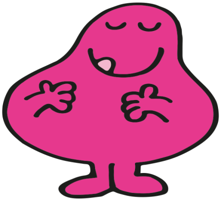
# Greedy maximal independent sets via local limits

Peleg Michaeli

Tel Aviv University

Workshop on Local Algorithms – WOLA 2019
ETH Zurich, July 21, 2019



Joint work with Michael Krivelevich, Tamás Mészáros and Clara Shikhelman

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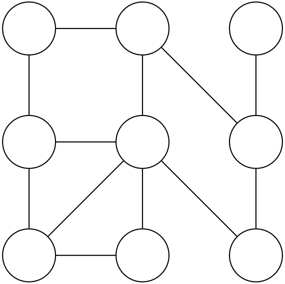

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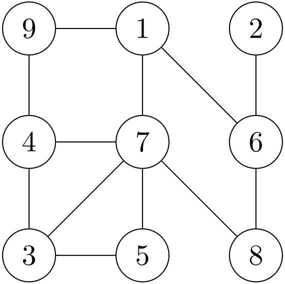

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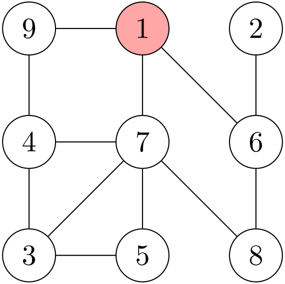

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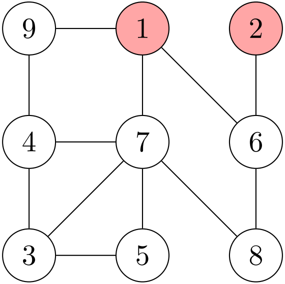

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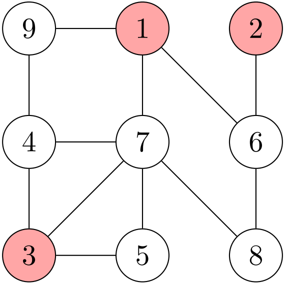

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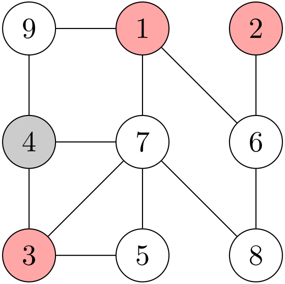

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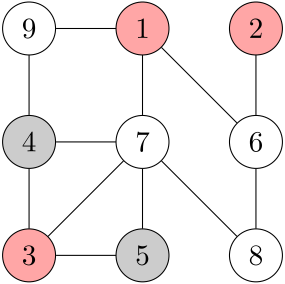

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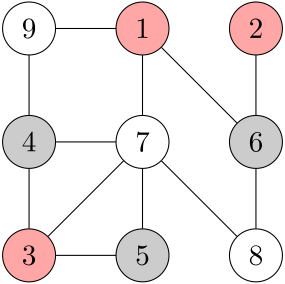

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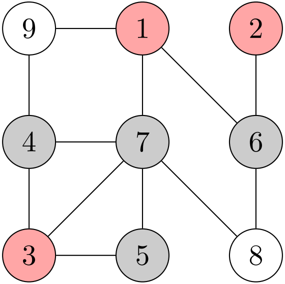

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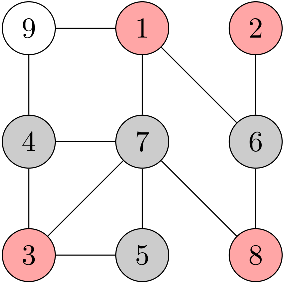

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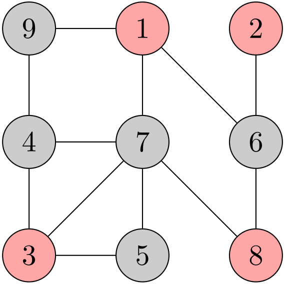

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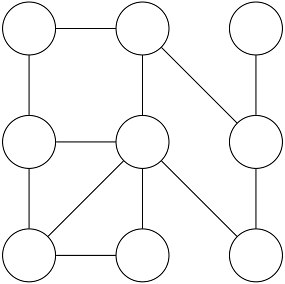

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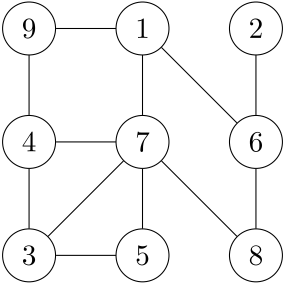

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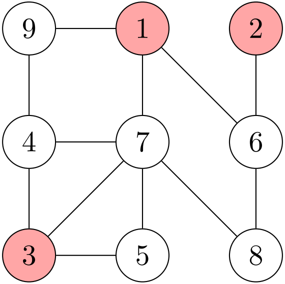

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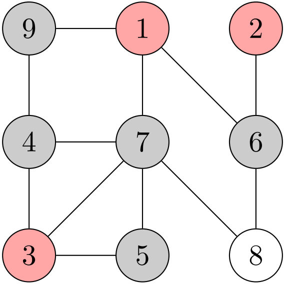

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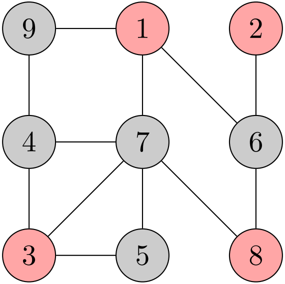

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| BJL '17, BJP '17 | $\iota$ of random graphs with given degree sequence |

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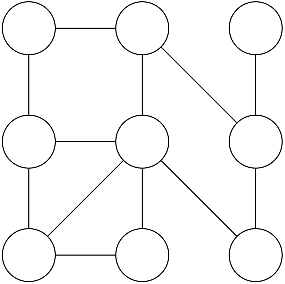

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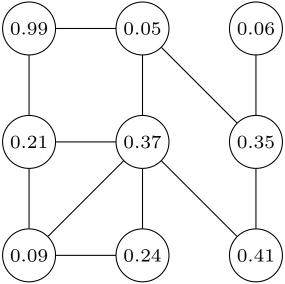

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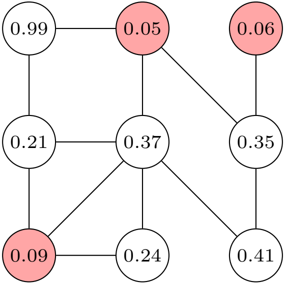

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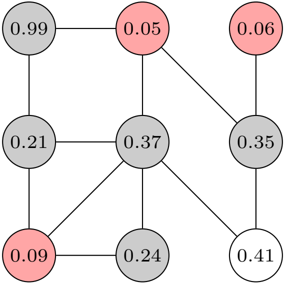

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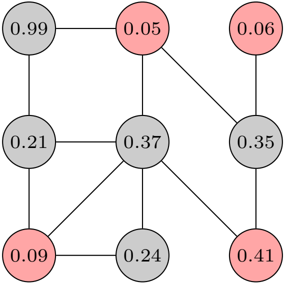

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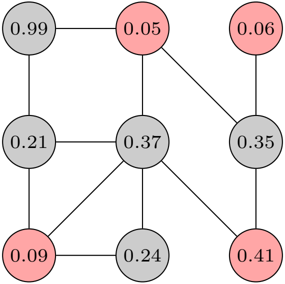


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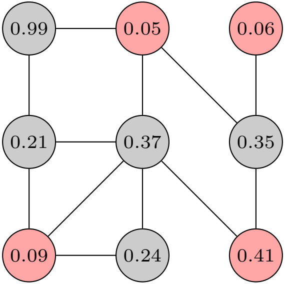


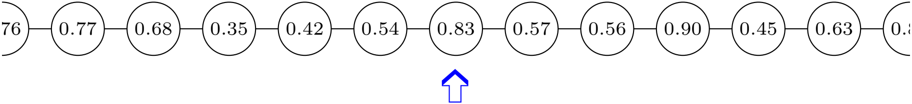

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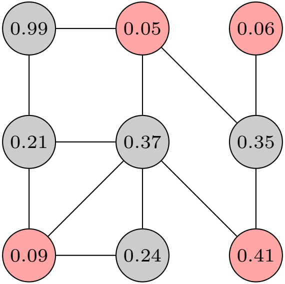


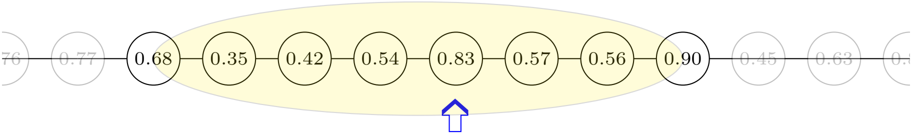

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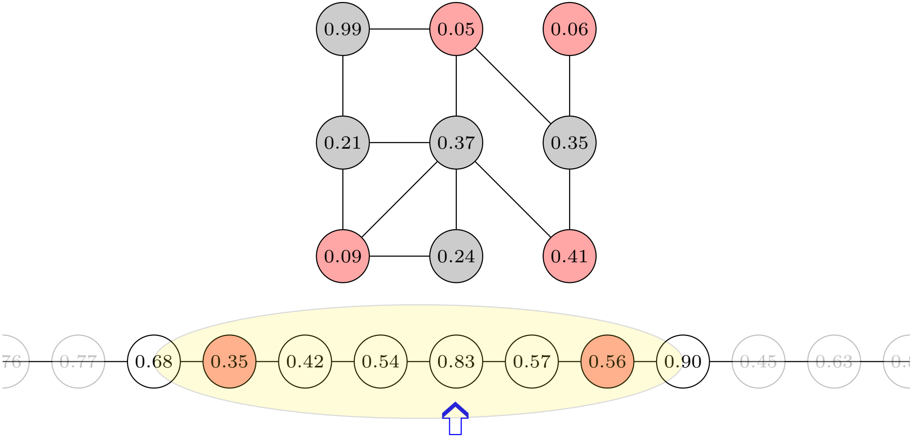

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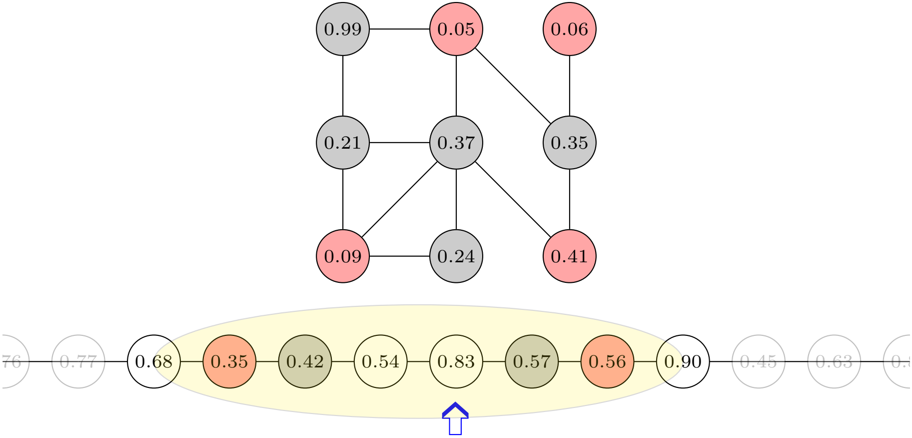

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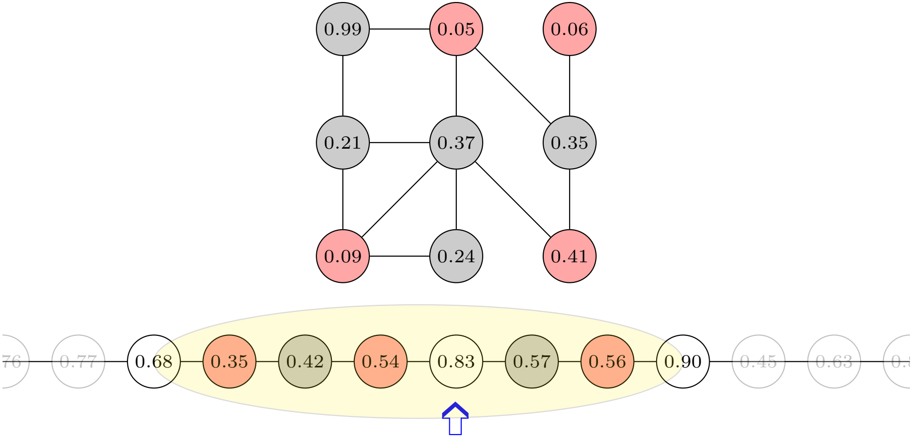

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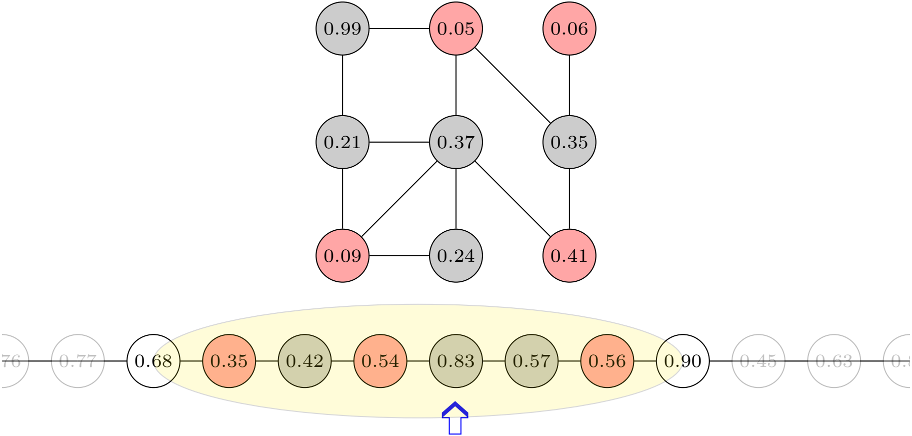

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- Develop a machinery to calculate the probability that the root is red.

# Local limits

We say that a (random) graph sequence $G_n$ locally converges to a random rooted graph $(U, \rho)$, if for every $r \geq 0$, the ball $B_r(G, \rho_n)$ converges in distribution to $B_r(U, \rho)$, where $\rho_n$ is a uniform vertex of $G_n$.

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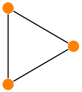

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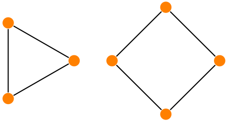

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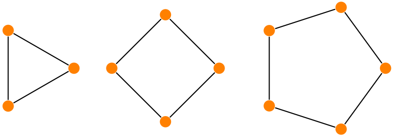

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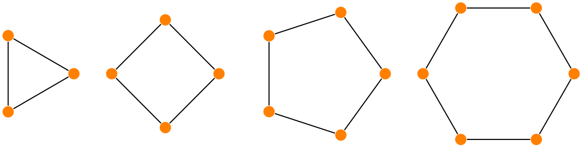

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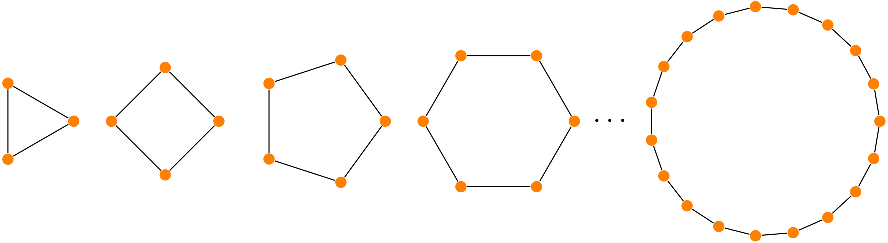

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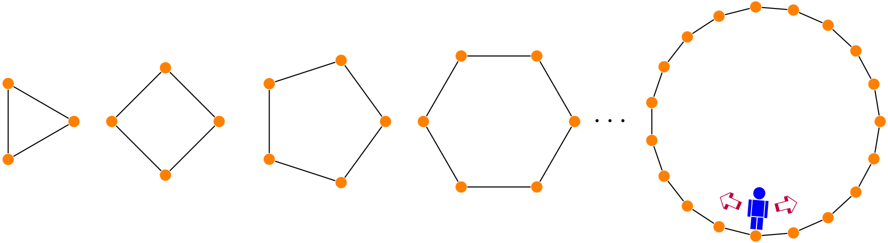

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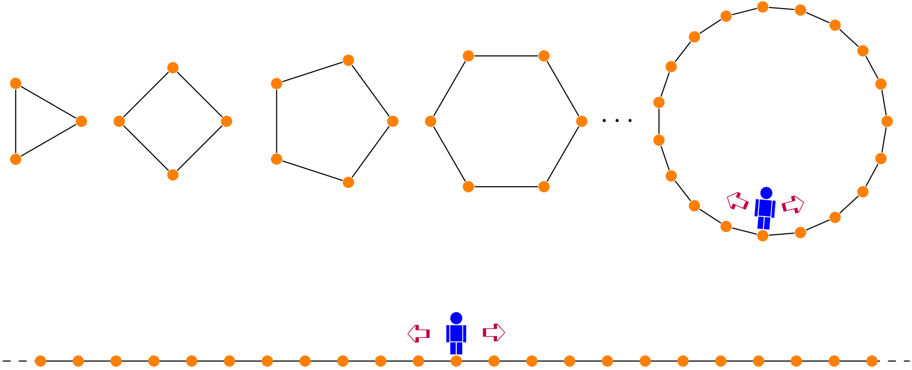

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## Examples

- $P_n, C_n \xrightarrow{\text{loc}} \mathbb{Z}$
- $[n]^d \xrightarrow{\text{loc}} \mathbb{Z}^d$
- $G(n, \lambda/n) \xrightarrow{\text{loc}} \mathcal{T}_\lambda$, a Galton-Watson Pois$(\lambda)$ tree
- $G_{n,d} \xrightarrow{\text{loc}}$ the $d$-regular tree
- Uniform random tree $T_n \xrightarrow{\text{loc}} \hat{\mathcal{T}}_1$, a size-biased GW Pois$(1)$ tree
- Finite $d$-ary balanced tree $\xrightarrow{\text{loc}}$ the canopy tree

# Convergence of the greedy independence ratio

**Theorem (Krivelevich, Mészáros, M., Shikhelman '19+)**

*Suppose* $G_n$ *has subfactorial growth.*
*If* $G_n \xrightarrow{\text{loc}} (U, \rho)$ *then* $\iota(G_n) \to \iota(U, \rho)$ *a.a.s.*

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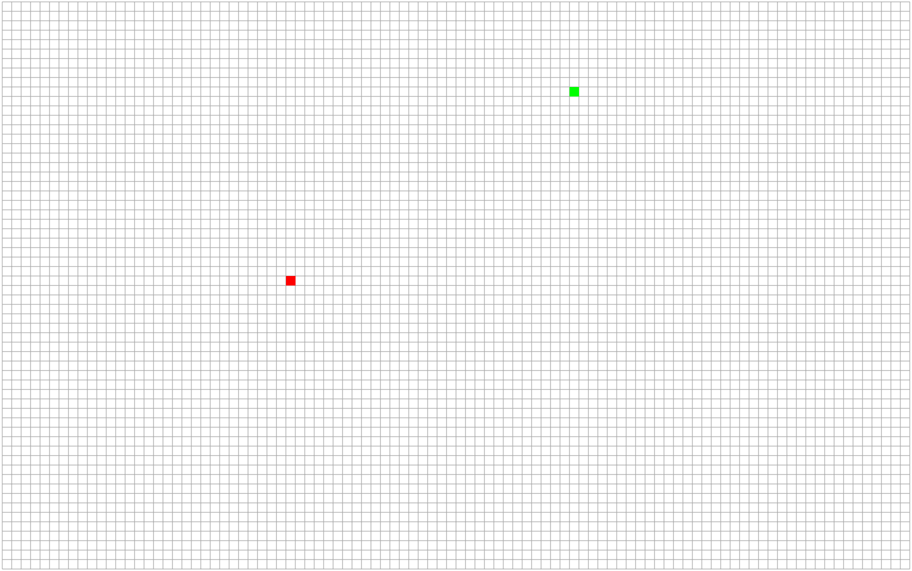

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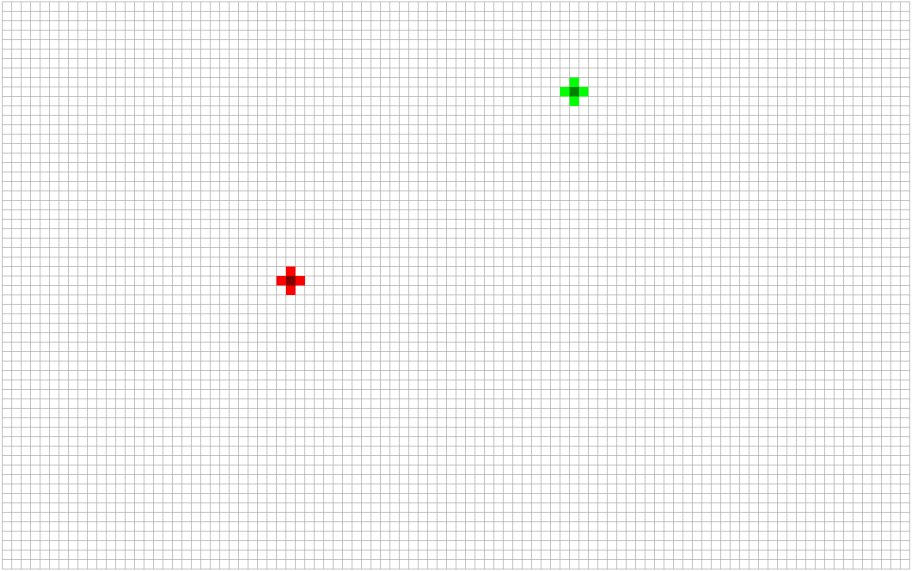

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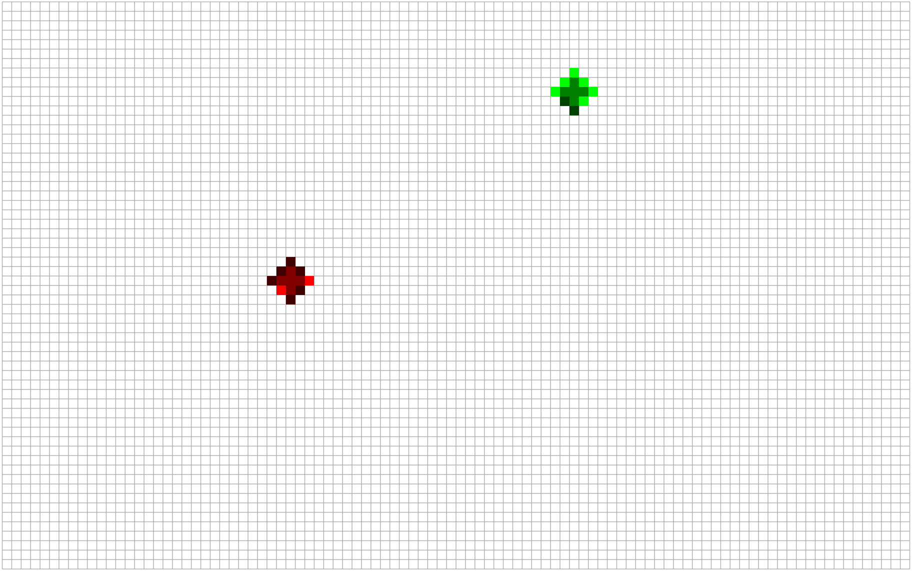

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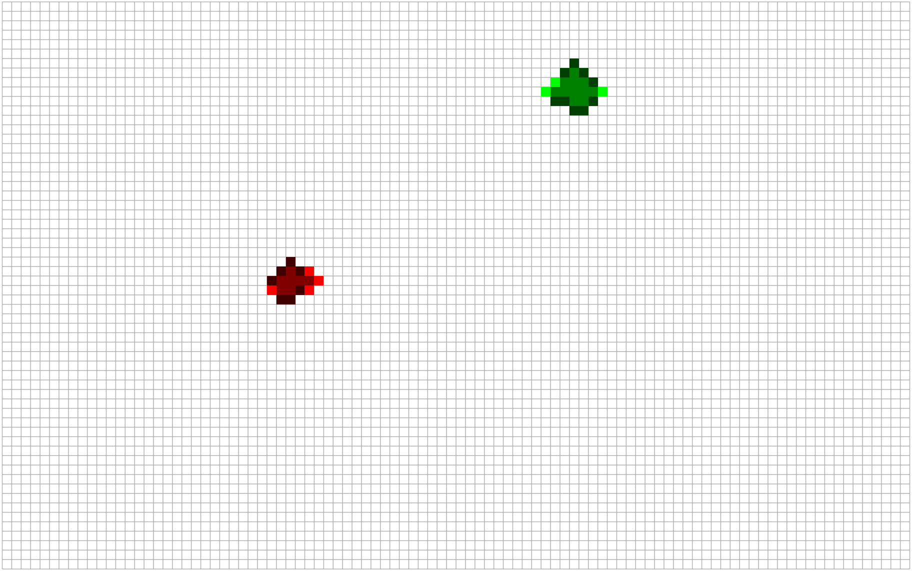

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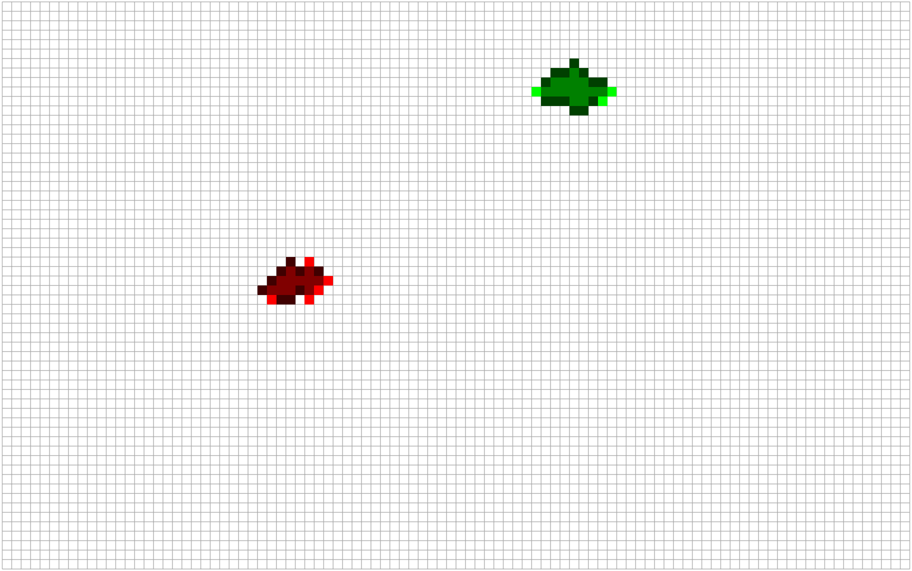

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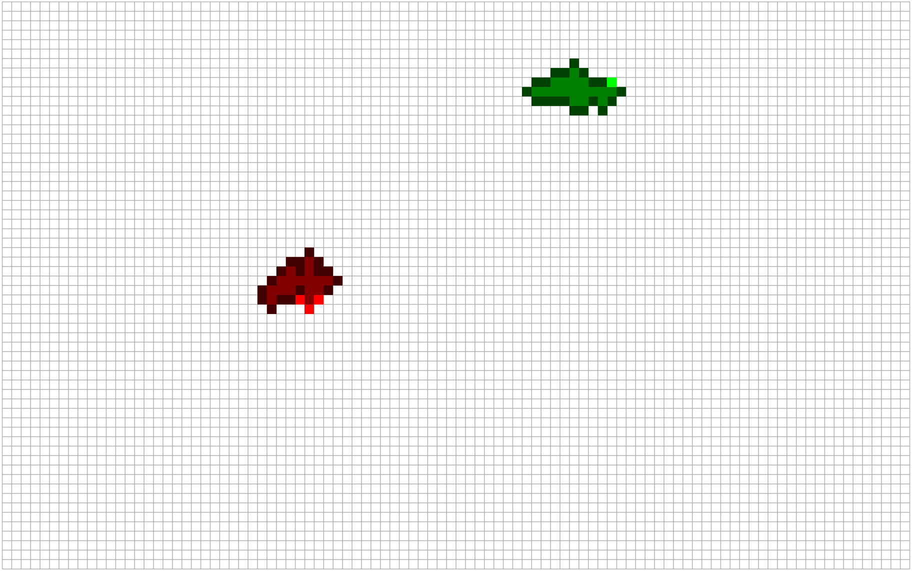

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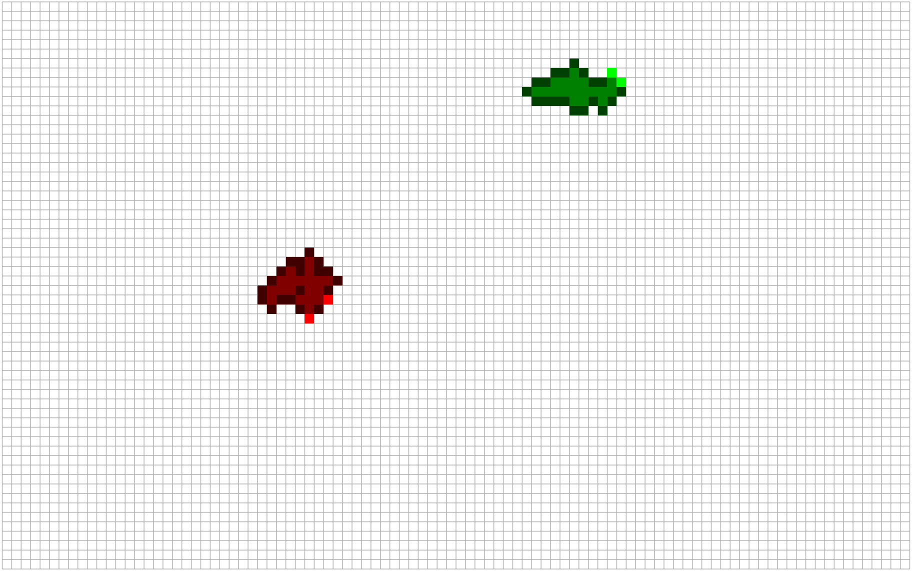

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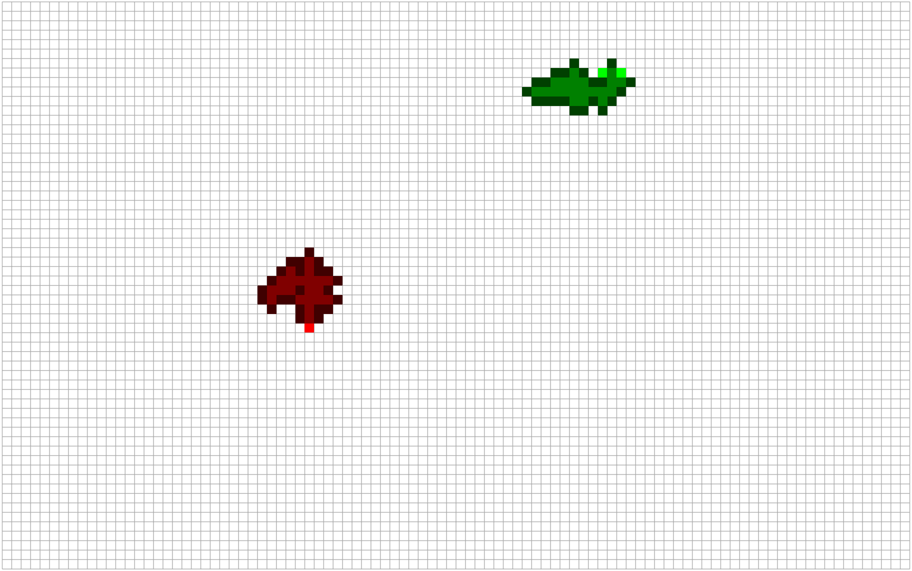

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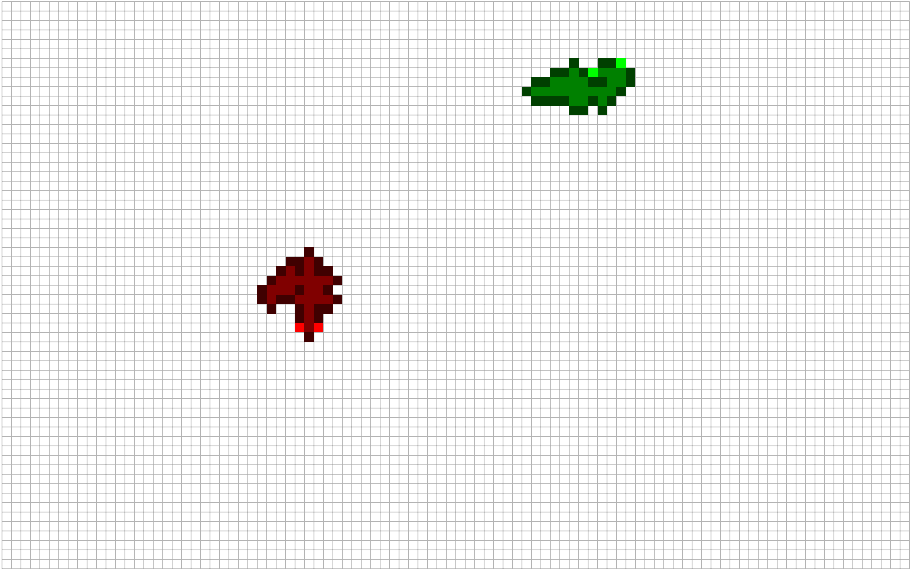

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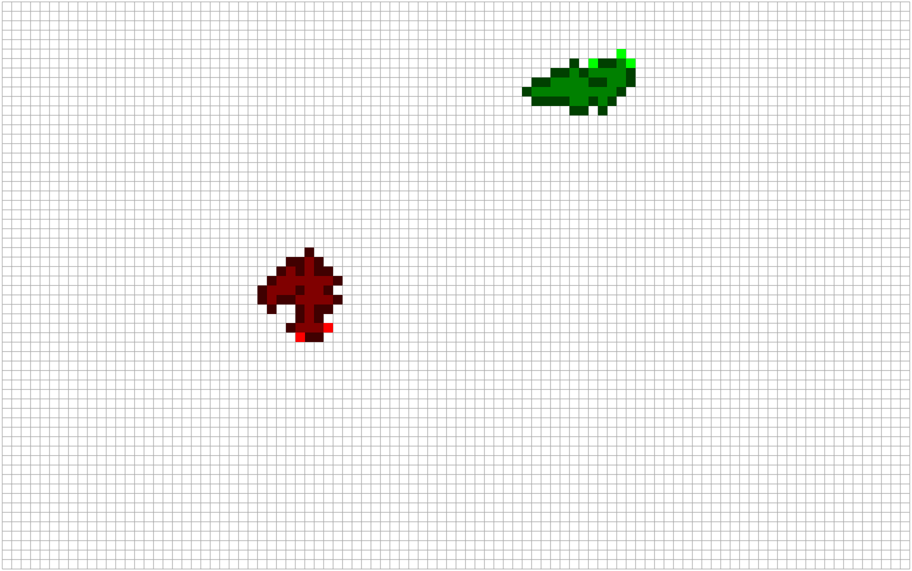

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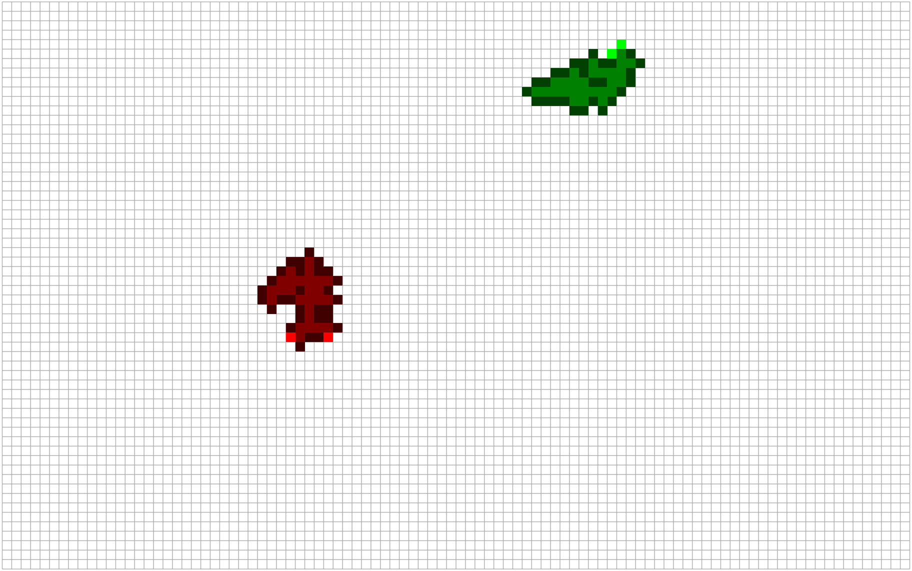

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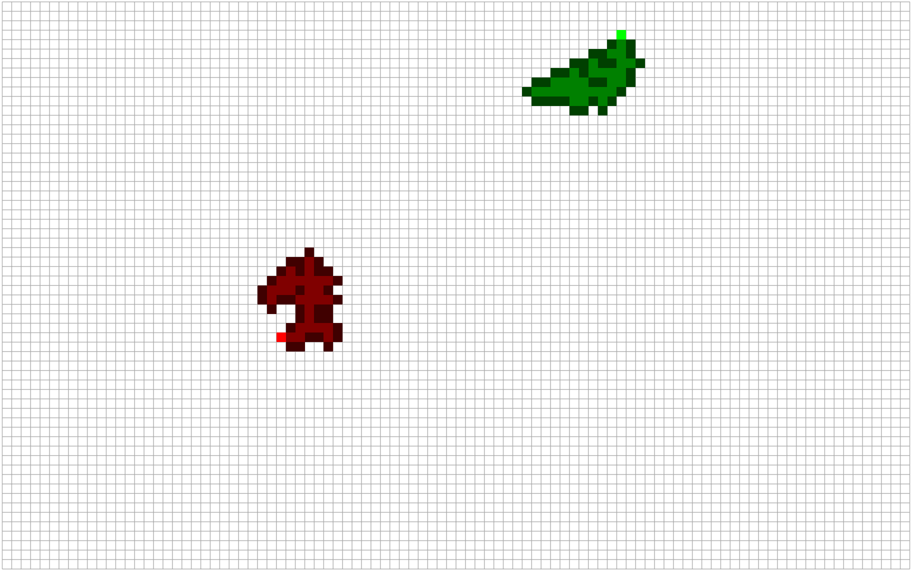

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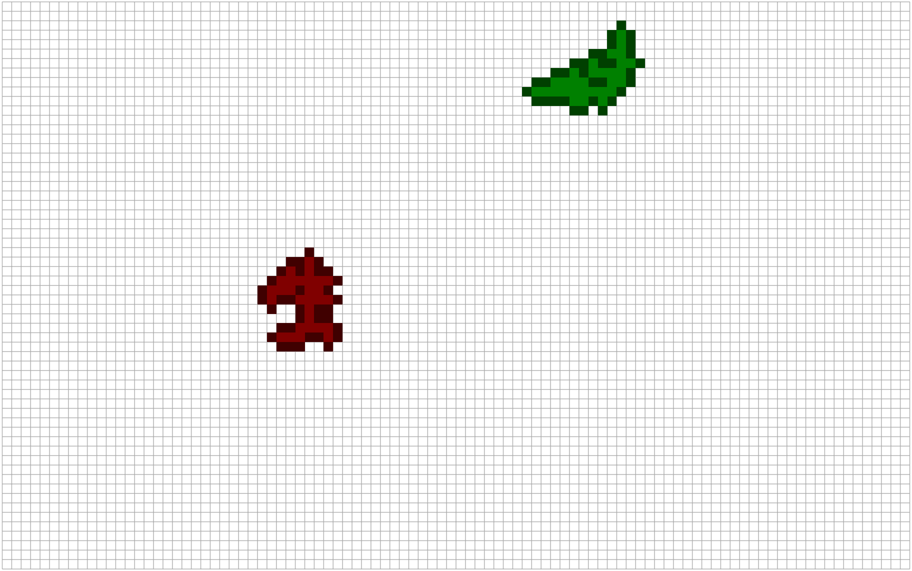

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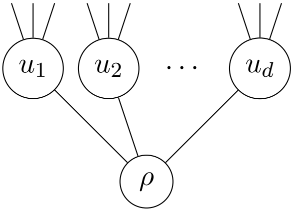

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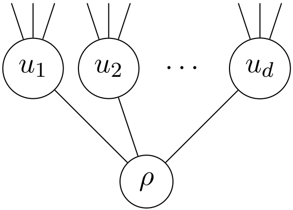


Children of the *past* are roots to independent subtrees.

# Systems of ordinary differential equations

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Thus, if $y$ is a unique solution of

$$y'(x) = \sum_{\ell \in \mathbb{N}} \mathbb{P}\big(\xi^{<x} = \ell\big) \left(1 - \frac{y(x)}{x}\right)^\ell, \qquad y(0) = 0,$$

then, $\iota(U, \rho) = y(1)$.

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Let $(U, \rho)$ be a multi-type branching process.

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Thus, if $y$ is a unique solution of

$$y'_k(x) = \sum_{\ell \in \mathbb{N}^T} \prod_{j \in T} \mathbb{P}\left(\xi_{k \to j}^{<x} = \ell_j\right) \left(1 - \frac{y_j(x)}{x}\right)^{\ell_j}, \qquad y_k(0) = 0,$$

then, $\iota(U, \rho) = \mathbb{E}(y_k(1))$.

# Size-biased Galton-Watson branching processes

Kolchin, Grimmett: the sequence of uniform random trees locally converges to the *size-biased Galton-Watson* Pois$(1)$ tree.

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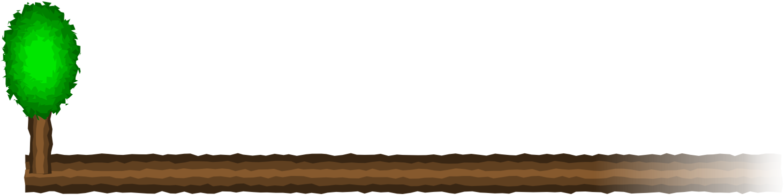

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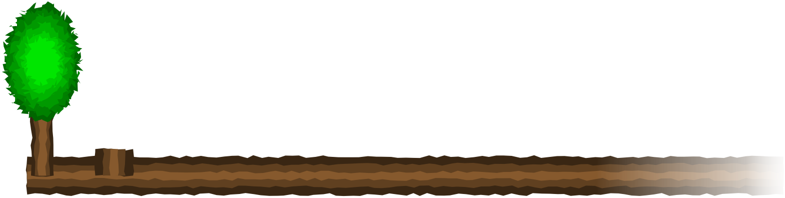

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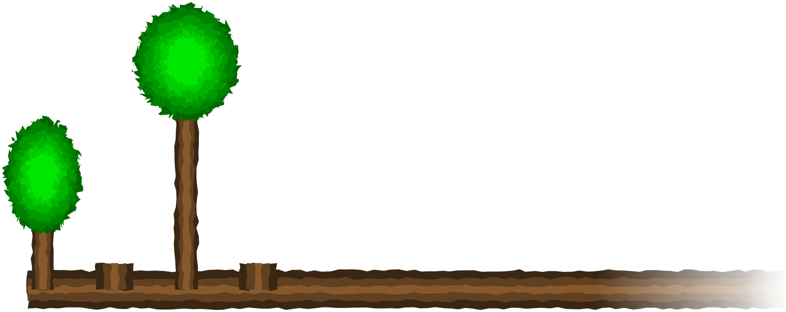

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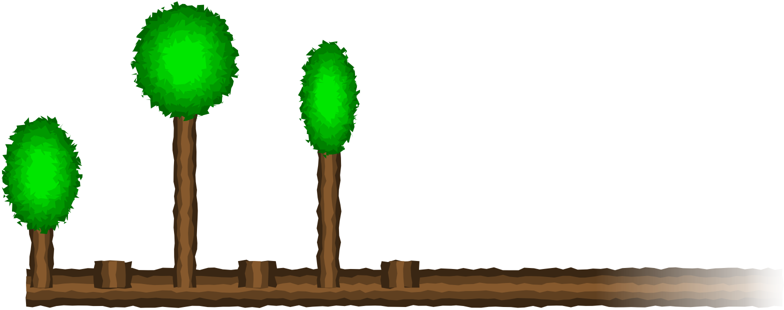

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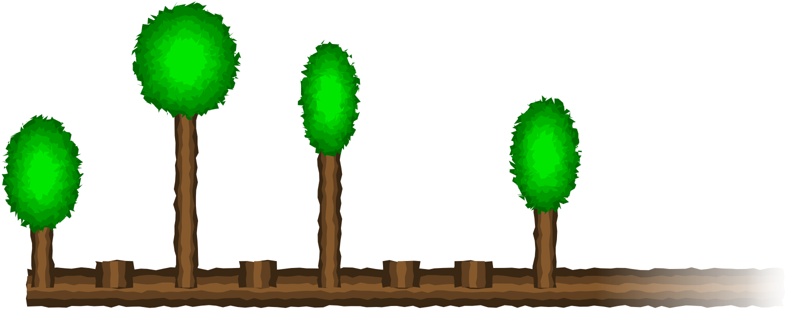

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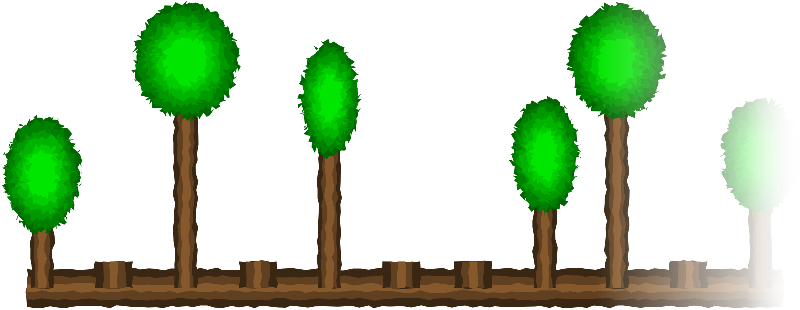

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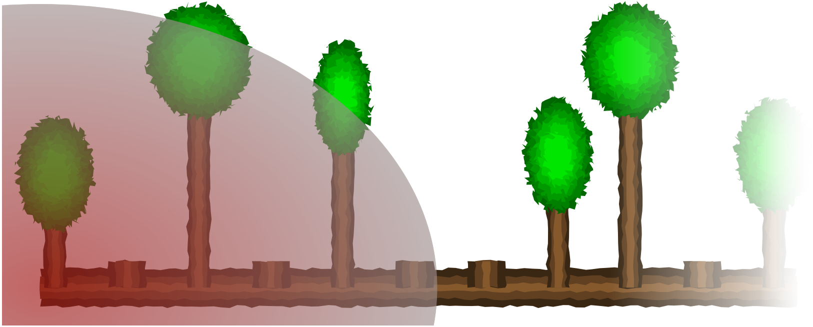

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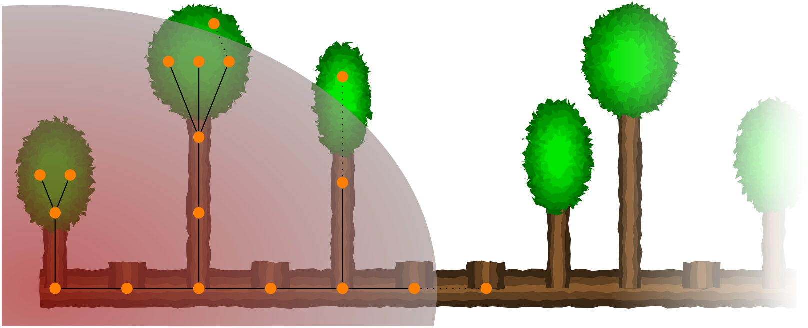

# Uniform random trees

$$y_{\mathsf{t}}'(x) = \sum_{d=0}^{\infty} \frac{(\lambda x)^d}{e^{\lambda x} d!} \left(1 - \frac{y_{\mathsf{t}}(x)}{x}\right)^d = e^{-\lambda y_{\mathsf{t}}(x)}.$$

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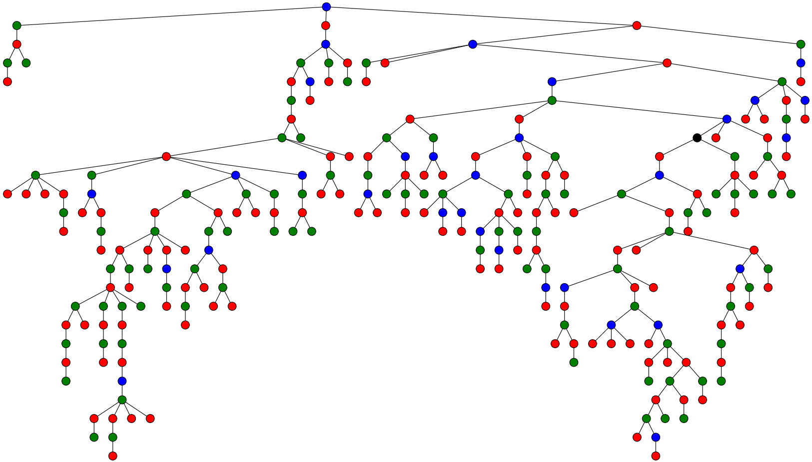
hence $y_{\mathsf{s}}(x) = 1 - (1+\lambda x)^{-1/\lambda}$, and for $\lambda = 1$, $y_{\mathsf{s}}(1) = 1 - (1+x)^{-1}$, and we get

$$\iota(T_n) \to \iota(\hat{\mathcal{T}}_1) = y_{\mathsf{s}}(1) = \frac{1}{2}.$$

# Simulations don't lie



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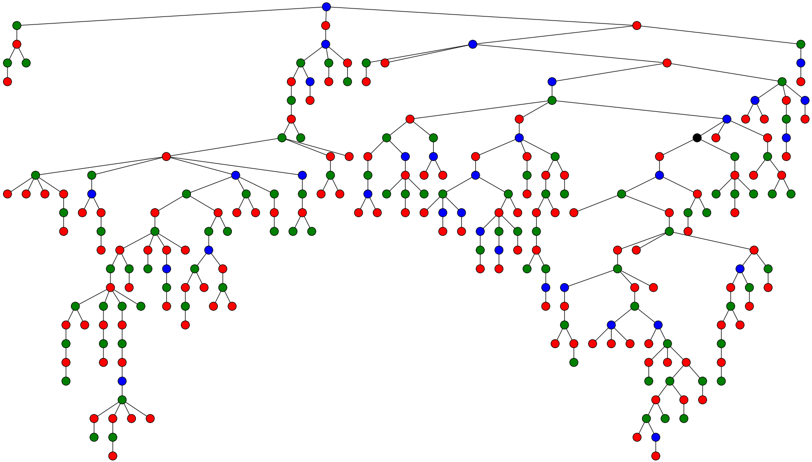
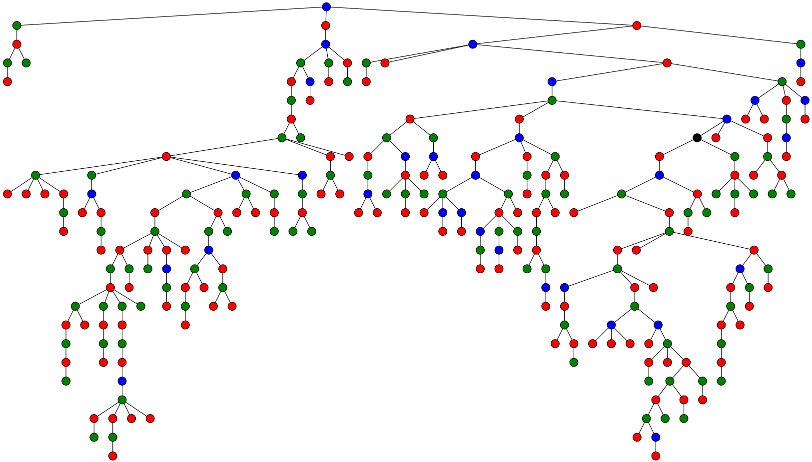


red: 125 (50%), green: 92 ($\approx$ 37%), blue: 32 ($\approx$ 13%), black: 1

# Simulations don't lie (but I do)



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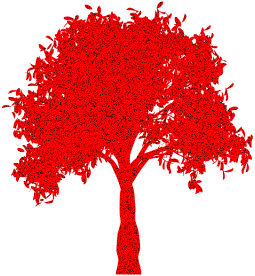

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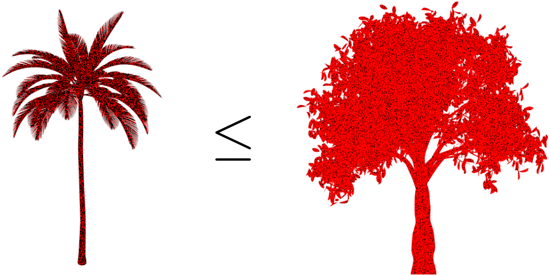

**Theorem (Krivelevich, Mészáros, M., Shikhelman '19+)**

*If $T$ is a tree on $n$ vertices, then $\iota(P_n) \leq \iota(T)$.*

# Thank You!