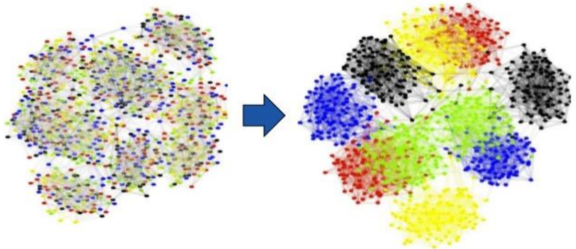
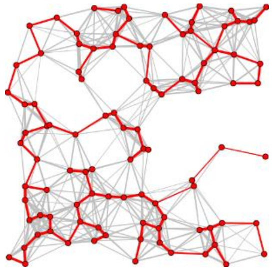
Walking Randomly, Massively, and Efficiently

Jakub Łącki Slobodan Mitrović Krzysztof Onak Piotr Sankowski

Why Random Walks?

- Web ratings [Page, Brin, Motwani, Winograd ’99] [Berkhin ‘05] [Chierichetti, Haddadan ‘17]
- Graph partitioning [Andersen, Chung, Lang ‘06]
- Random spanning trees [Kelner, Mądry ‘09]
- Laplacian solvers [Andoni, Krauthgamer, Pogrow ‘18]
- Connectivity [Reif ’85] [Halperin, Zwick ’94]
- Matching [Goel, Kapralov, Khanna ‘13]
- Property testing [Goldreich, Ron ’99] [Kaufman, Krivelevich, Ron ‘04] [Czumaj, Sohler ’10] [Nachmias, Shapira ‘10] [Kale, Seshadhri ‘11] [Czumaj, Peng, Sohler ’15] [Chiplunkar, Kapralov, Khanna, Mousavifar, Peres ‘18] [Kumar, Seshadhri, Stolman ‘18] [Czumaj, Monemizadeh, Onak, Sohler ’19]





How to Compute Random Walks?

- Centralized [direct implementation]
- Streaming [Sarma, Gollapudi, Panigrahy ’11, Jin ‘19]
- Distributed (CONGEST) [Sarma, Nanongkai, Pandurangan, Tetali ’13]
- MPC, undirected graphs *(non-independent walks)* [Bahmani, Chakrabarti, Xin ’11]

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Our result (undirected graphs):
Independent random walks in MPC
with sublinear memory per machine.

Our Results

Input: Undirected graph G; length L

Output: An L-length random walk per vertex;
walks mutually independent

Rounds: O(log L)

Space per machine: sublinear in n

Total space: O(m L log n).

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- Approximate connectivity and MST
- PageRank for **directed** graph

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Conditional lower-bound of Ω(log L)

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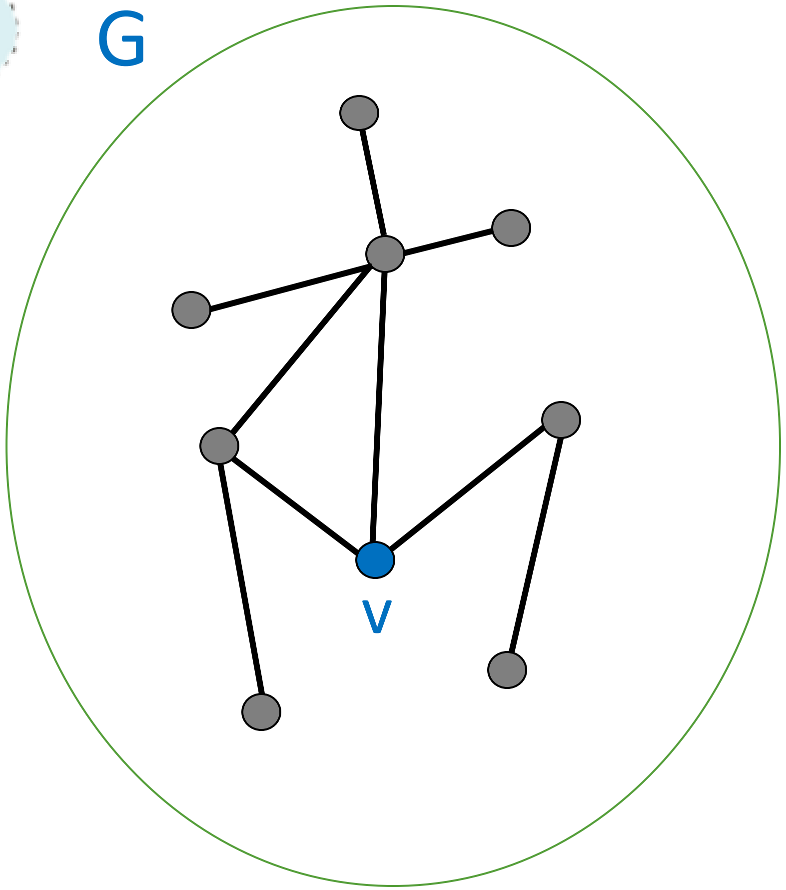
Random Walks in Undirected Graphs

Random Walks: Doubling by Stitching

Output: deg(v) L-length random walk per v; walks mutually independent

Track spare random walks. Use spare to **double** wanted ones.



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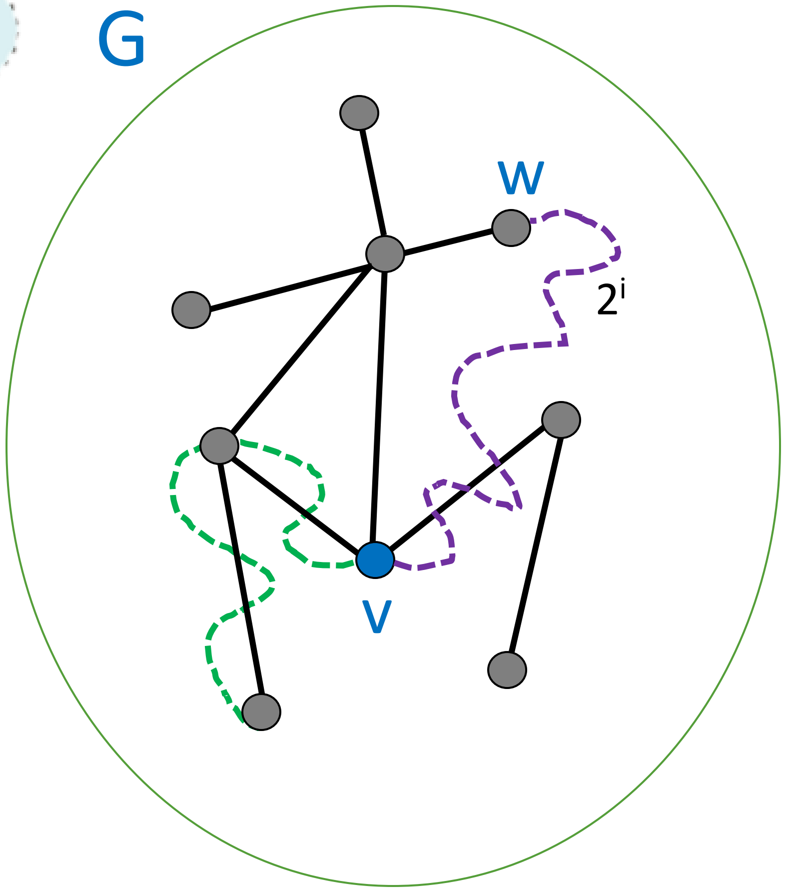

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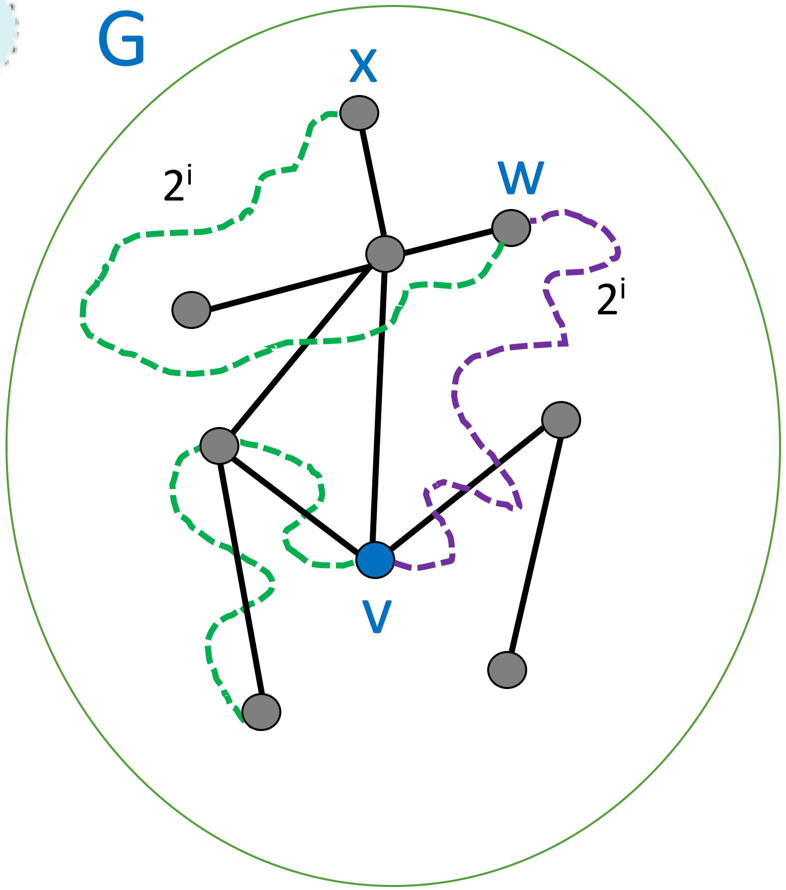

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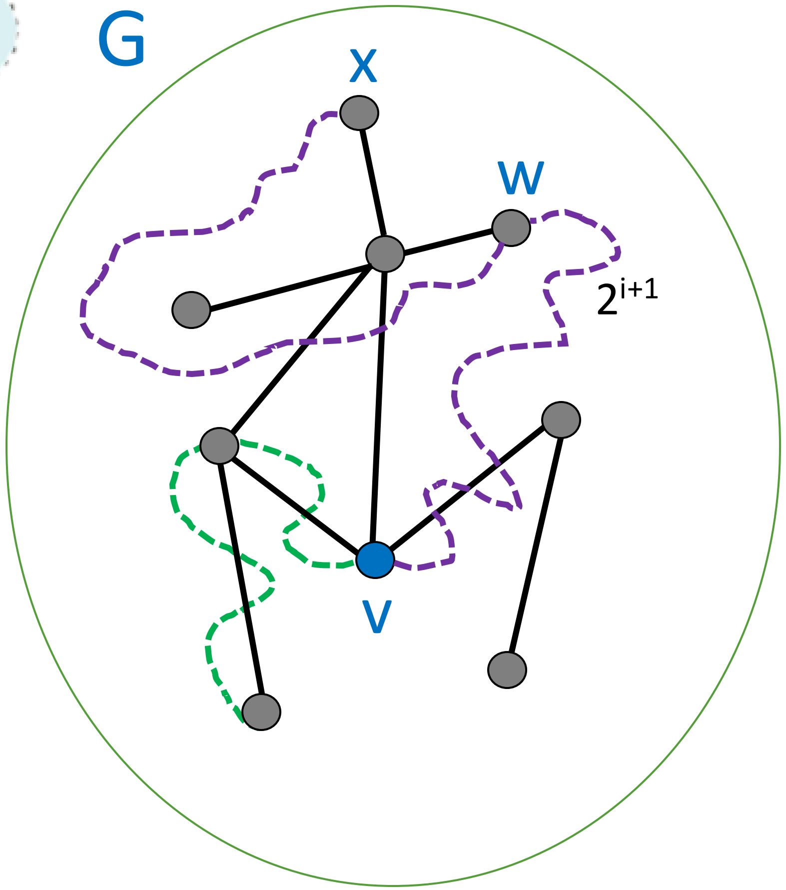

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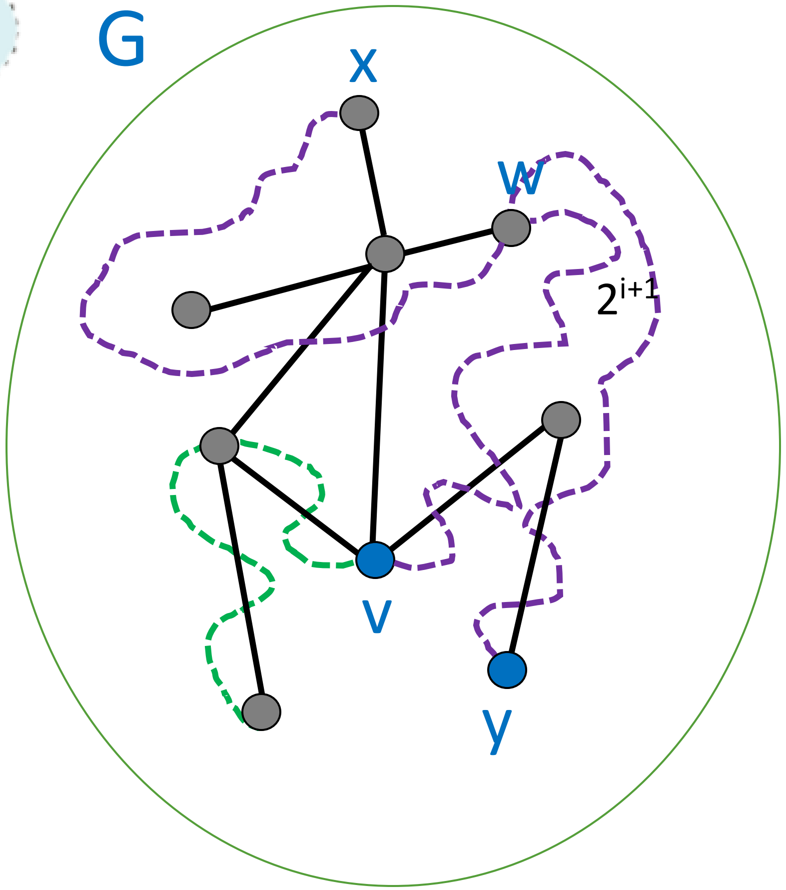

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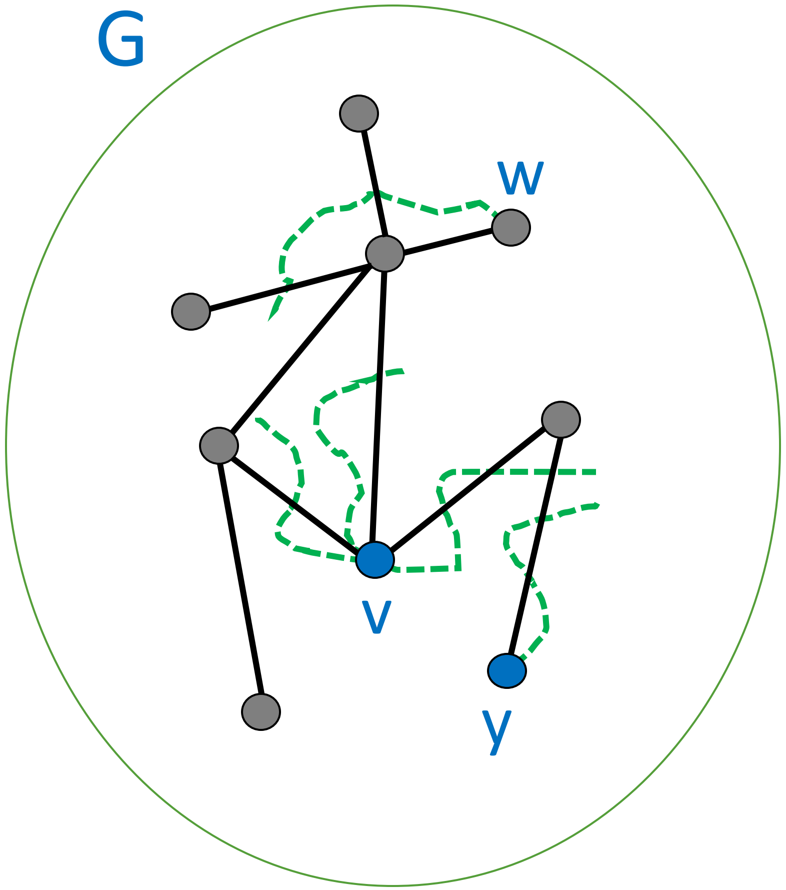

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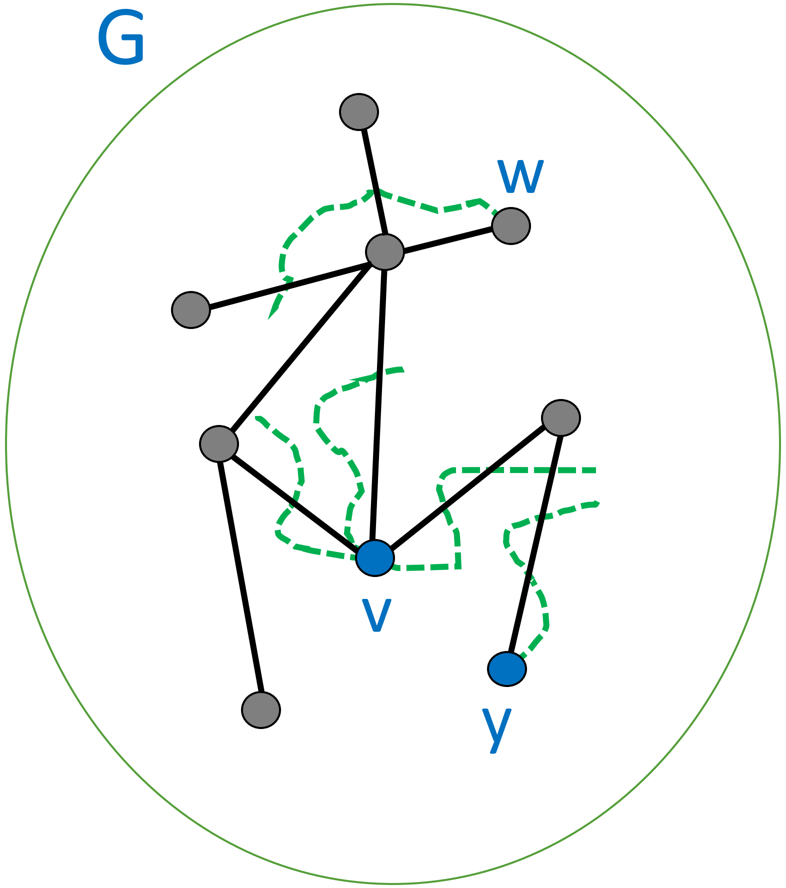

In **expectation**, after t steps there are proportionally to deg(v) walks ending at v.

Random Walks: Takeaway

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2. The memory requirement is inversely proportional to the min entry of the stationary distribution.

>=1/(2m)

PageRank for Directed Graphs

Input: Directed graph G^D

Output: $(1+\alpha)$-approximate PageRank;

ε is the jumping probability

Rounds: $\tilde{O}(\varepsilon^{-1} \log \log n)$

Space per machine: sublinear in n

Total space: $\tilde{O}((m + n^{1+o(1)})\, \varepsilon^{-4}\, \alpha^{-2})$.

(Prelude) Random Walks: Undirected vs Directed

Undirected graphs

Directed graphs

vs

(Prelude) Random Walks: Undirected vs Directed

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Directed graphs

Stationary distribution is easy to compute: deg(v) / (2m).

VS

Stationary distribution of v is "nicely" lower-bounded.

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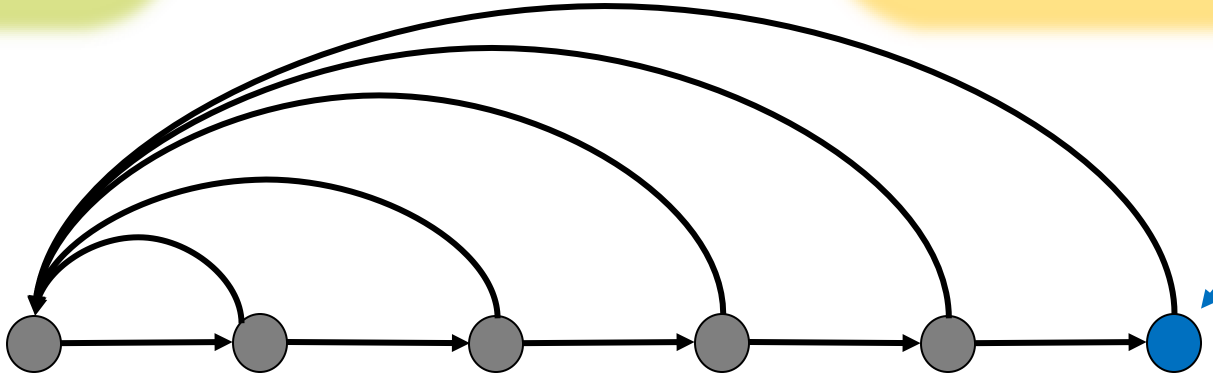
Stationary distribution of v can be $O(1/2^n)$.



PageRank: Undirected vs Directed Graphs

Input: $P = GD^{-1}$

$$T = (1-\epsilon)P + \frac{\epsilon}{n}\vec{1}\vec{1}^T$$

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Walk matrix of G.

Following P with prob. $1-\epsilon$.

Jumping to a random vertex

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We do not know stationary distribution of T.

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Undirected graphs

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Stationary distribution of v w.r.t. to P can be O(1/2^n).

vs

Directed graphs

We do not know stationary distribution of T.

Stationary distribution of v w.r.t. T at least ε/n.


Improvise. Adapt. Overcome

PageRank: Molding Undirected to Directed

PageRank for undirected G.

PageRank for directed G^D.

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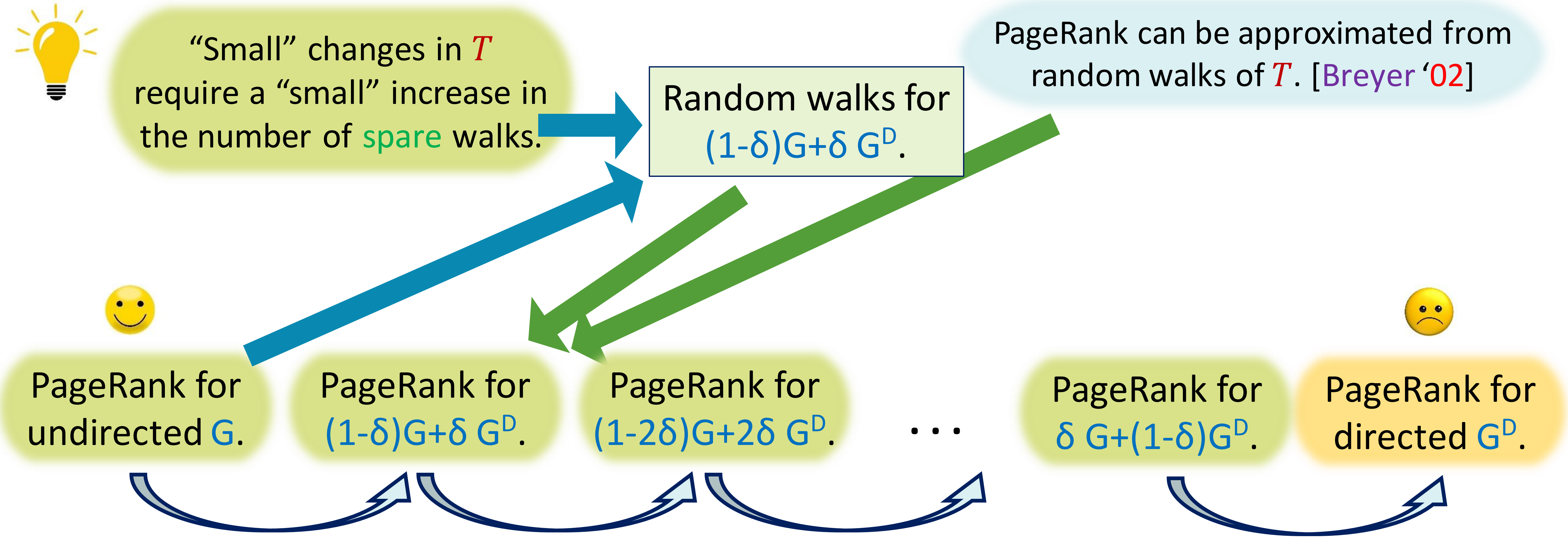

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PageRank for $\delta\, G+(1-\delta)G^D$.

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PageRank: Takeaway

1. The stationary distribution is lower-bounded by ε/n.

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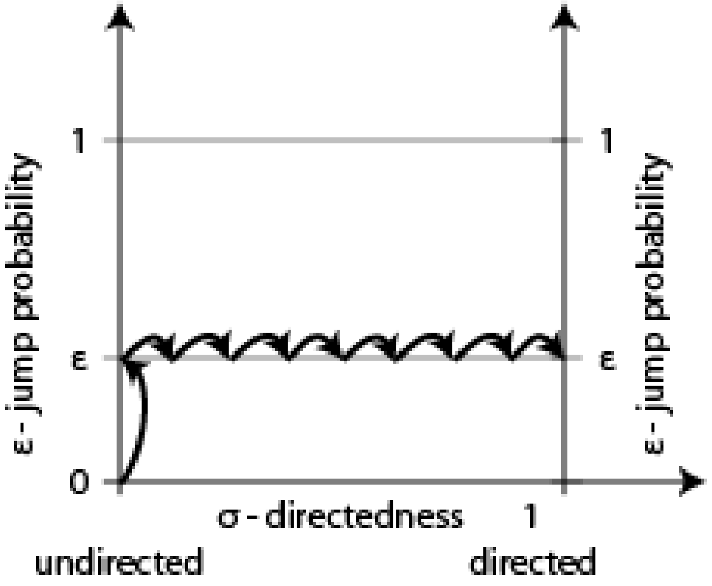
2. "Small" changes in a walk matrix affect the stationary distribution by little.

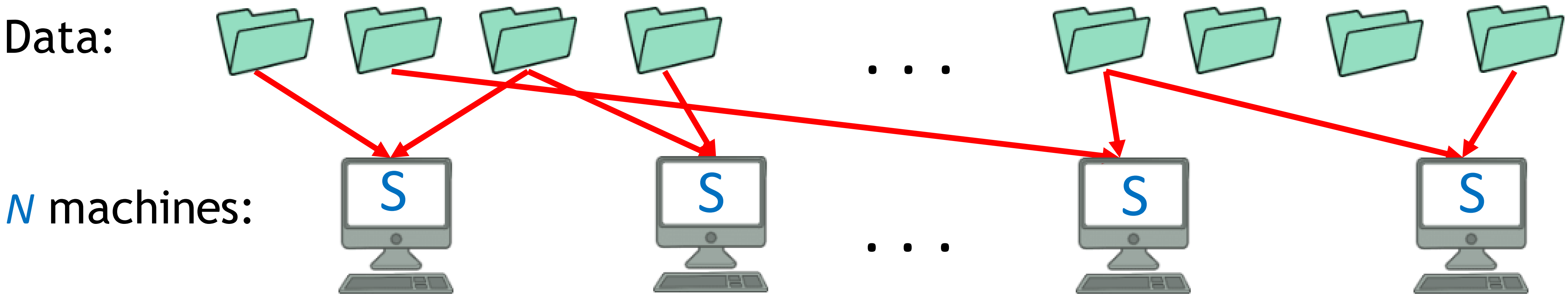


Massively Parallel Computation (MPC) round

Data:

N machines:

Massively Parallel Computation (MPC) round



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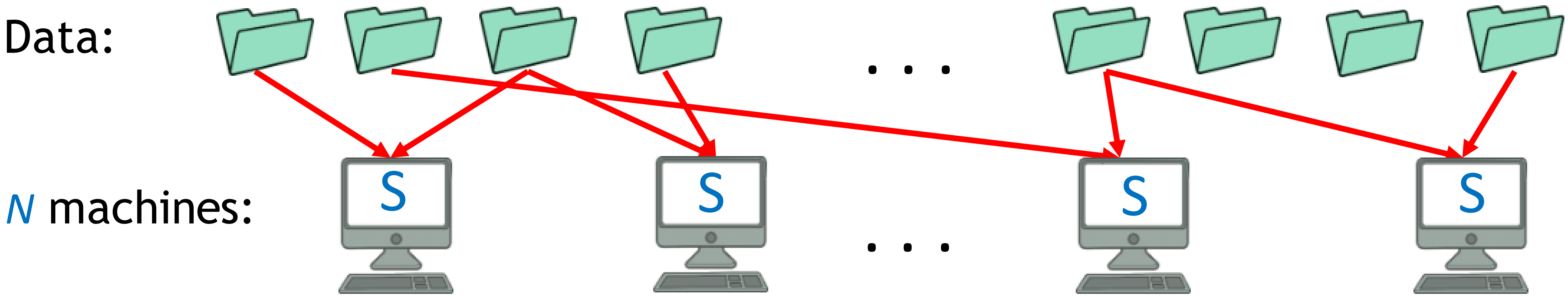

process data locally

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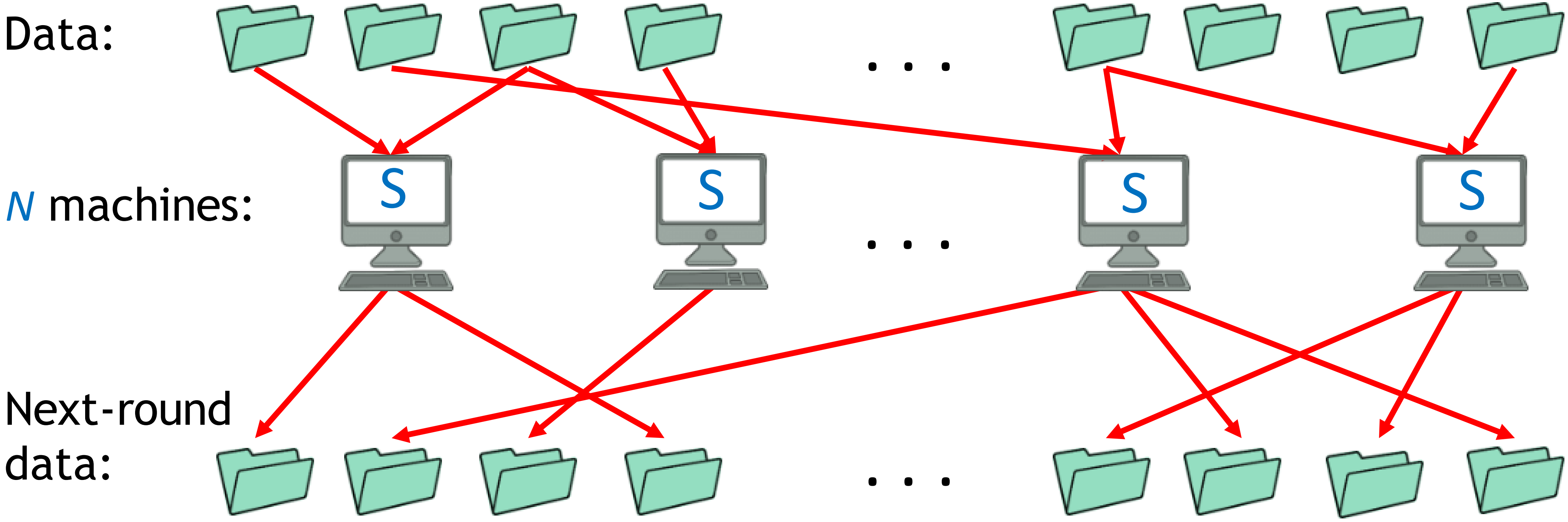

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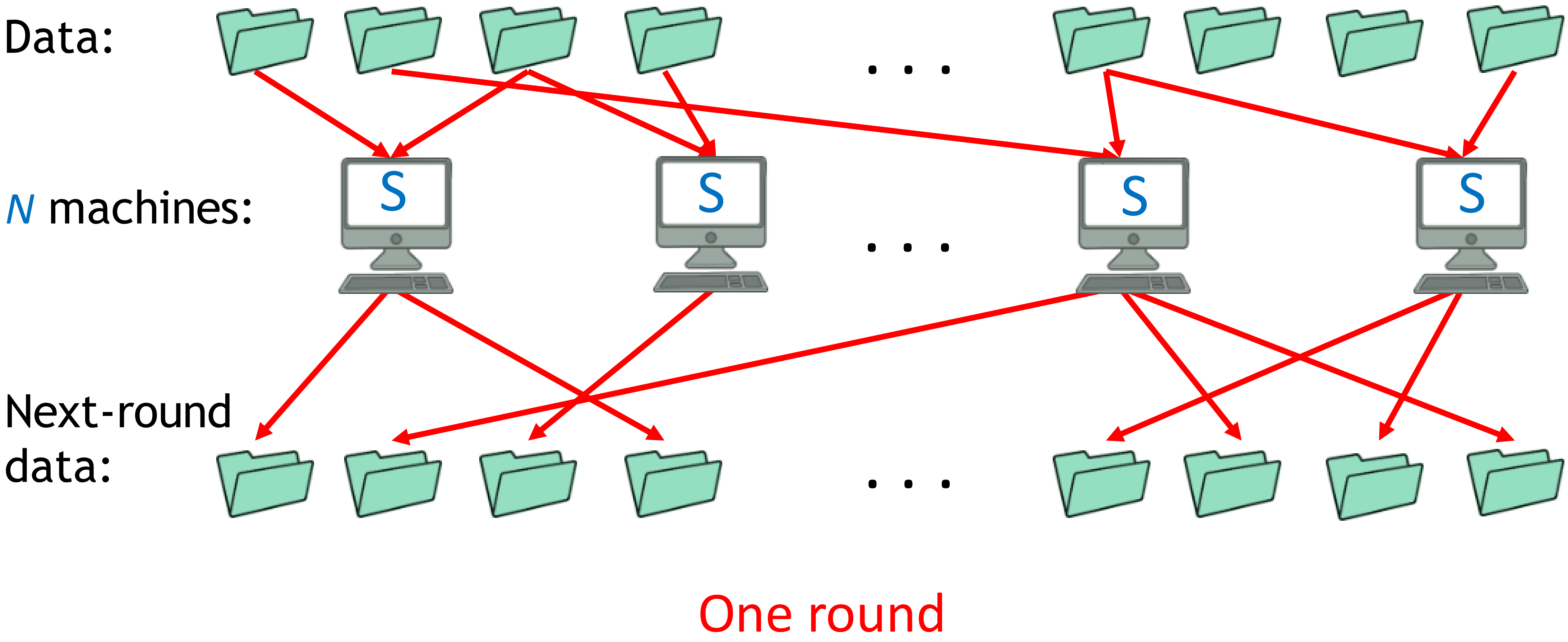

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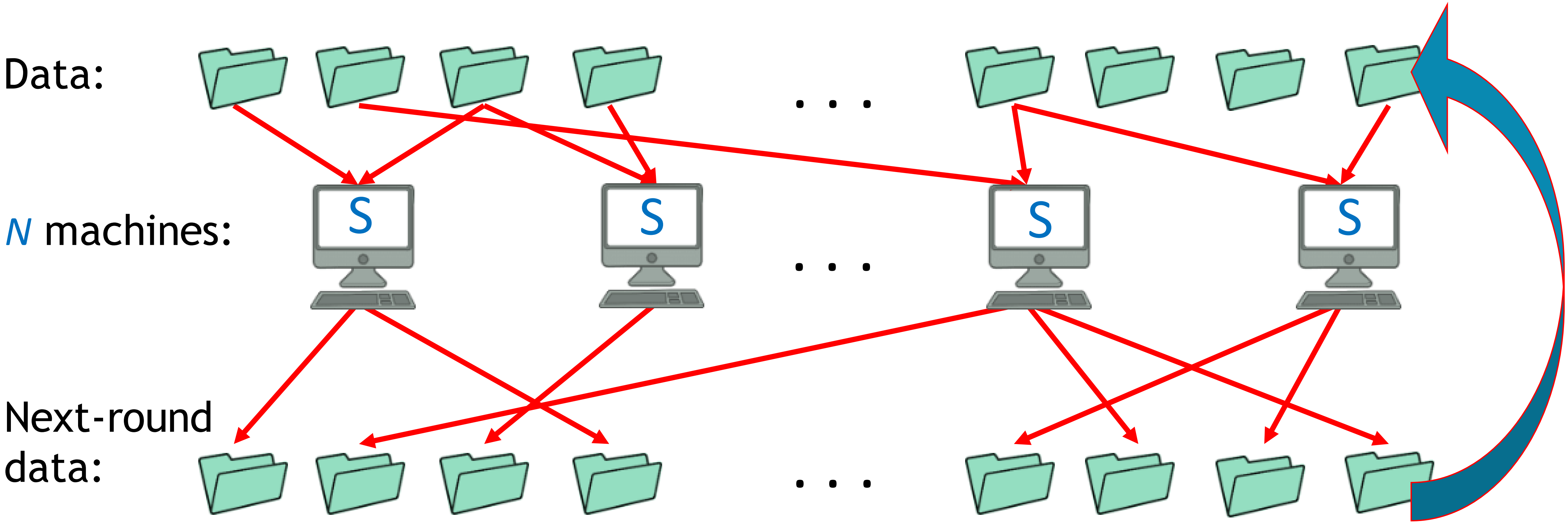


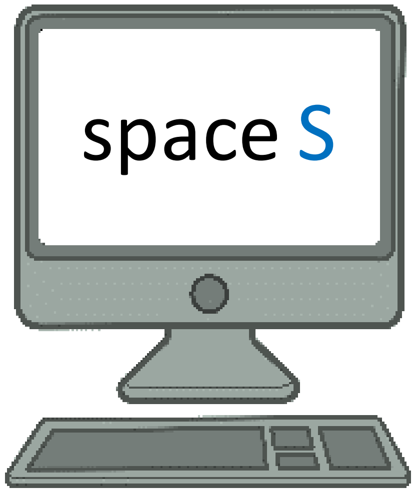
One round

Massively Parallel Computation (MPC) parameters

N = # of machines

S = space per machine

For graphs, $N * S = \Theta$(# of edges)



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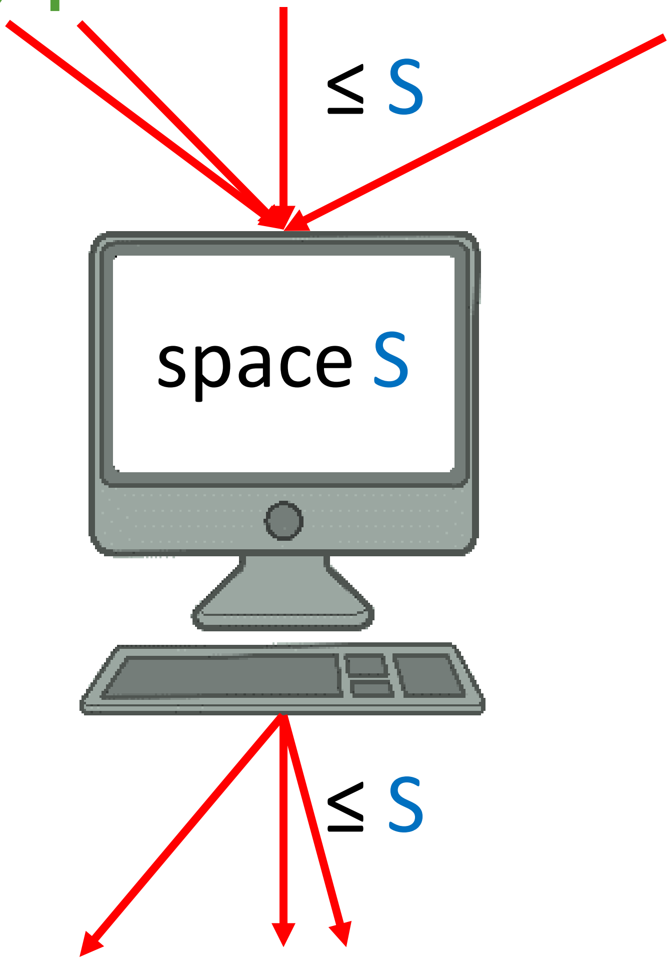

<u>Interesting case</u>:

S much smaller than the input size

<u>Goal</u>:

make the small # of rounds