Greedy maximal independent sets via local limits

Peleg Michaeli

Tel Aviv University

Workshop on Local Algorithms – WOLA 2019
ETH Zurich, July 21, 2019

Joint work with Michael Krivelevich, Tamás Mészáros and Clara Shikhelman
Finding maximum independent sets is very hard.
Finding maximal independent sets is very easy.
Finding **maximum** independent sets is very hard 😞
Independent sets

- Finding **maximum** independent sets is very hard 😞
- Finding **maximal** independent sets is very easy 😊
Random greedy MIS – sequential

![Network Diagram]

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Random greedy MIS – sequential

Graph representation:

- Vertices: 1, 2, 3, 4, 5, 6, 7, 8, 9
- Edges: 1-2, 1-7, 2-6, 2-7, 3-7, 3-8, 4-7, 5-7, 5-9

The vertex 1 is selected as a part of the MIS.
Random greedy MIS – sequential
Random greedy MIS – sequential

Greedy MIS

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Random greedy MIS – sequential

Diagram of a graph with nodes 1, 2, 3, 4, 5, 6, 7, 8, 9.
Random greedy MIS – sequential

![Graph showing a random greedy MIS](image)

- Nodes: 1, 2, 3, 4, 5, 6, 7, 8, 9
- Color: Red for selected nodes, Gray for non-selected nodes.

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Greedy MIS

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Random greedy MIS – sequential
Random greedy MIS – sequential

![Graph diagram]

9 - 1 - 2
4 - 7 - 6
3 - 5 - 8
Random greedy MIS – sequential

Greedy MIS

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Random greedy MIS – sequential
Random greedy MIS – parallel

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Random greedy MIS – parallel

![Graph diagram]

Nodes 1 and 2 in red, indicating they are part of the maximum independent set (MIS).
Random greedy MIS – parallel
Greedy independence ratio – previous work

Let $I(G)$ be the yielded independent set, and let $\iota(G) = \frac{|I(G)|}{|V(G)|}$.
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Flory '39, Page '59

$\iota(P_n) \rightarrow \frac{1}{2}(1 - e^{-2})$
Let $I(G)$ be the yielded independent set, and let $\nu(G) = |I(G)|/|V(G)|$.

- Flory '39, Page '59: $\nu(P_n) \to \frac{1}{2}(1 - e^{-2})$
- McDiarmid '84: $\nu(G(n, \lambda/n)) \to \ln(1 + \lambda)/\lambda$
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$$\iota(G(n, \lambda/n)) \rightarrow \ln(1 + \lambda)/\lambda$$

Wormald '95

$$\iota(G_{n,d}) \rightarrow \frac{1}{2} \left( 1 - \frac{(d - 1)^{-2}}{(d-2)} \right)$$
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- Lauer & Wormald '07
  (same for $d$-regular graphs with girth $\to \infty$)
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- Lauer & Wormald '07: (same for $d$-regular graphs with girth $\to \infty$)
- BJL '17, BJP '17: $\iota$ of random graphs with given degree sequence
Random labelling
Random labelling

![Graph Image]

- Nodes labeled with values: 0.99, 0.05, 0.06, 0.21, 0.37, 0.35, 0.09, 0.24, 0.41.
Random labelling
Random labelling

Greedy MIS
Random labelling

0.99 -- 0.05 -- 0.06

0.21 -- 0.37 -- 0.35

0.09 -- 0.24 -- 0.41
Random labelling

![Graph with nodes and edges with values]

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Random labelling
Random labelling
Random labelling
Random labelling
Random labelling
We wish to calculate the asymptotics of $\nu(G_n)$. 

We first calculate $E(\nu(G_n)) = P(\rho_n \in I(G_n))$ for $\rho_n$ chosen u.a.r.

We hope that this is determined by a small neighbourhood of $\rho_n$. This local view of $\rho_n$ is captured by the local limit of $G_n$.

$\Rightarrow \nu(G_n) \sim E(\nu(G_n))$ a.a.s.

Develop a machinery to calculate the probability that the root is red.
General framework

- We wish to calculate the asymptotics of $\iota(G_n)$.
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1. We first calculate \( \mathbb{E}(\nu(G_n)) = \mathbb{P}(\rho_n \in I(G_n)) \) for \( \rho_n \) chosen u.a.r.
2. We hope that this is determined by a small neighbourhood of \( \rho_n \).
3. This local view of \( \rho_n \) is captured by the *local limit* of \( G_n \).
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We hope that this is determined by a small neighbourhood of $\rho_n$.

This local view of $\rho_n$ is captured by the local limit of $G_n$.

Decay of correlation $\implies \iota(G_n) \sim E(\iota(G_n))$ a.a.s.
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Develop a machinery to calculate the probability that the root is red.
Local limits

We say that a (random) graph sequence $G_n$ locally converges to a random rooted graph $(U, \rho)$, if for every $r \geq 0$, the ball $B_r(G, \rho_n)$ converges in distribution to $B_r(U, \rho)$, where $\rho_n$ is a uniform vertex of $G_n$. 
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[Diagram of different types of graphs converging to a circular arrangement]
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Examples

- $P_n, C_n \xrightarrow{\text{loc}} \mathbb{Z}$
- $[n]^d \xrightarrow{\text{loc}} \mathbb{Z}^d$
- $G(n, \lambda/n) \xrightarrow{\text{loc}} T_\lambda$, a Galton-Watson Pois($\lambda$) tree
- $G_{n,d} \xrightarrow{\text{loc}}$ the $d$-regular tree
- Uniform random tree $T_n \xrightarrow{\text{loc}} \hat{T}_1$, a size-biased GW Pois(1) tree
- Finite $d$-ary balanced tree $\xrightarrow{\text{loc}}$ the canopy tree
Theorem (Krivelevich, Mészáros, M., Shikhelman ’19+)

Suppose $G_n$ has subfactorial growth.

If $G_n \xrightarrow{\text{loc}} (U, \rho)$ then $\iota(G_n) \rightarrow \iota(U, \rho)$ a.a.s.
Decay of correlation
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Locally tree-like

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graph sequences for which \((U, \rho)\) is a.s. a tree.

\[
\begin{align*}
\rho & \quad \downarrow \\
\cdots & \quad \downarrow \\
u_1 & \quad \downarrow \\
\quad & \quad \downarrow \\
u_2 & \quad \downarrow \\
\quad & \quad \downarrow \\
u_d & \quad \downarrow \\
\end{align*}
\]

Children of the past are roots to independent subtrees.
We need to calculate $\nu(U, \rho)$, but even $\nu(\mathbb{Z}^2)$ is still unknown...
Let us therefore restrict ourselves to \textit{locally tree-like} graph sequences, i.e., graph sequences for which $(U, \rho)$ is a.s. a tree.

Children of the \textit{past} are roots to independent subtrees.
Let \((U, \rho)\) be a single-type branching process.
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\[ y(x) = \mathbb{P}(\rho \in I(U, \rho) \land \sigma_\rho < x) \]
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y(x) = \mathbb{P}(\rho \in I(U, \rho) \land \sigma_\rho < x) \\
= x \cdot \mathbb{P}(\rho \in I(U, \rho)|\sigma_\rho < x) \\
= \int_0^x \mathbb{P}(\rho \in I(U, \rho)|\sigma_\rho = z) \, dz
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y'(x) = \mathbb{P}(\rho \in I(U, \rho) | \sigma_\rho = x)
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y'(x) = \mathbb{P}(\rho \in I(U, \rho) | \sigma_\rho = x)
\]

Thus, if \(y\) is a unique solution of

\[
y'(x) = \sum_{\ell \in \mathbb{N}} \mathbb{P}(\xi < x = \ell) \left(1 - \frac{y(x)}{x}\right)^\ell,
\quad y(0) = 0,
\]

then, \(\iota(U, \rho) = y(1)\).
Let \((U, \rho)\) be a multi-type branching process.

\[
y(x) = \mathbb{P}(\rho \in I(U, \rho) \land \sigma_\rho < x) \\
= x \cdot \mathbb{P}(\rho \in I(U, \rho) | \sigma_\rho < x) \\
= \int_0^x \mathbb{P}(\rho \in I(U, \rho) | \sigma_\rho = z) dz
\]

\[
y'(x) = \mathbb{P}(\rho \in I(U, \rho) | \sigma_\rho = x)
\]

Thus, if \(y\) is a unique solution of

\[
y'_k(x) = \sum_{\ell \in \mathbb{N}^T} \prod_{j \in T} \mathbb{P}(\xi_{k \rightarrow j}^< x = \ell_j) \left(1 - \frac{y_j(x)}{x}\right)^{\ell_j}, \quad y_k(0) = 0,
\]

then, \(\iota(U, \rho) = \mathbb{E}(y_k(1))\).
Kolchin, Grimmett: the sequence of uniform random trees locally converges to the *size-biased Galton-Watson* Pois(1) tree.
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Uniform random trees

\[ y_t'(x) = \sum_{d=0}^{\infty} \frac{(\lambda x)^d}{e^\lambda x d!} \left(1 - \frac{y_t(x)}{x}\right)^d = e^{-\lambda y_t(x)}. \]

hence \( y_t(x) = \ln(1 + \lambda x)/\lambda \). Thus

\[ \nu(G(n, \lambda/n)) \to \nu(T_\lambda) = y_t(1) = \frac{\ln(1 + \lambda)}{\lambda}. \]
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\[ y_s'(x) = (1 - y_s(x)) y_t'(x) = (1 - y_s(x)) e^{-\lambda y_t(x)} = \frac{1 - y_s(x)}{1 + \lambda x}, \]

hence \( y_s(x) = 1 - (1 + \lambda x)^{-1/\lambda} \), and for \( \lambda = 1 \), \( y_s(1) = 1 - (1 + x)^{-1} \), and we get

\[ \iota(T_n) \to \iota(\hat{T}_1) = y_s(1) = \frac{1}{2}. \]
Simulations don’t lie

Greedy MIS
Simulations don’t lie

red: 125 (50%), green: 92 (≈ 37%), blue: 32 (≈ 13%), black: 1
Simulations don’t lie (but I do)

red: 125 (50%), green: 92 (≈ 37%), blue: 32 (≈ 13%), black: 1
Greedy independence ratio – results

Flory ’39, Page ’59
\[ \nu(P_n) \to \frac{1}{2} (1 - e^{-2}) \]

McDiarmid ’84
\[ \nu(G(n, \lambda/n)) \to \ln(1 + \lambda)/\lambda \]

Wormald ’95
\[ \nu(G_{n,d}) \to \frac{1}{2} (1 - (d - 1)^{-2}/(d-2)) \]

Lauer & Wormald ’07
\( (d\text{-regular graphs with girth } \to \infty) \)

(Please note: The paper refers to Flory’s work in 1939 and Page’s work in 1959, which is a common stylization to denote pages. However, Page ’59 typically refers to the page number, not a year. This could be a typographical error.)
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McDiarmid '84 \[ \nu(G(n, \lambda / n)) \rightarrow \ln(1 + \lambda) / \lambda \] ✓

Wormald '95 \[ \nu(G_{n,d}) \rightarrow \frac{1}{2} \left(1 - (d - 1)^{-2} / (d - 2)\right) \] ✓

Lauer & Wormald '07 (\(d\)-regular graphs with girth \(\rightarrow \infty\))
Greedy independence ratio – results

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✓

Lauer & Wormald ’07

(d-regular graphs with girth \( \rightarrow \infty \))

✓

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Greedy independence ratio – results

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\[ \iota(G(n, \lambda/n)) \to \ln(1 + \lambda)/\lambda \]
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✓

Lauer & Wormald ’07
\((d\text{-regular graphs with girth } \to \infty)\)
✓

KMMS ’19+
\[ \iota(T_n) \to \frac{1}{2} \]
★

(same for functional digraphs)
Greedy independence ratio – results

Flory '39, Page '59
\[ \nu(P_n) \to \frac{1}{2}(1 - e^{-2}) \] ✓

McDiarmid '84
\[ \nu(G(n, \lambda/n)) \to \ln(1 + \lambda)/\lambda \] ✓

Wormald '95
\[ \nu(G_{n,d}) \to \frac{1}{2}\left(1 - (d - 1)^{-2}/(d-2)\right) \] ✓

Lauer & Wormald '07
\(d\)-regular graphs with girth \(\to \infty\) ✓

KMMS '19+
\[ \nu(T_n) \to \frac{1}{2} \]
(same for functional digraphs) ⭐️
Bonus: paths are the worst trees

\[ \pi(P_n) \to 1 \leq 1 - e^{-2} \approx 0.43233 \ldots \]

\[ \pi(S_n) \to 1 \leq \text{Theorem (Krivelevich, Mészáros, M., Shikhelman '19+)} \]

If \( T \) is a tree on \( n \) vertices, then \( \pi(P_n) \leq \pi(T) \).
Bonus: paths are the worst trees

\[ \iota(P_n) \rightarrow \frac{1}{2}(1 - e^{-2}) \approx 0.43233 \ldots \]
Bonus: paths are the worst trees

- $\nu(P_n) \rightarrow \frac{1}{2}(1 - e^{-2}) \approx 0.43233\ldots$
- $\nu(S_n) \rightarrow 1$
Bonus: paths are the worst trees

- $\iota(P_n) \rightarrow \frac{1}{2}(1 - e^{-2}) \approx 0.43233\ldots$
- $\iota(S_n) \rightarrow 1$
Bonus: paths are the worst trees

- $\iota(P_n) \to \frac{1}{2}(1 - e^{-2}) \approx 0.43233\ldots$
- $\iota(S_n) \to 1$
**Theorem (Krivelevich, Mészáros, M., Shikhelman '19+)**

*If $T$ is a tree on $n$ vertices, then $\nu(P_n) \leq \nu(T)$.***
Thank You!