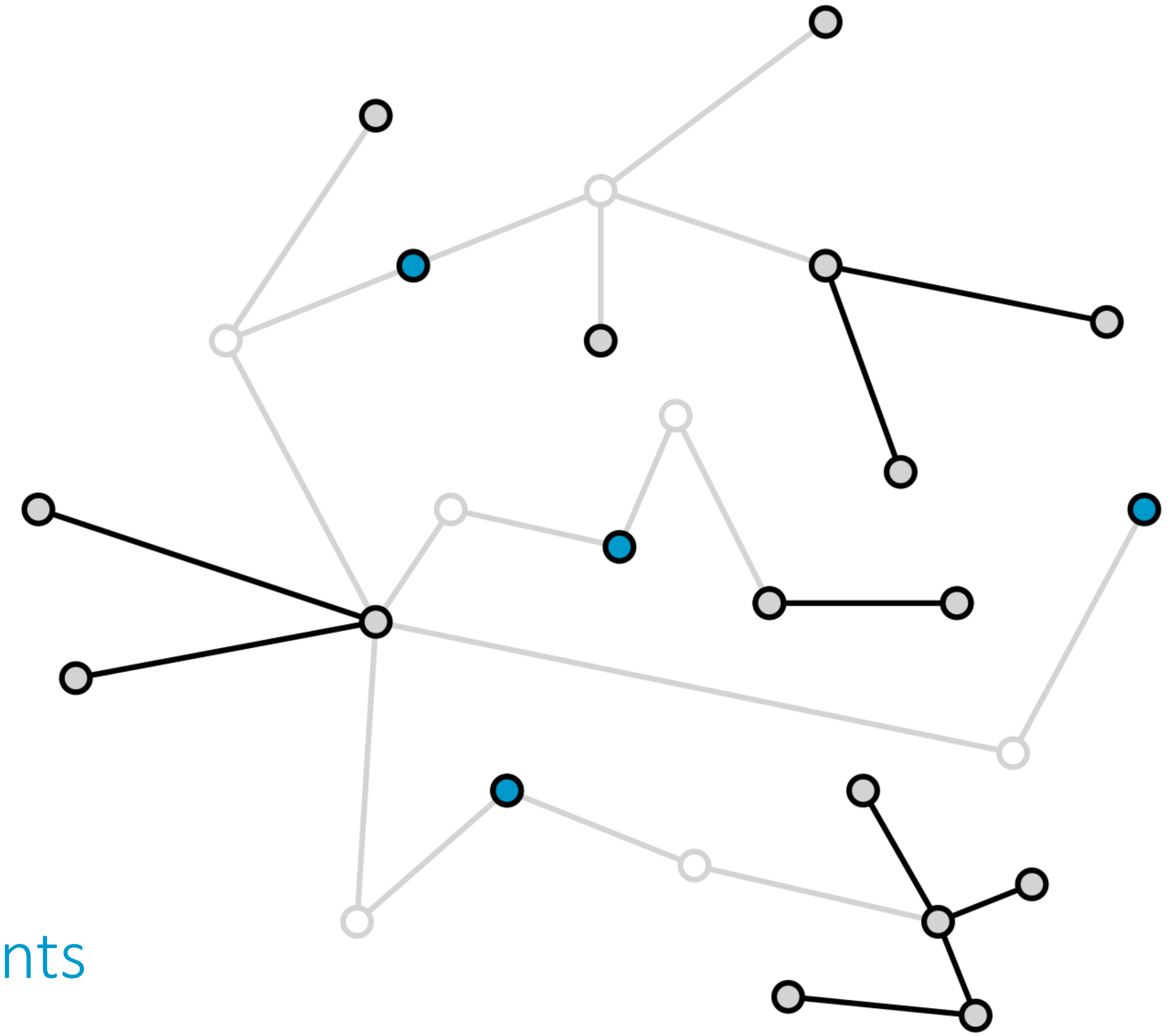

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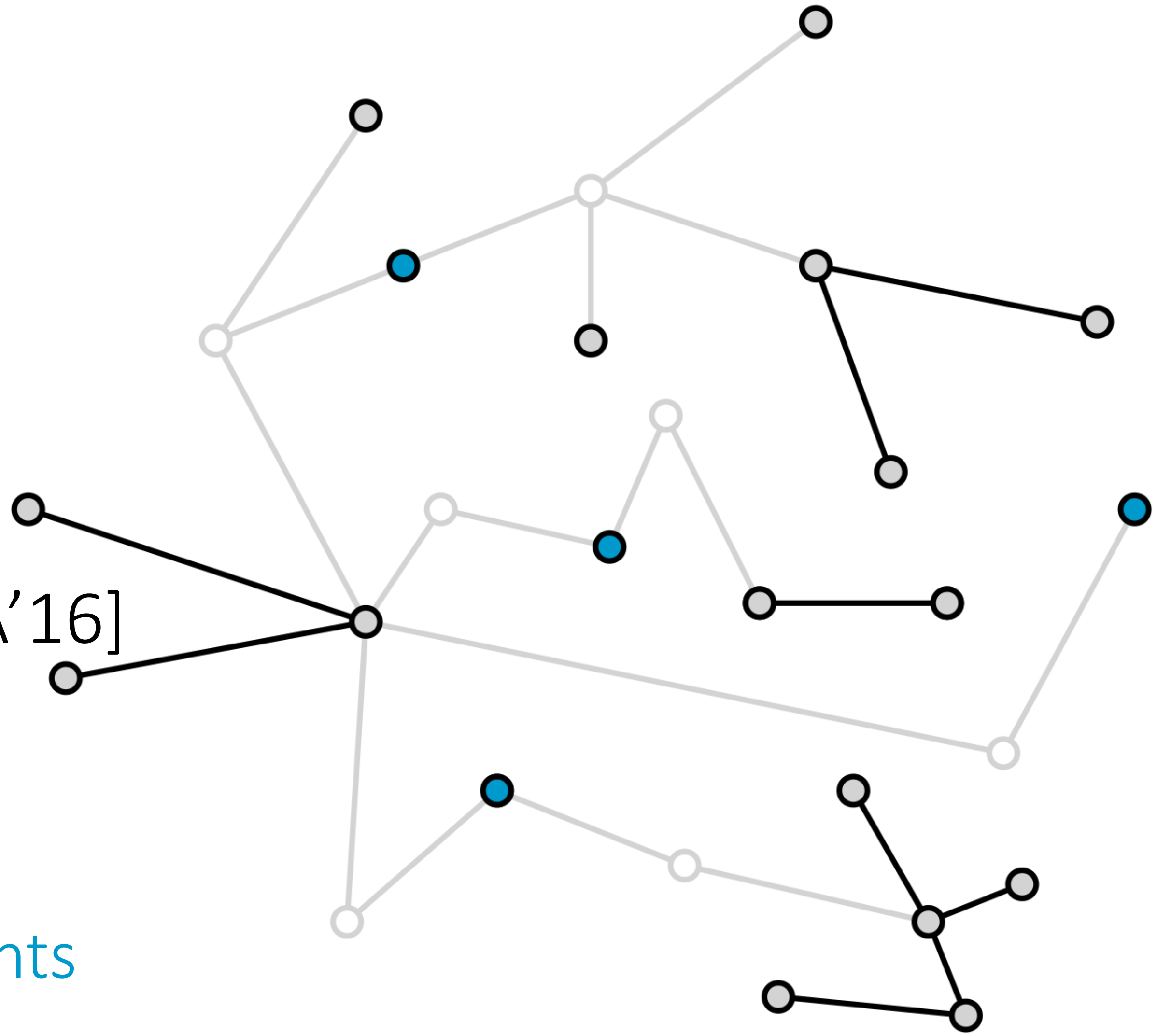

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# Algorithm Outline

## 1) Shattering

break graph into small components

**i) Degree Reduction**

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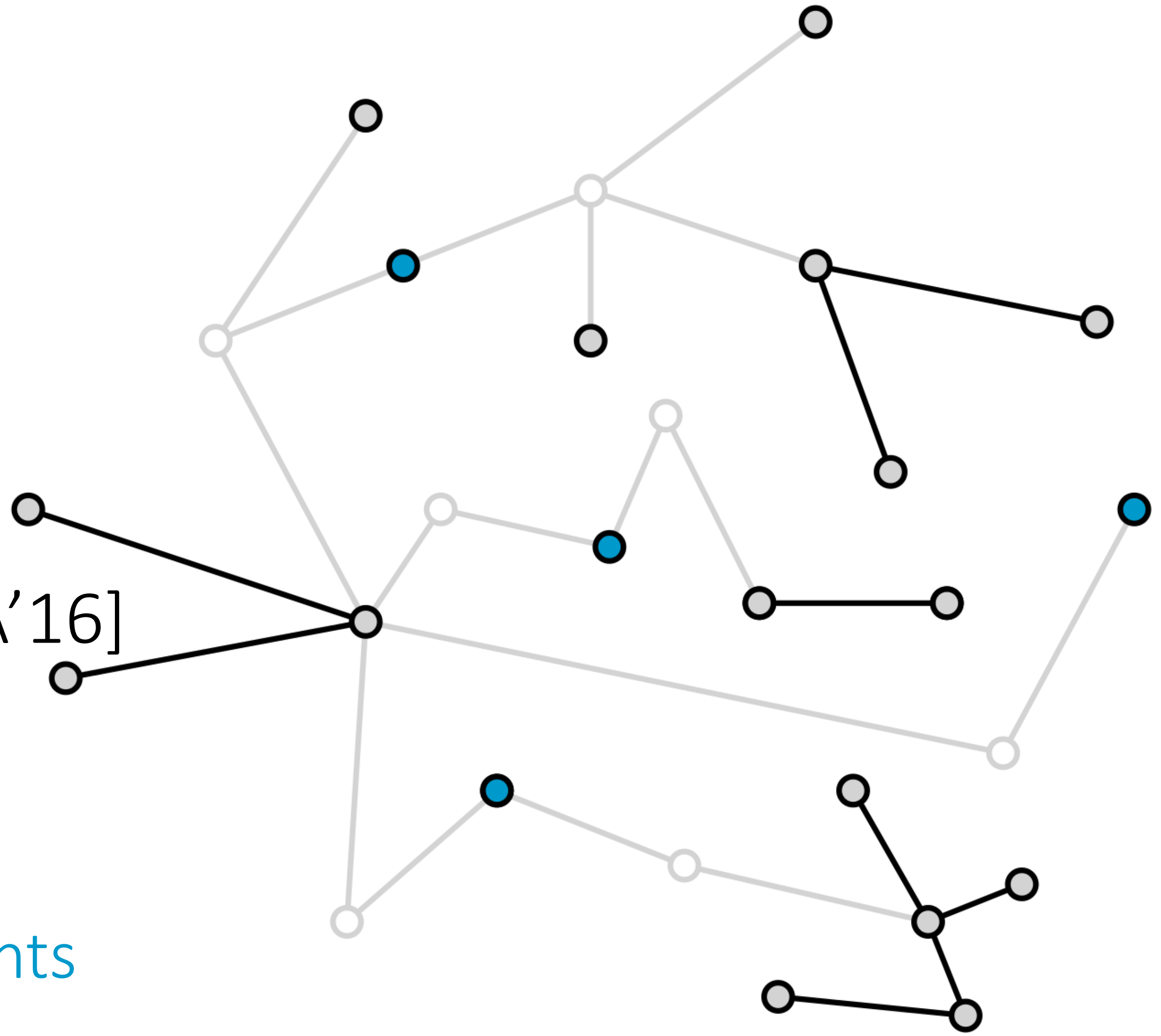

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# Polynomial Degree Reduction:
# **Subsample-and-Conquer**

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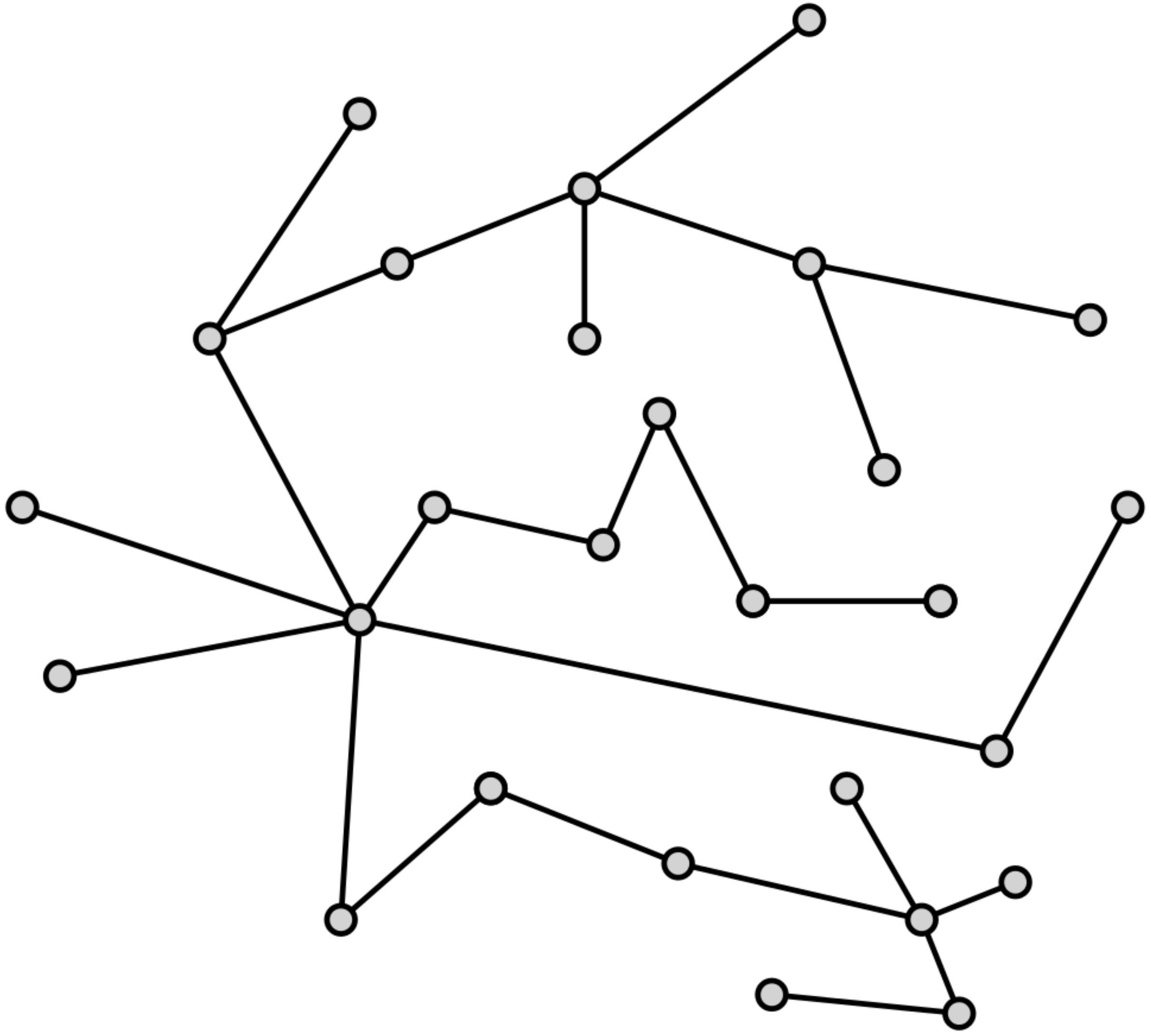
**Subsample**

**Conquer**

# Polynomial Degree Reduction:
# **Subsample-and-Conquer**



**Subsample**

**Conquer**

## Polynomial Degree Reduction:
# Subsample-and-Conquer

**Subsample**

subsample **nodes** independently

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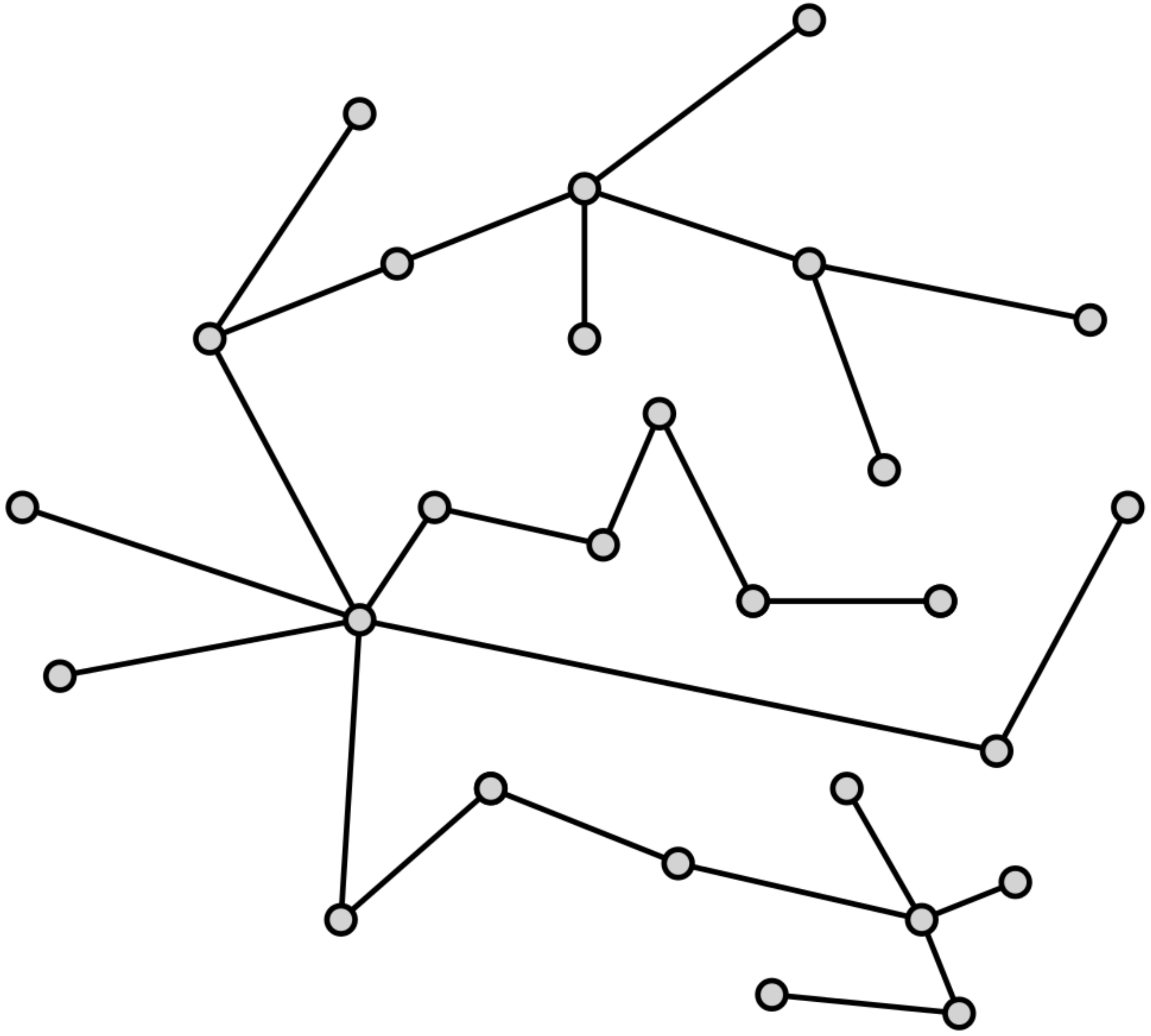

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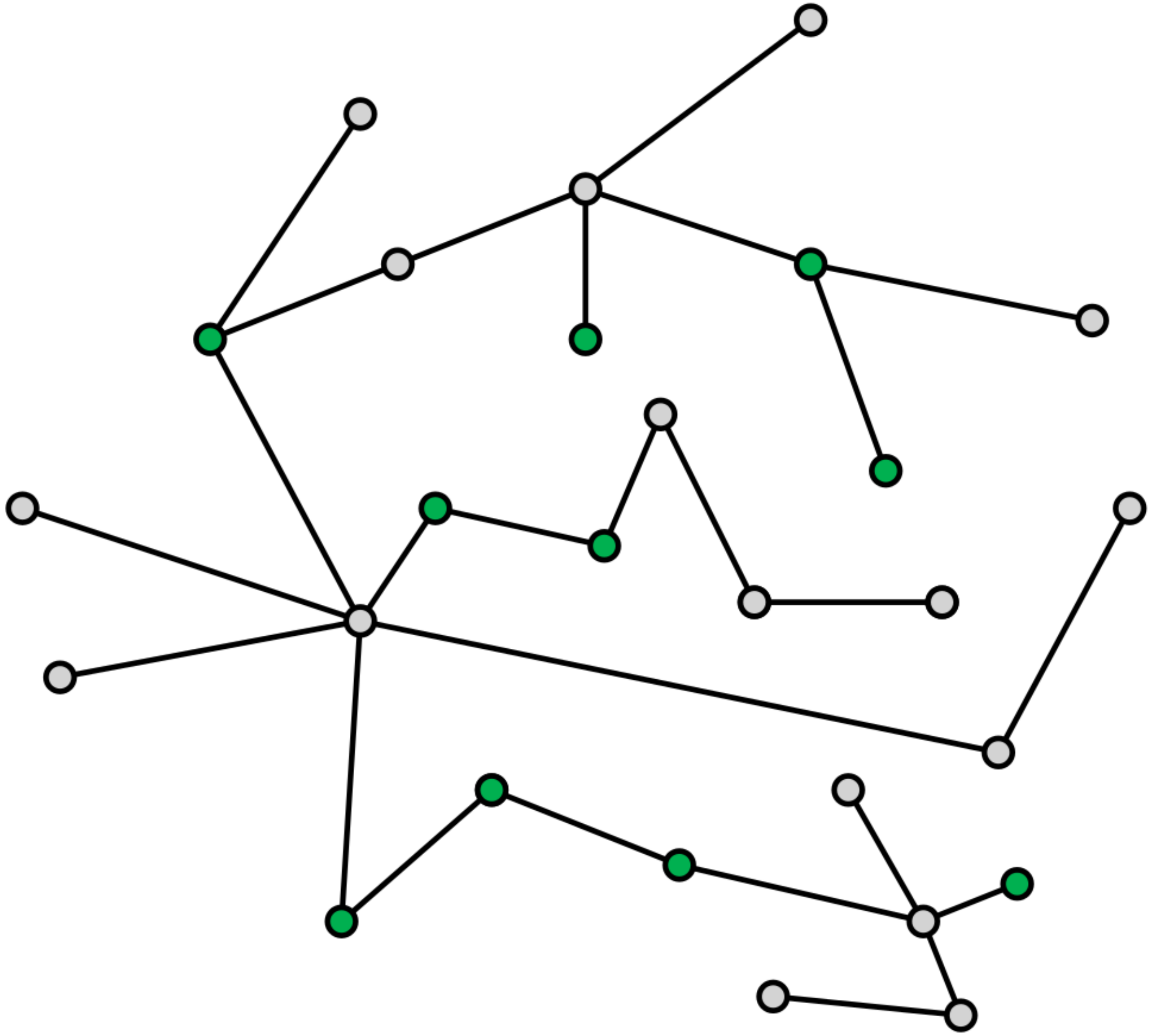

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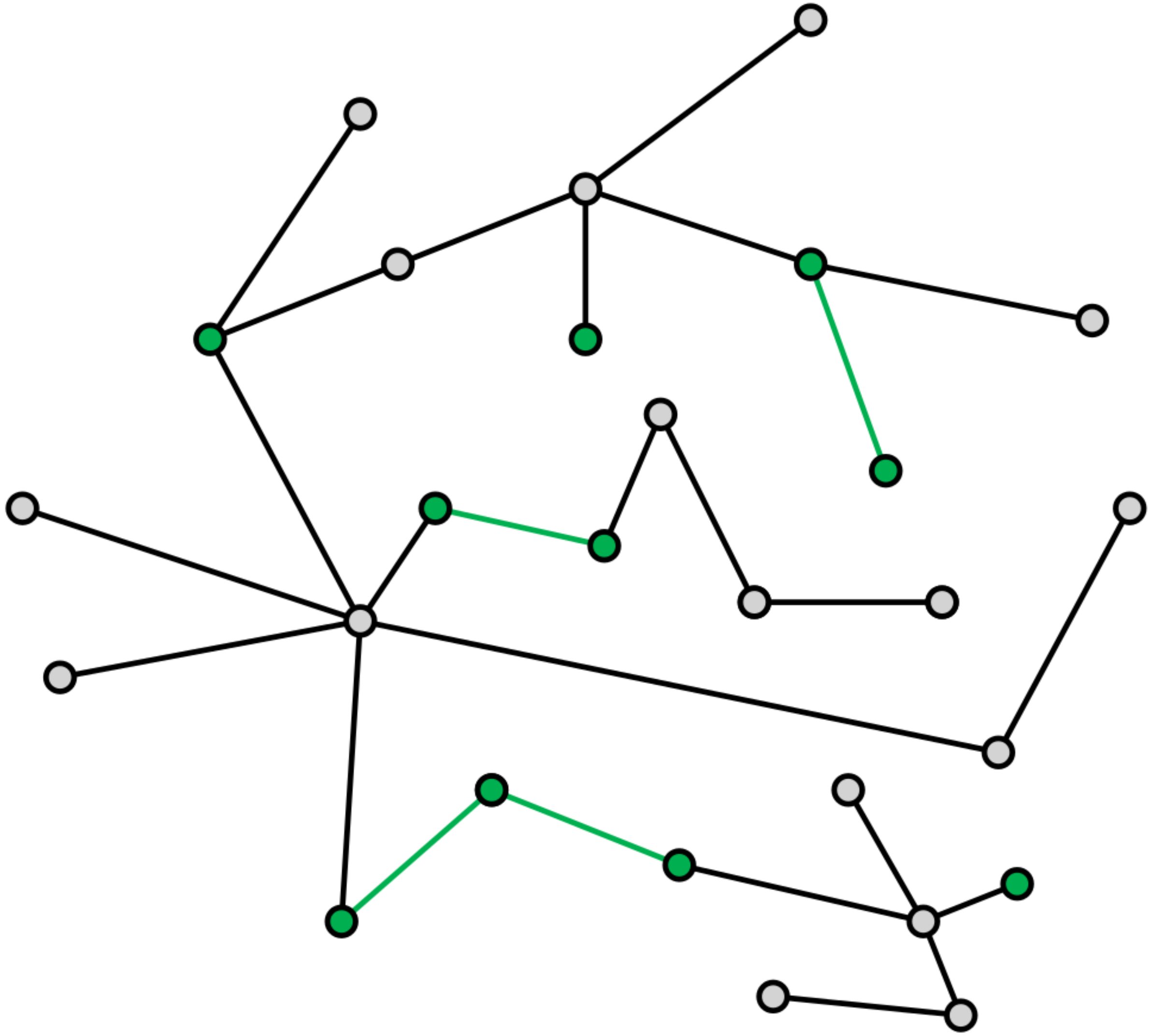

# Polynomial Degree Reduction:
# **Subsample-and-Conquer**

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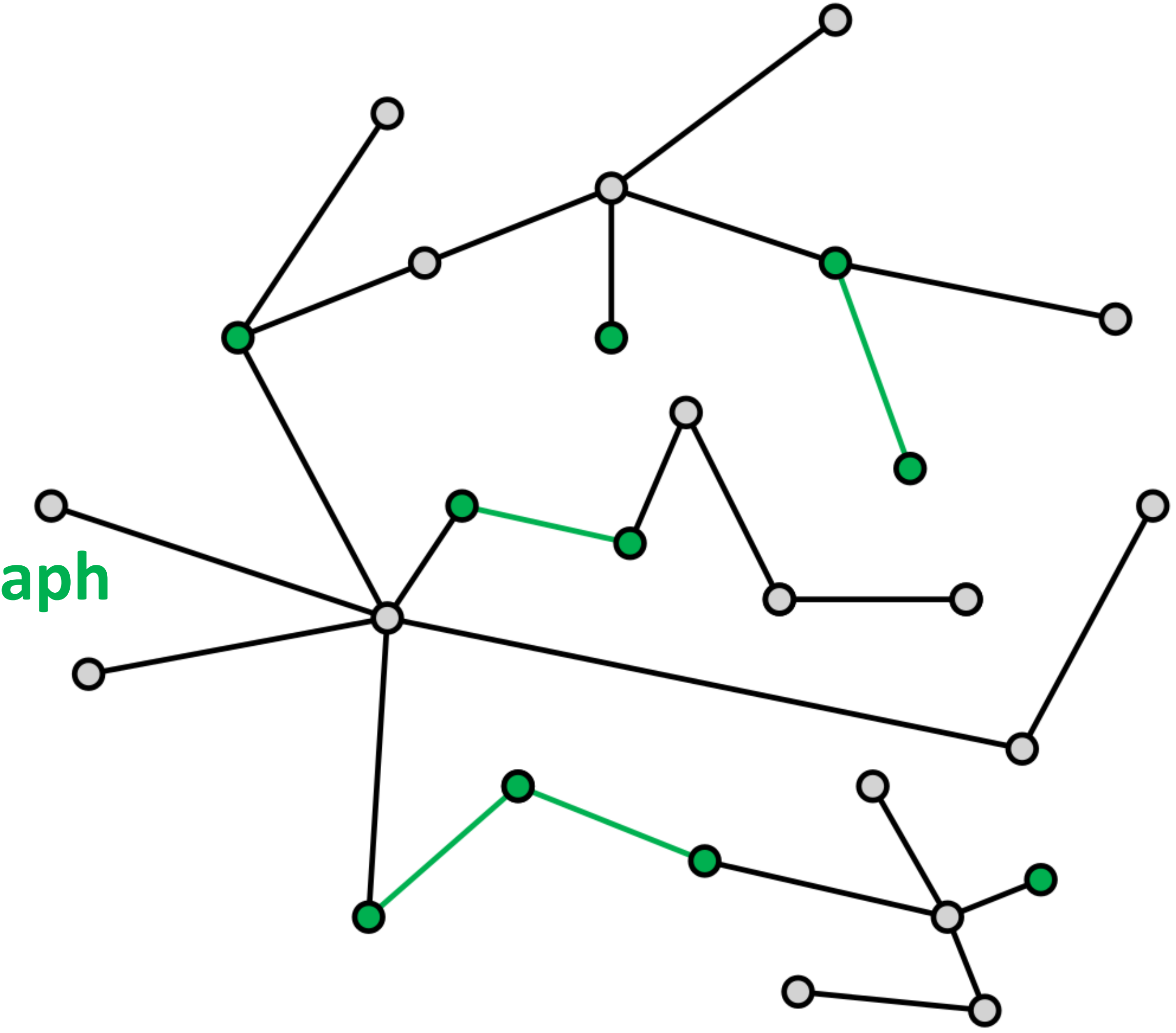
**Conquer**

compute **random MIS** in **subsampled graph**



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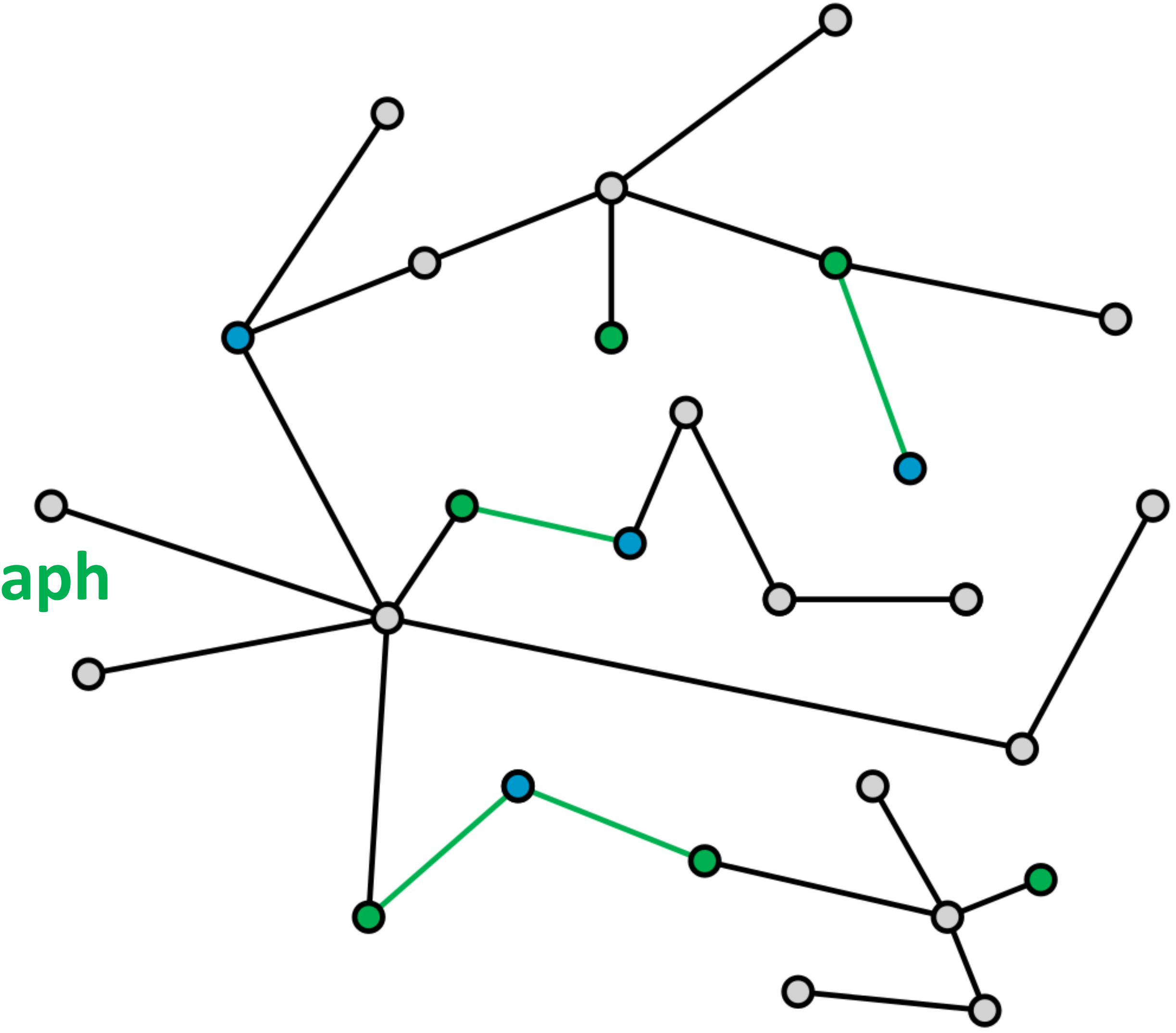

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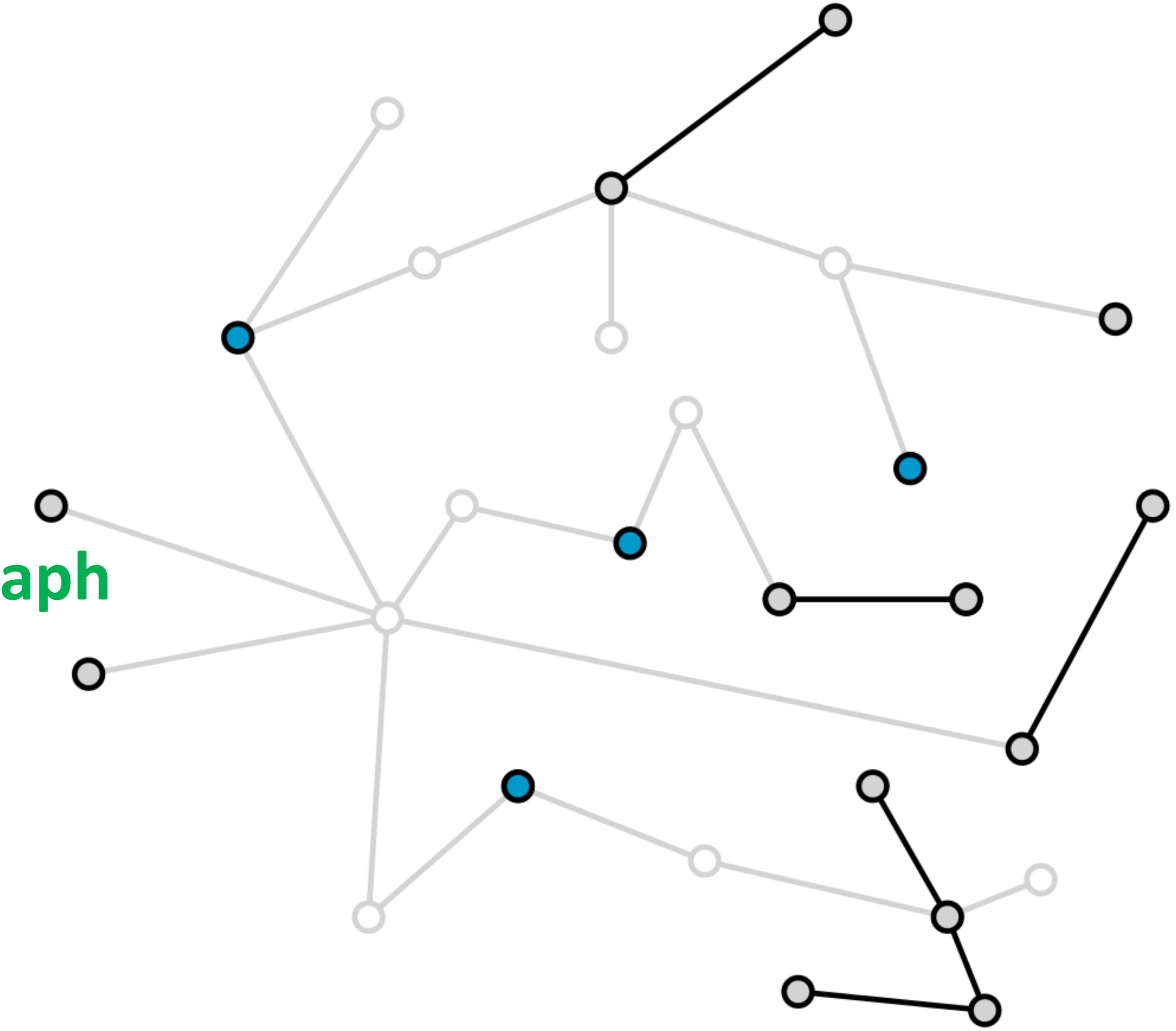

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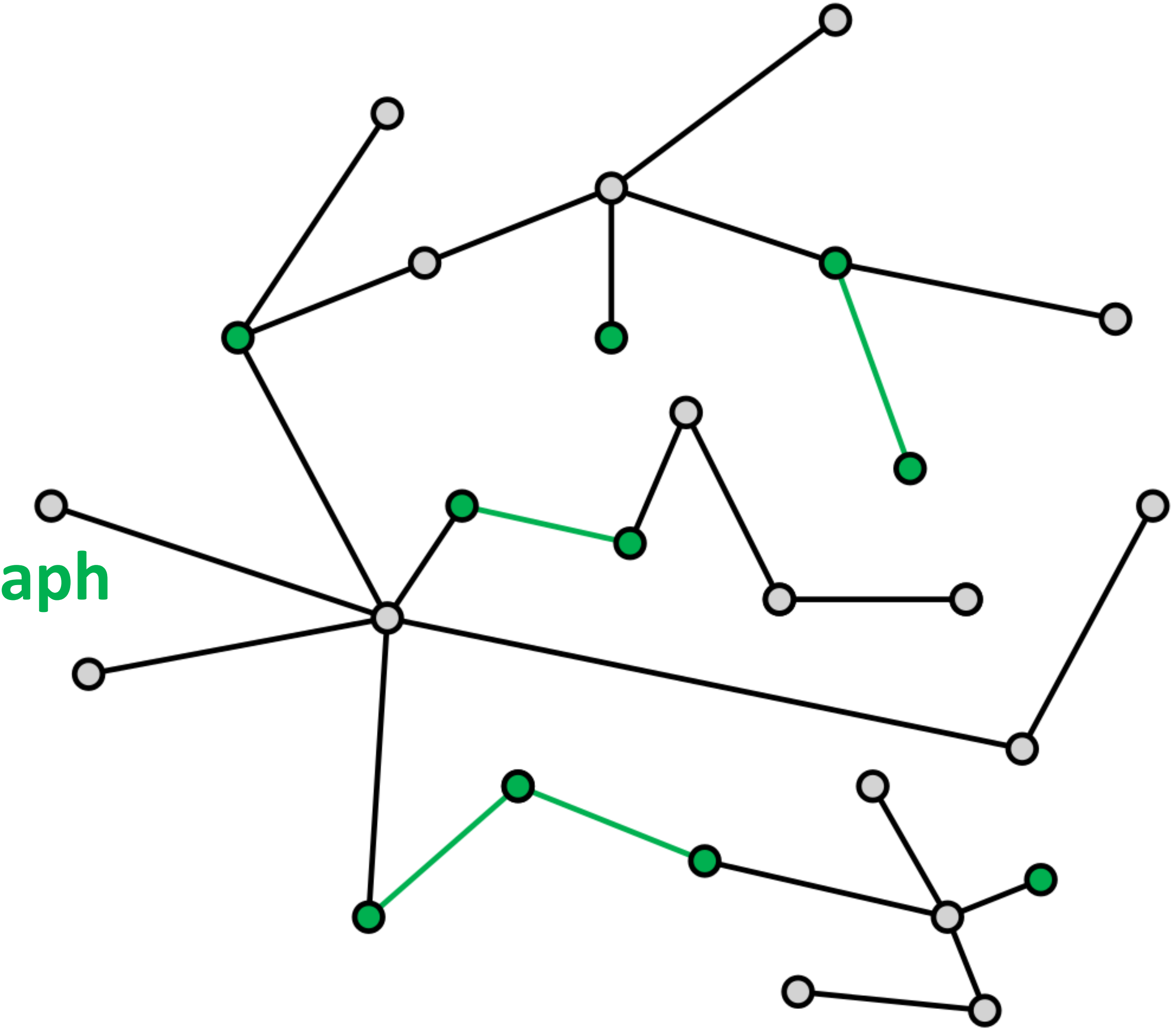

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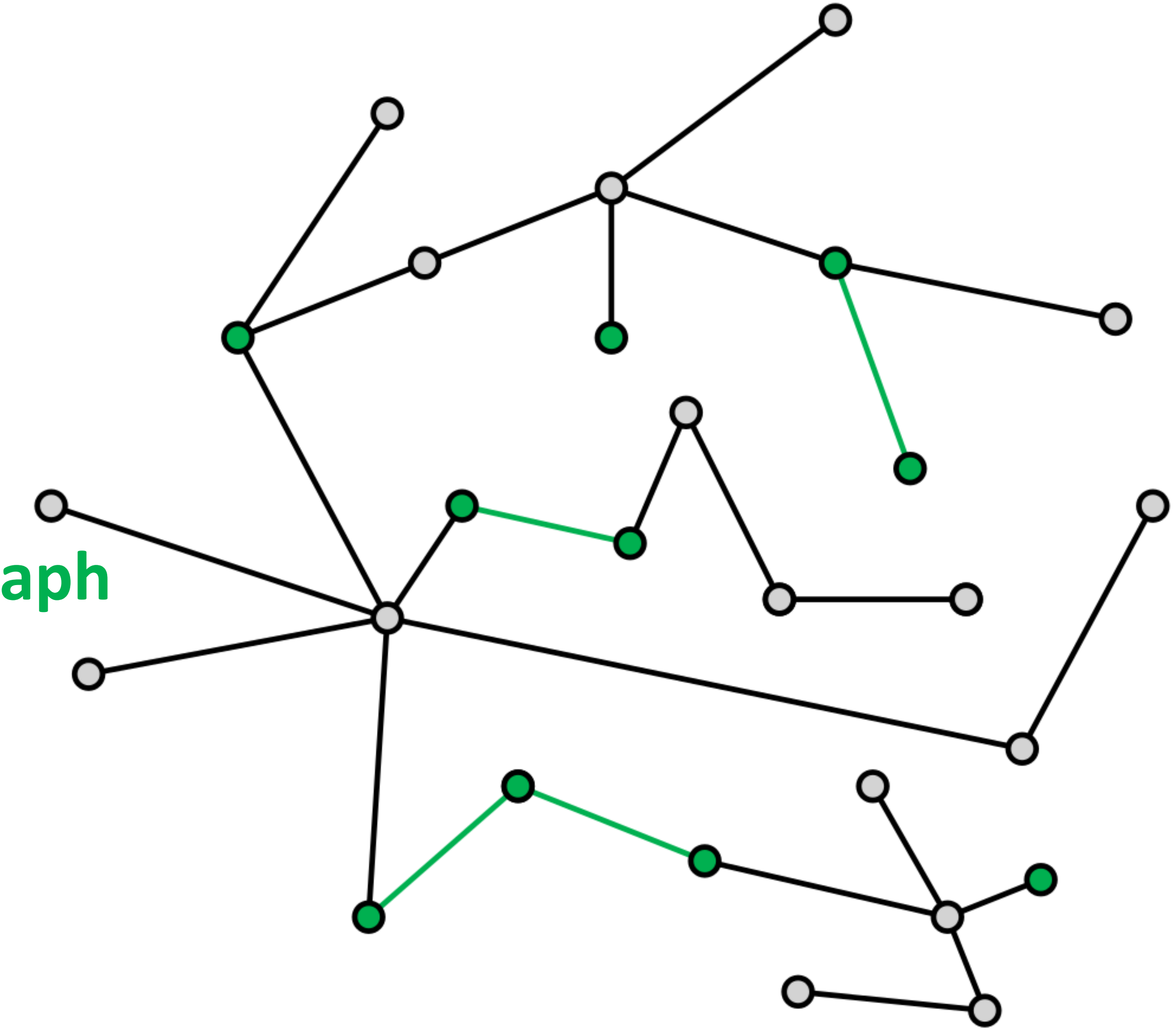

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- gather connected components

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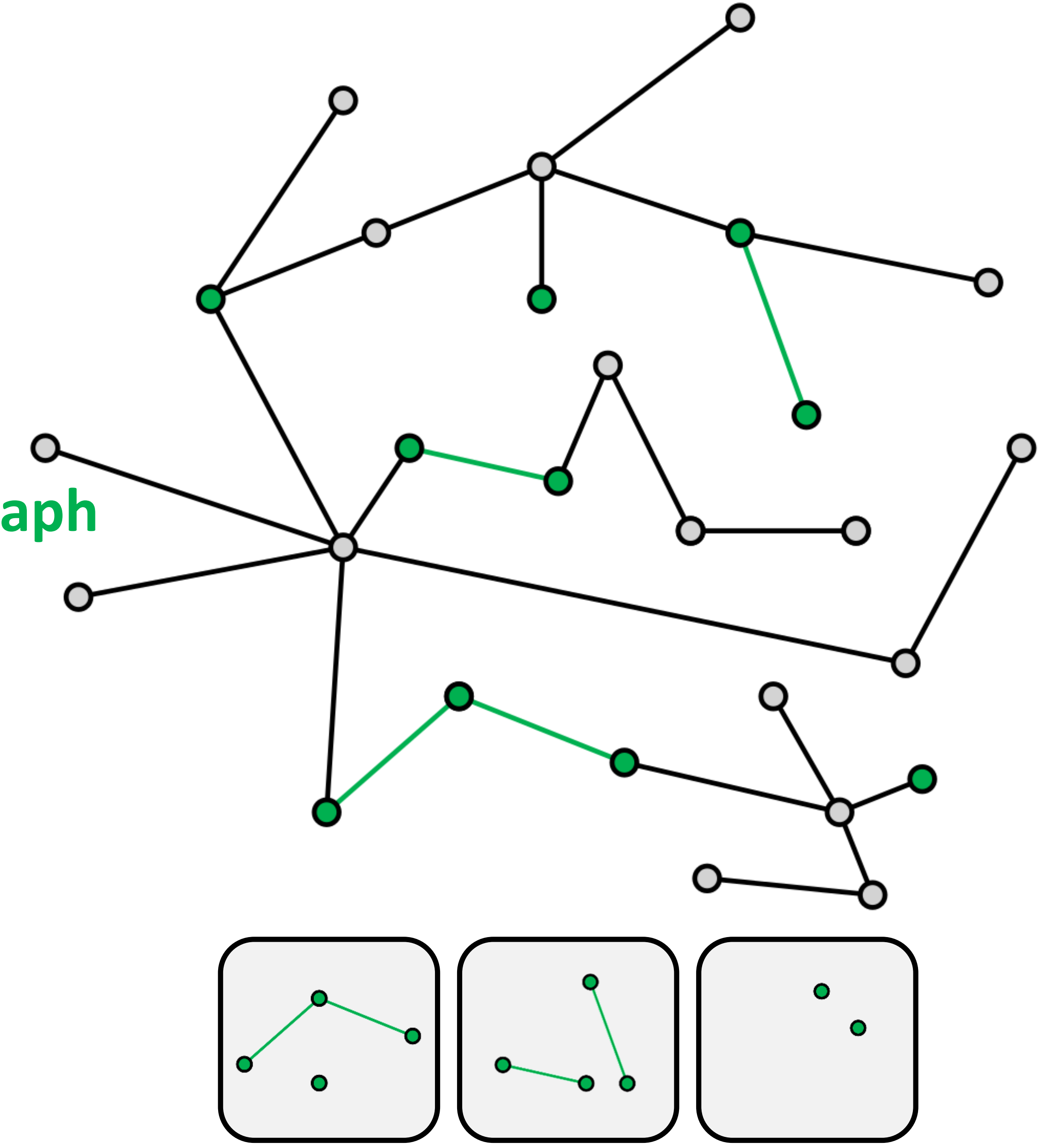


# Polynomial Degree Reduction:
# **Subsample-and-Conquer**

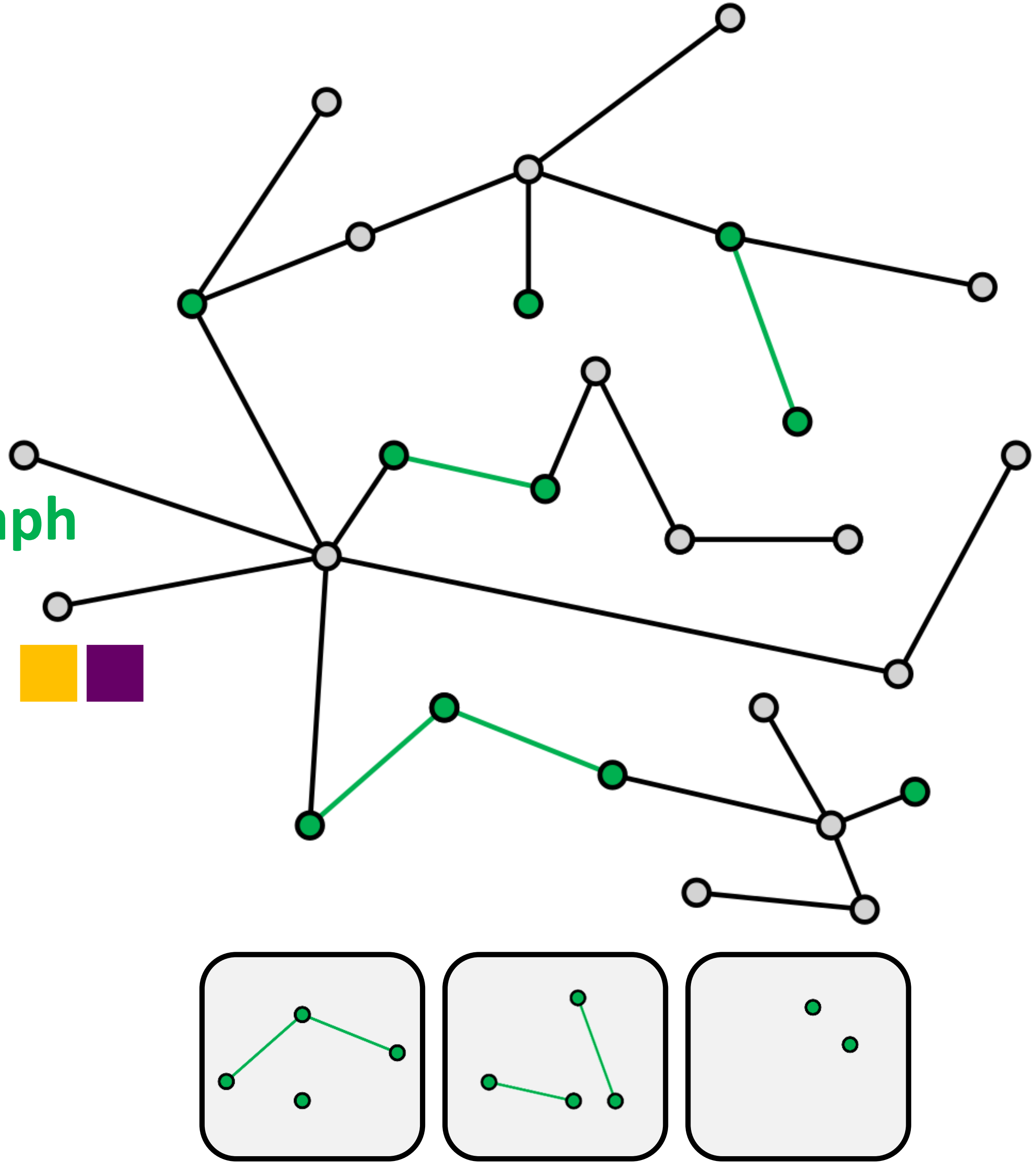
### Subsample

subsample **nodes** independently

### Conquer

compute **random MIS** in **subsampled graph**

- gather connected components
- locally compute random 2-coloring 🟨🟪



# Polynomial Degree Reduction:
# **Subsample-and-Conquer**

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subsample **nodes** independently

## **Conquer**

compute **random MIS** in **subsampled graph**

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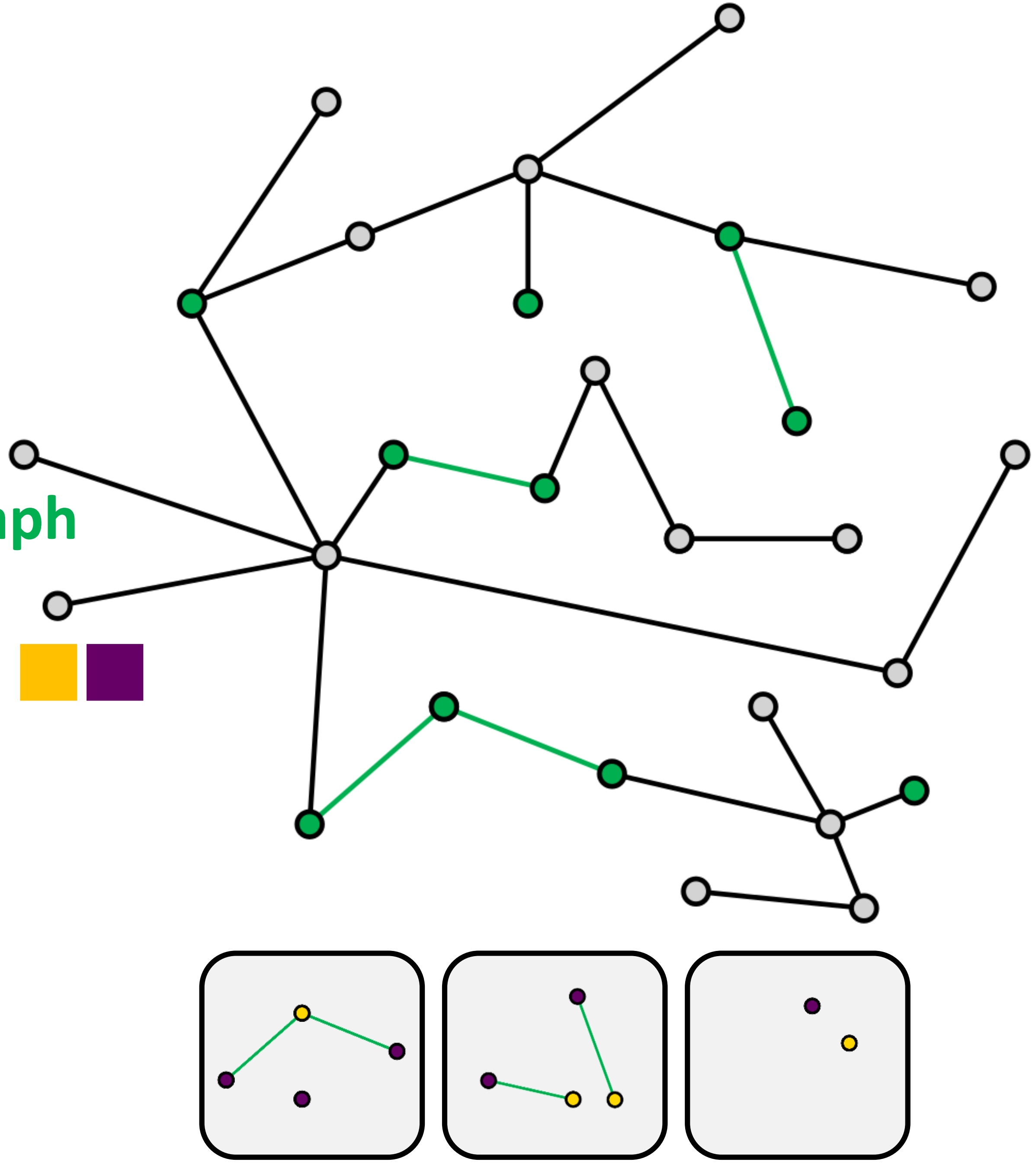

Polynomial Degree Reduction:

# Subsample-and-Conquer

### Subsample

subsample **nodes** independently

### Conquer

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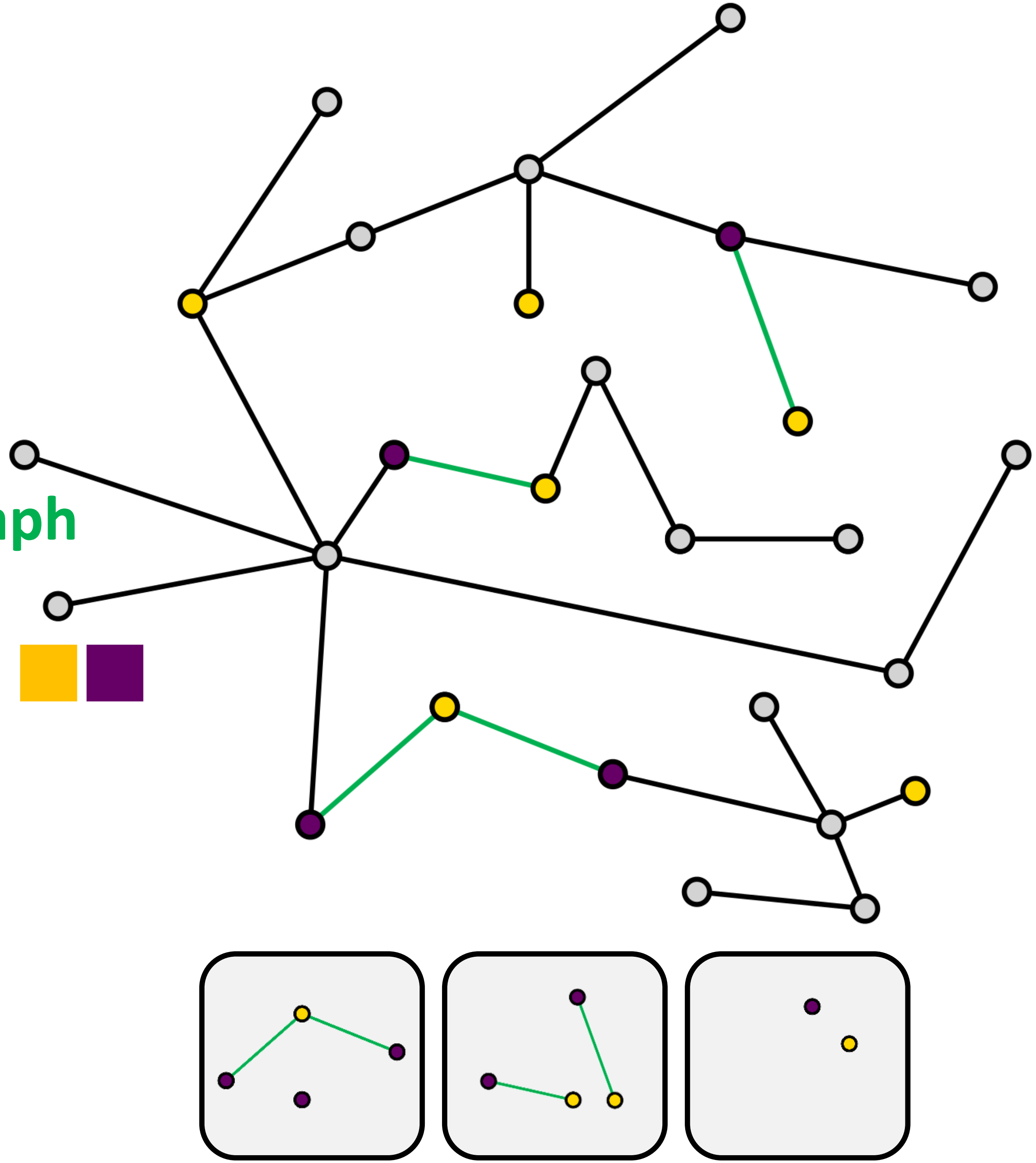

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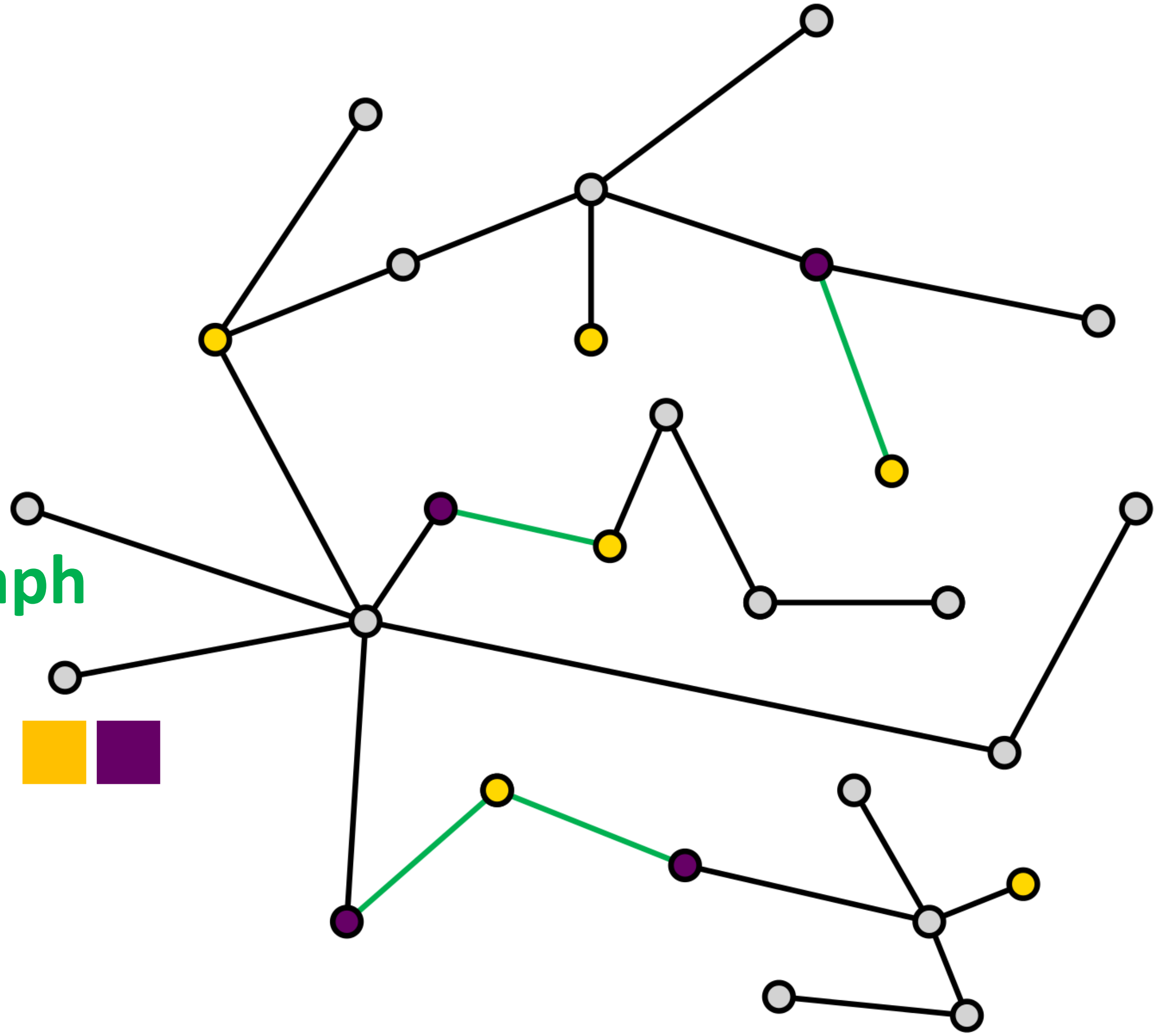

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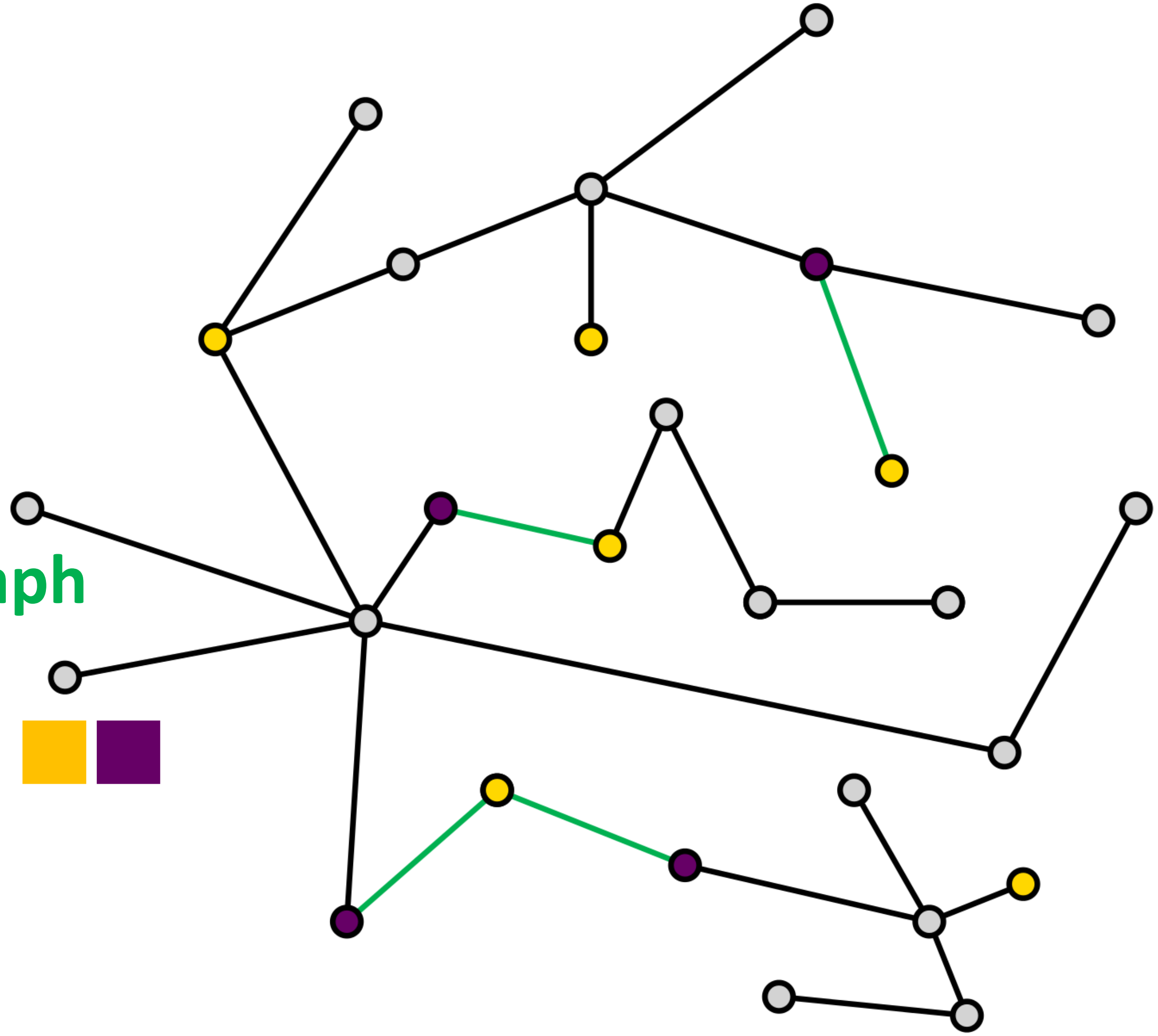

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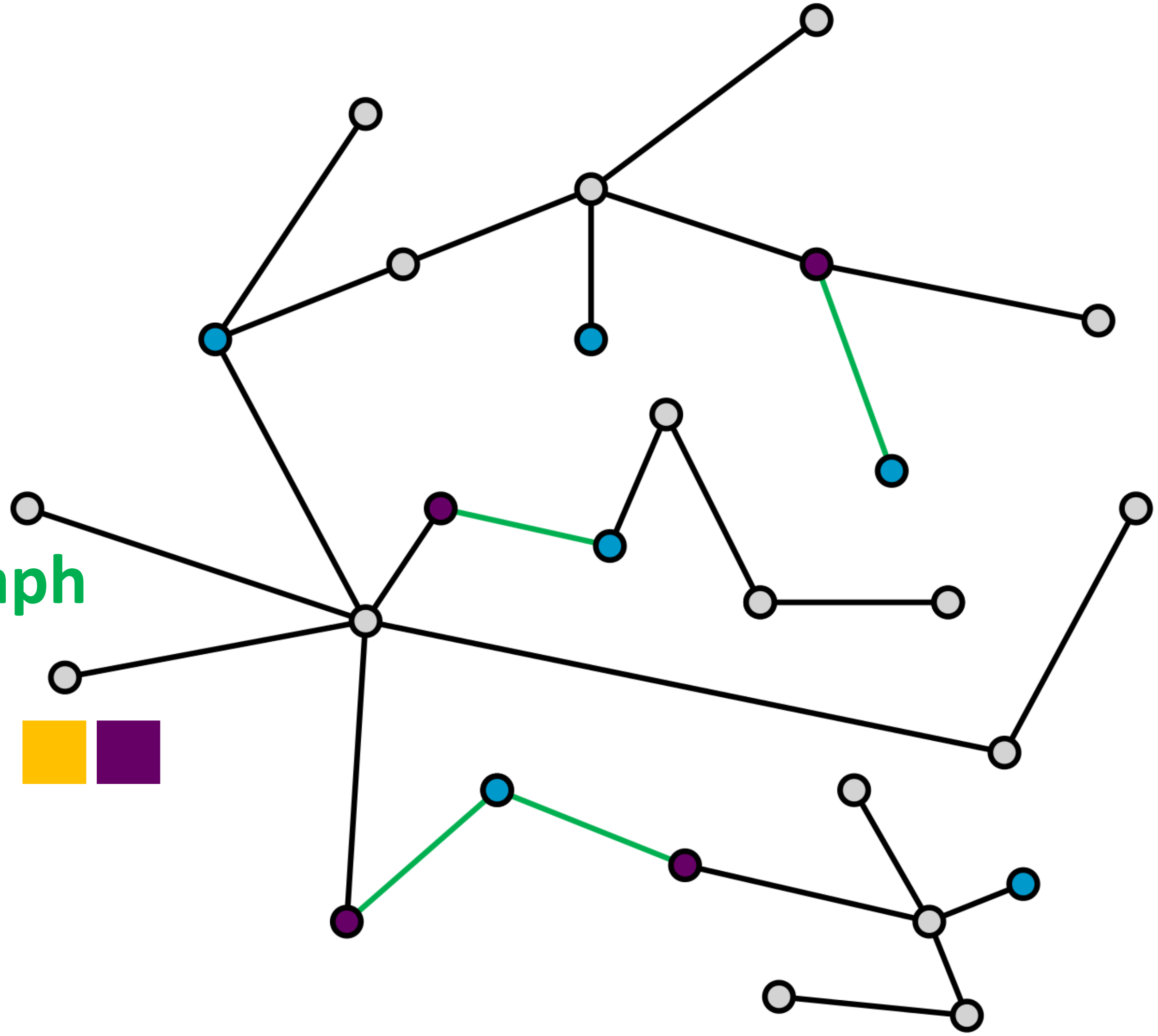

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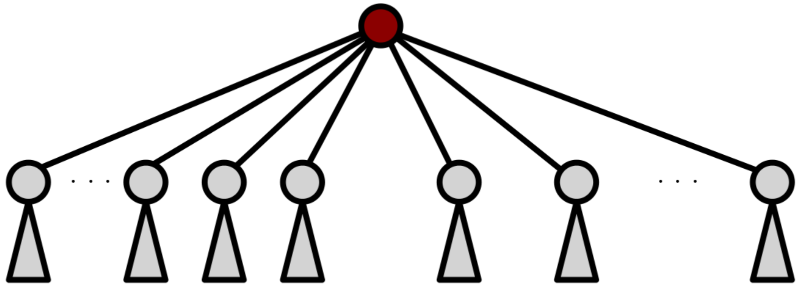

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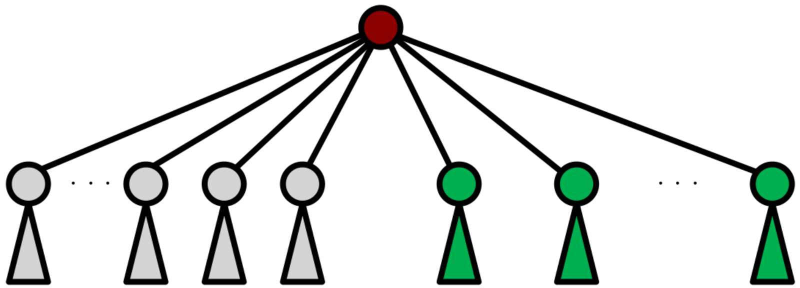

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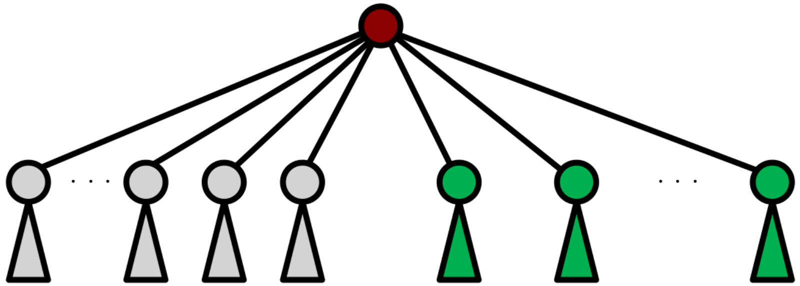

**independence due to restriction to trees!**

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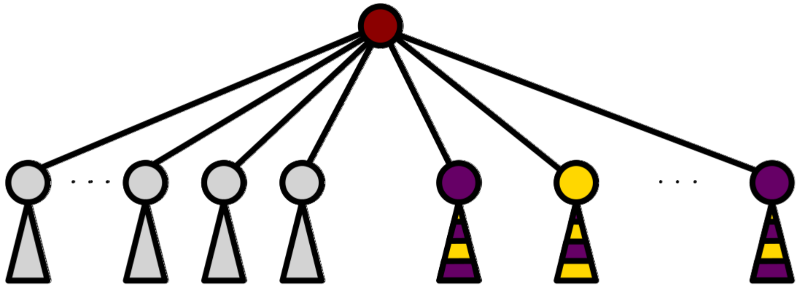

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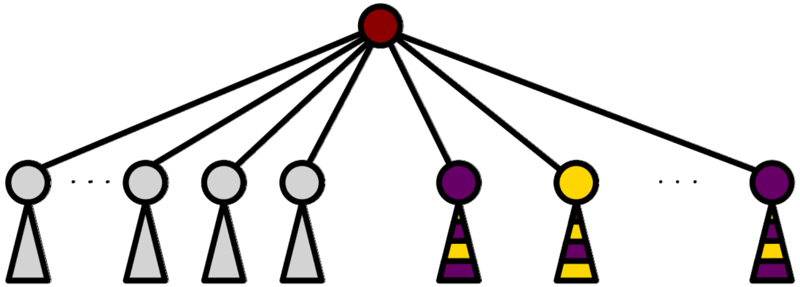

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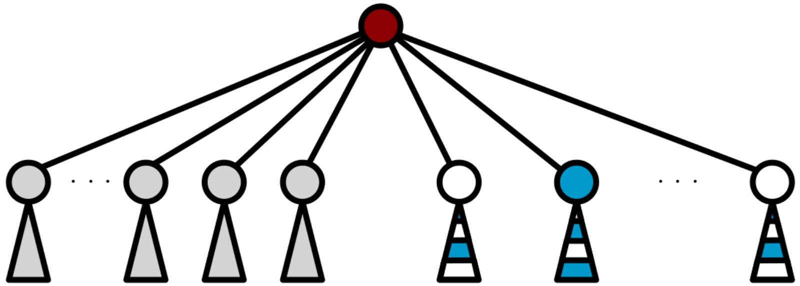

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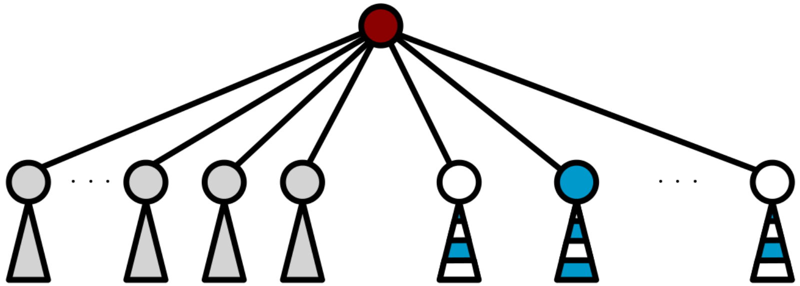
## Non-subsampled **High-Degree Node**

- w.h.p. has many **subsampled neighbors**
- thus w.h.p. has at least one **MIS neighbor**
- hence will be removed from the graph



# Algorithm Outline

## 1) Shattering

break graph into small components

i) **Degree Reduction** *Iterated Subsample-and-Conquer*

ii) **LOCAL Shattering** *Ghaffari* [SODA'16]

## 2) Post-Shattering

solve problem on remaining components

i) **Gathering of Components** *Distributed Union-Find*

ii) **Local Computation**

# Conclusion
and
Open Questions

MODEL: **Sublinear-Memory MPC**

$S = \tilde{O}(n^\delta)$ local memory

$\text{poly} \log \log n$ rounds

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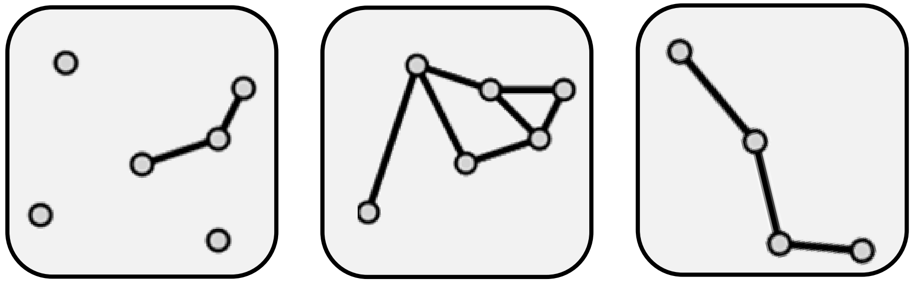

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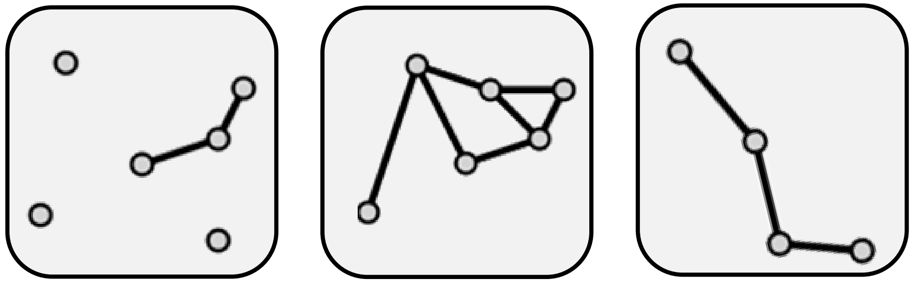

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LOCAL algorithms &
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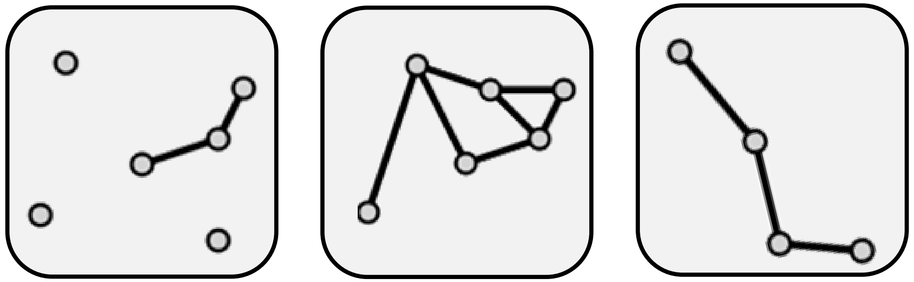

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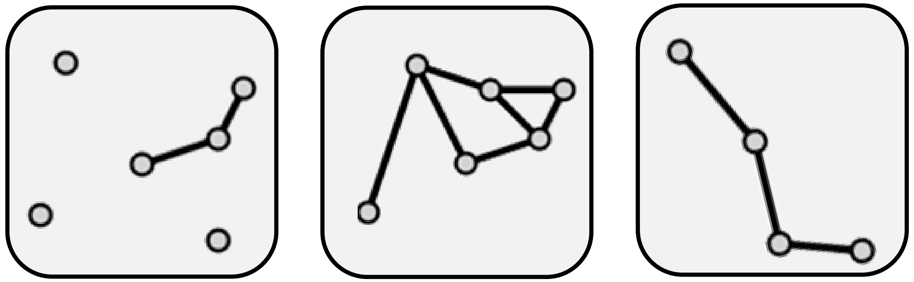

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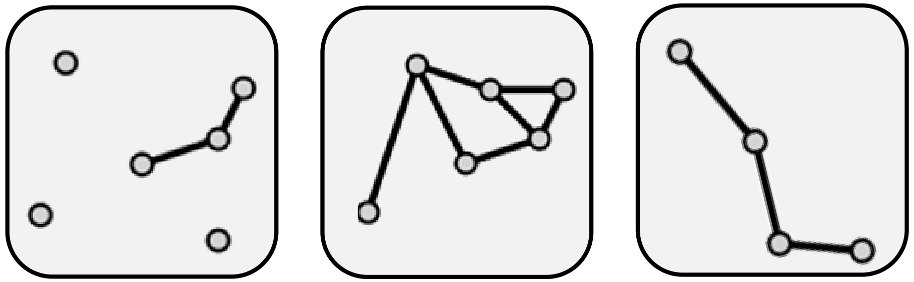

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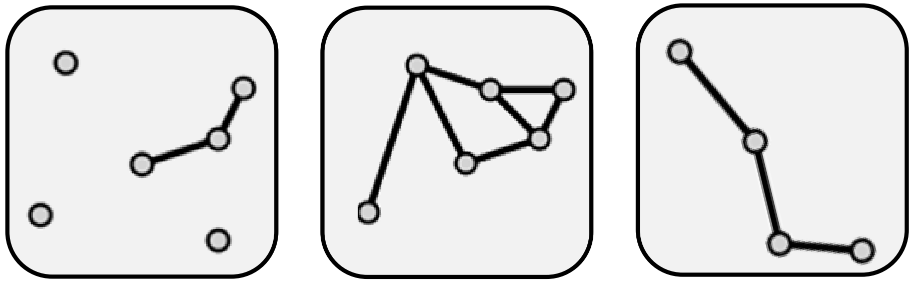

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MIS & Matching for locally sparse graphs
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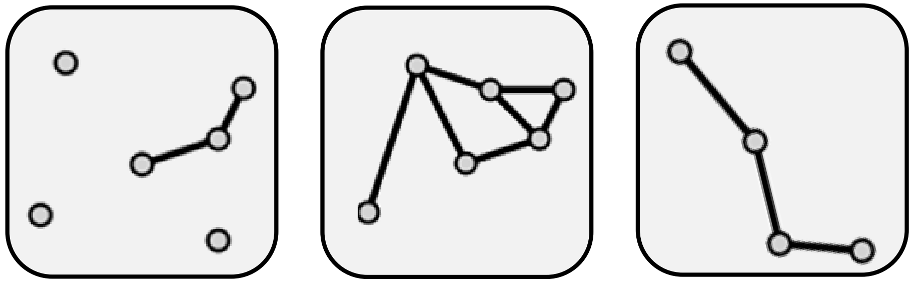

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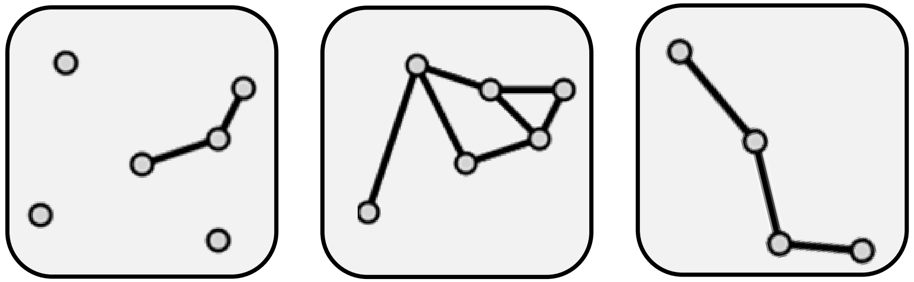

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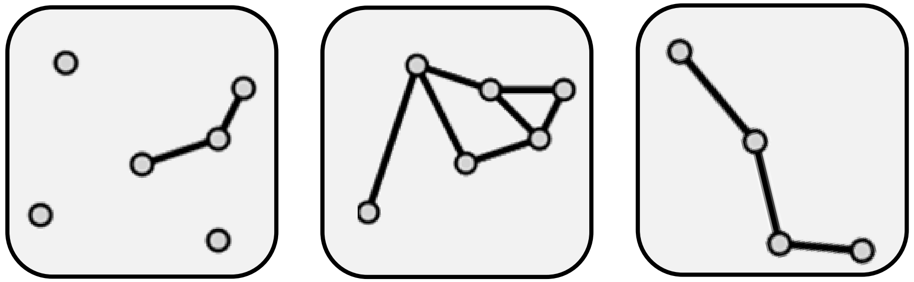
## Thank you!



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