Additive Approximation of Generalized Turán Questions

Joint work with Noga Alon

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GRAPH MODIFICATION

For an input graph G find the minimum possible number of edges/vertices we need to add/remove/edit to get a graph with given property.

INTRO
NP-HARD EDGE MODIFICATION PROBLEMS

Yannakakis ‘81 being outerplanar, transitively orientable, and line-invertible.

Asano and Hirata ‘82, Asano ’87 Certain properties expressible by forbidding minors or topological minors.

Natanzon, Shamir and Sharan ‘01 Hereditary properties such as being Perfect and Comparability.

INTRO
APPROXIMATION OF EDGE MOD PROBLEMS

Fernandez de la Vega '96, Arora, Karger and Karpinski '95 several NP-complete problems such as MAX-CUT and MAX-3-CNF

Frieze and Kannan '99 Graph theo. properties

Arora, Frieze and Kaplan '02 Quadratic assignment problems and other

Alon, Vega, Kannan and Karpinski '02 Constraint-Satisfaction-Problem

INTRO

SOME DEFINITIONS

A graph property is called ***monotone*** if it can be defined by forbidding a family of graphs.

The only relevant edge modification for monotone properties is edge deletion.

For a graphs G, T and a family of graphs $\mathcal{F}$ let $ex(G, T, \mathcal{F})$ be the maximum possible number of copies of T in an $\mathcal{F}$-free subgraph of G.

INTRODUCTION $ex(G, K_2, \mathcal{F})$

Alon, Shapira and Sudakov '05 For any $\epsilon > 0$ and $\mathcal{F}$ there is a polynomial time algorithm that approximates $ex(G, K_2, \mathcal{F})$ up to an additive error of ϵn^2.

A significantly $(n^{2-\epsilon})$ better approximation is possible iff there is a bipartite graph in $\mathcal{F}$.

ALGORITHM FOR GENERAL T

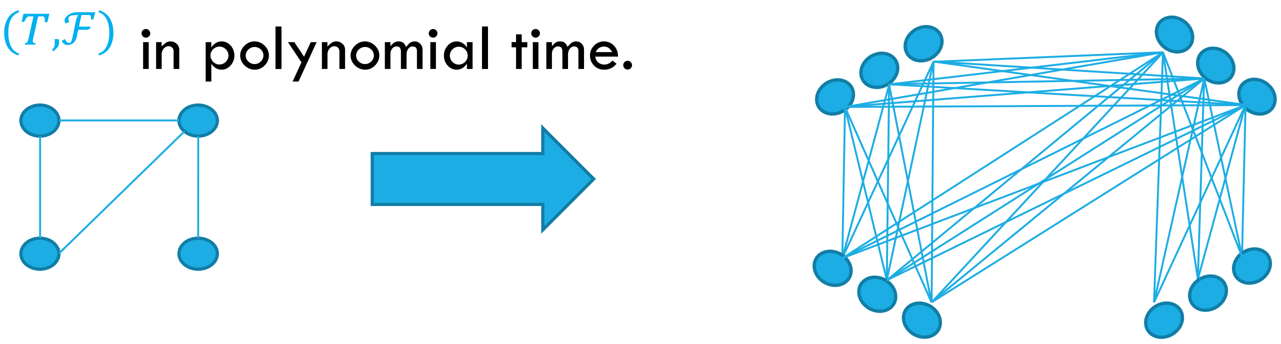
Alon, Sh. '18+ For any graph T, finite family of graphs $\mathcal{F}$ and $\epsilon > 0$ there is a polynomial time algorithm that approximates $ex(G,T,\mathcal{F})$ up to an additive error of $\epsilon n^{v(T)}$.

Can we do better?

CAN WE DO BETTER?

$\mathcal{B}(T)$ - The family of blow ups is all the graphs obtained from T by replacing vertices with independent sets and every edge with complete bipartite graph.

Proposition (Alon, Sh. '18+) Let T be a graph and $\mathcal{F}$ a family of graphs s.t. there is a graph $H \in \mathcal{F} \cap \mathcal{B}(T)$. Then $ex(G, T, \mathcal{F})$ can be calculated up to an additive error of $n^{v(T)-c(T,\mathcal{F})}$ in polynomial time.



CAN WE DO BETTER?

Conjecture It is NP-hard to approximate $ex(G, T, \mathcal{F})$ up to an additive error of $n^{v(T)-\epsilon}$ iff $\mathcal{F} \cap \mathcal{B}(T) = \emptyset$.

Proved for:

1. Both T and $\mathcal{F}$ are complete graphs
2. Both T and $\mathcal{F}$ are 3-connected, NP-hard up to an additive error of $n^{v(T)-2-\epsilon}$
3. In progress – additive error of $n^{v(T)-\epsilon}$

INTUITION FOR THE ALGORITHM
REGULAR PAIRS

Given a graph G and a pair of disjoint sets V_1, V_2,

we say that (V_1, V_2) is an ***ϵ-regular pair*** if for

every $U_i \subseteq V_i$ s.t. $|U_i| > \epsilon |V_i|$

$$\left|d(V_i, V_j) - d(U_i, U_j)\right| \leq \epsilon$$

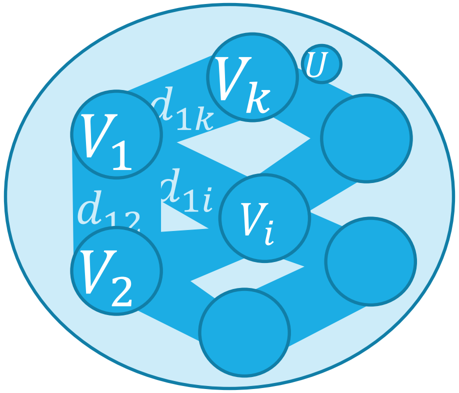
Where $d(V_i, V_j) = \frac{e(V_i, V_j)}{|V_i| \cdot |V_j|}$

INTUITION FOR THE ALGORITHM

REGULAR PARTITIONS

For a graph G, a partition of its vertices $V = U \cup_{i=1}^{k} V_i$ is called **strongly ϵ-regular** if

1. For every i, j $|V_i| = |V_j|$ and $|U| < k$
2. Every pair (V_i, V_j) is ϵ-regular



Can be found in pol-time by changing $o(n^2)$ edges.

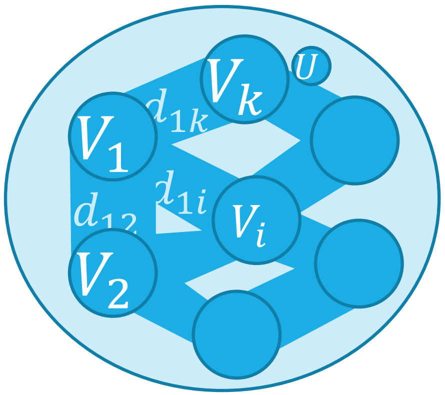
INTUITION FOR THE ALGORITHM
CONVENTIONAL SUBGRAPH

A conventional subgraph of G if it is obtained by deleting

1. All of edge inside the sets V_i

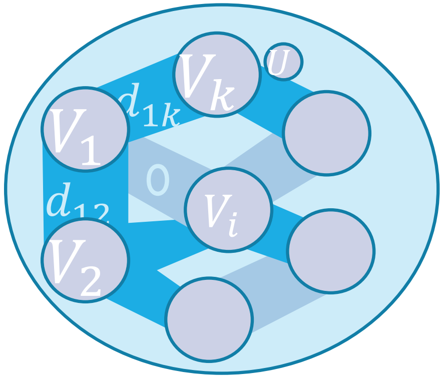
2. All of the edges with endpoint in U

3. All of the edges between some (V_i, V_j)



How many conventional subgraphs are there?

$2^{\binom{k}{2}}$



THE (SIMPLE) ALGORITHM

For a graph T, a finite family of graphs $\mathcal{F}$ and input graph G:

1. Find a strong ϵ-regular partition of G.
2. Find a conventional subgraph of G which is $\mathcal{F}$-homorphism-free and has max. possible T density.

Main Lemma

This graph has $\geq ex(G, T, \mathcal{F}) - \epsilon n^{v(T)}$ copies of T.

THANKS FOR YOUR ATTENTION!