Breaking Quadratic Time for Small Vertex Connectivity and an Approximation Scheme

Danupon Nanongkai
KTH

Thatchaphol Saranurak
TTIC

Sorrachai Yingchareonthawornchai
Aalto University

@WOLA'19

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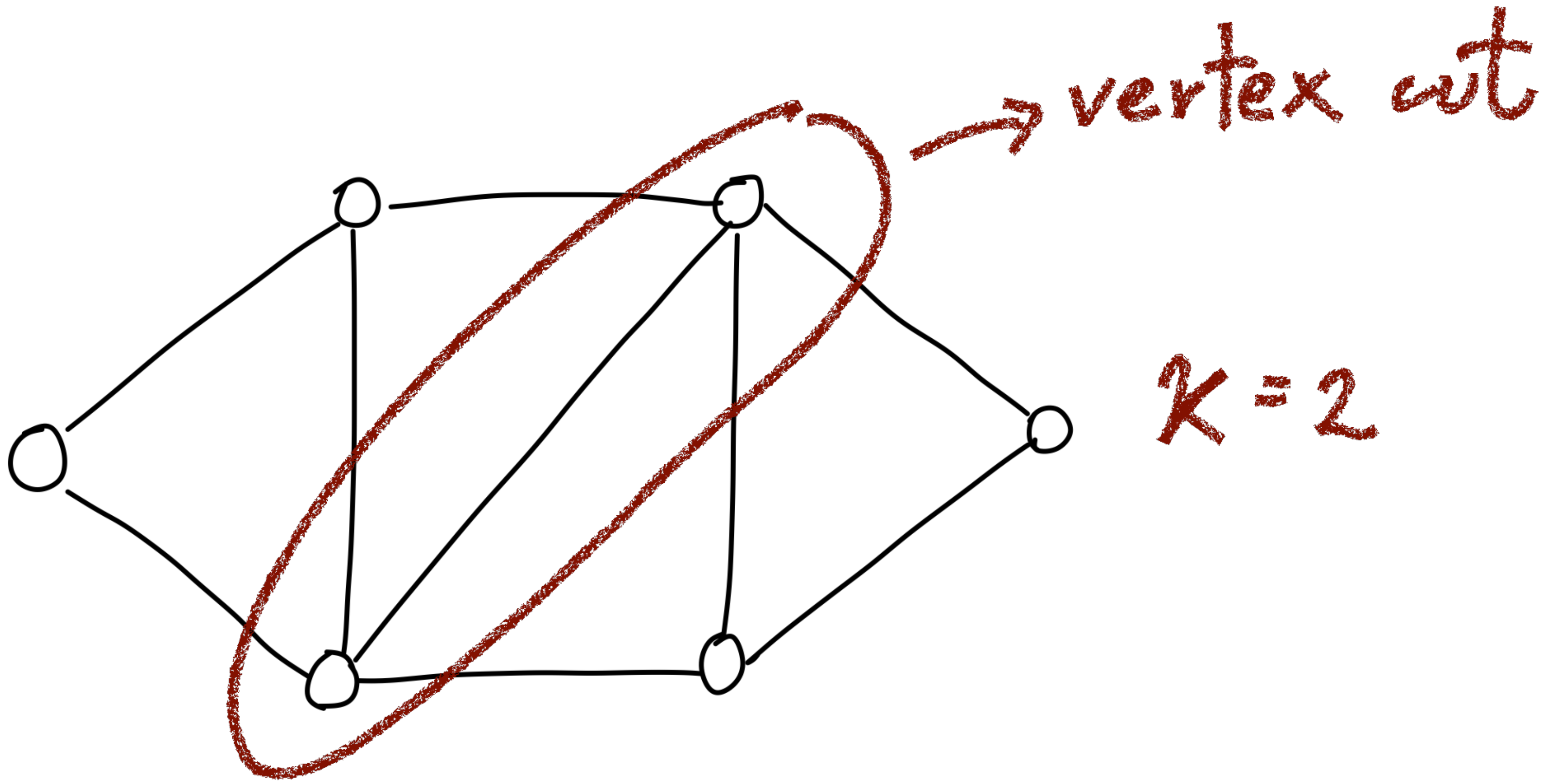
Take Home Message:

“Local algorithms are useful for vertex connectivity”

Vertex Connectivity

$$G = (V, E), |V| = n, |E| = m$$



Compute κ the minimum number of vertices to be removed to disconnect the graph

Part 1: Introduction

Background and Results

$G = (V, E), |V| = n, |E| = m, k = \kappa$ *omit polylog factor

	Reference	General	$m = O(n)$
	Kleitman'69, Podderyugin'73, Even Tarjan'75	mnk^2	n^2
	Even'75, Galil'80, Esfahanian Hakimi'84, Matula'87	$(n + k^2)mk$	n^2
	Becker et al'82 (Randomized)	mnk	n^2
	Nagamochi Ibaraki'92	Can assume $m = O(nk)$	
*	Linial Lovasz Wigderson'88, Cheriyan Reif'91	$n^\omega + nk^\omega$	n^ω
*	Henzinger Rao Gabow'00 (Randomized)	mn	n^2

*** state-of-the-art**

$G = (V, E), |V| = n, |E| = m, k = \kappa$ *omit polylog factor

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*	Henzinger Rao Gabow'00 (Randomized)	mn	n^2
	Gabow'06 (Deterministic)	$mn + m\min\{k^{2.5} + n^{3/4}k\}$	n^2

Even for $m = O(n), k = O(1)$,
$O(n^2)$ time remains since 1969

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Our work (STOC'19)	$m + n^{4/3}k^{7/3}$	$n^{4/3}$
Our follow-up work'19, Forster Young'19	$m + nk^3$	n

Near-linear time when k = O(polylog(n))

Approximation

$O(n^{2.5})$ **when** $k = \Omega(n^{0.64..})$

Reference	Time	Approx
Linial Lovasz Wigderson'88, Cheriyan Reif'91, Henzinger'97	$\min\{n^{2.5}, n^2k, n^\omega + nk^\omega\}$	2
Censor-Hillel Ghaffari Kuhn'14	m	$\log n$
Our work* (STOC'19)	$\min\{n^{2.2+o(1)}, n^\omega\}$	$(1+\epsilon)$

For any k

*Actual bound is $m + \frac{1}{\textbf{poly}(\epsilon)} \min\left\{nk^2, n^{5/3+o(1)}k^{2/3}, \frac{n^{3+o(1)}}{k}, n^\omega\right\}$

Remark: use fast approximate s,t-vertex connectivity from Chuzhoy Khanna (STOC'19)

Contributions

(including our follow-up work)

Breaking 50-year-old $O(n^2)$ time for small κ

Near-linear time for $\kappa = O(1)$

Fastest up to $\kappa = O(n^{0.457})$

Fastest $(1+\epsilon)$-approximation for any κ

Follow-up'19:
Testing *k*-Vertex Connectivity

One-sided tester: *k*-vertex connected vs. ϵ-far

Reference	Query Complexity (Simplified)	
Goldreich Ron'02, Orenstein Ron'11, Yoshida Ito'10, Yoshida' Ito 12	$\tilde{O}(\frac{2^{k+1}}{\epsilon^{k+1}})$	**Open**: polynomial #queries?
Forster Young'19	$\tilde{O}(\frac{k^3}{\epsilon^2})$	
Our follow-up '19	$\tilde{O}(\frac{k}{\epsilon^2})$	

Part 2: Techniques

$$G = (V, E), |V| = n, |E| = m, k = \kappa$$ *omit polylog factor

Reference	General	$m = O(n)$
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Barrier: Single-Source Vertex Connectivity

No $o(n^2)$ algorithm is known

Fix a node *s*, compute the smallest vertex-cut separating *s* from some node

Part 2.1:
Local Vertex Connectivity

to replace Single-Source Vertex Connectivity

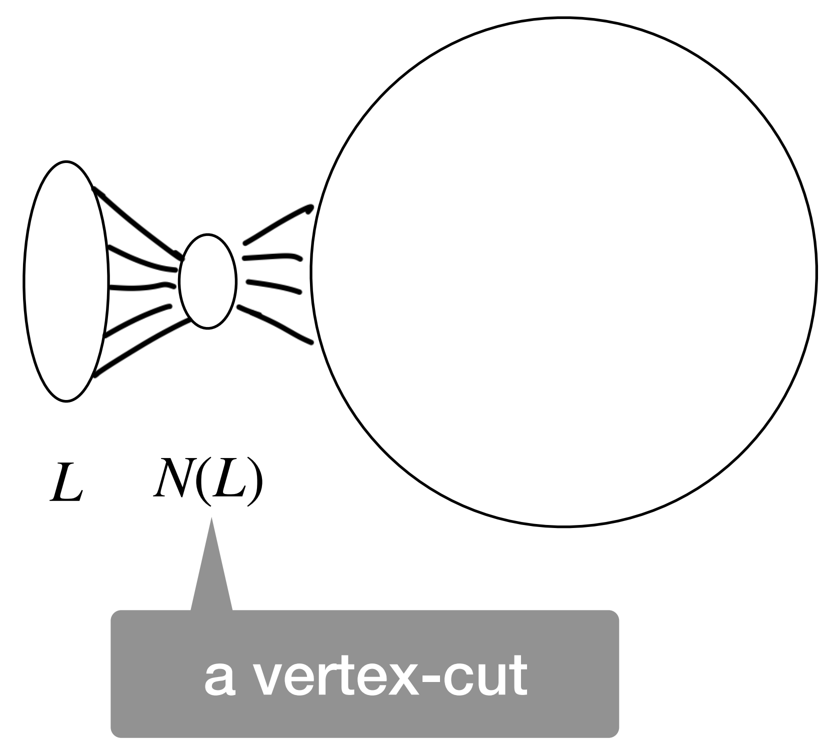
Fix a node *s*, is there a vertex-cut "near" *s* ?
(in sub-linear time)

Explore around s

Assumption: G **is regular,**

$$m = O(n), \kappa = O(1)$$

Notation: $N(L) =$ **neighbors of** $L \subset V$



Local Vertex Connectivity

Explore around x

Input: LocalVC$(x \in V, \nu, k)$ **where** $\nu < \frac{m}{k}$

Target vertex-cut

Output: Either

(1) declare that no $L \subset V$ **exists where** $L \ni x, |L| \leq \nu, |N(L)| < k,$

OR

(2) a vertex-set L **where** $|N(L)| < k$

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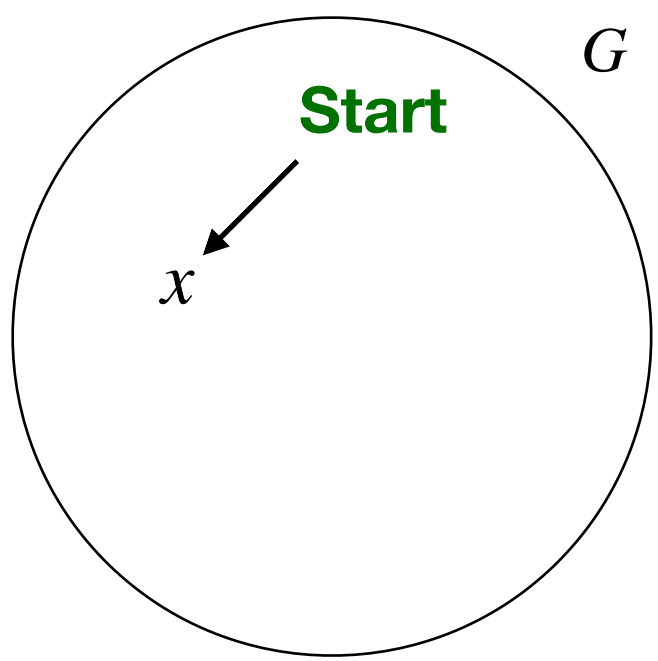

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(2) a vertex-set L **where** $|N(L)| < k$

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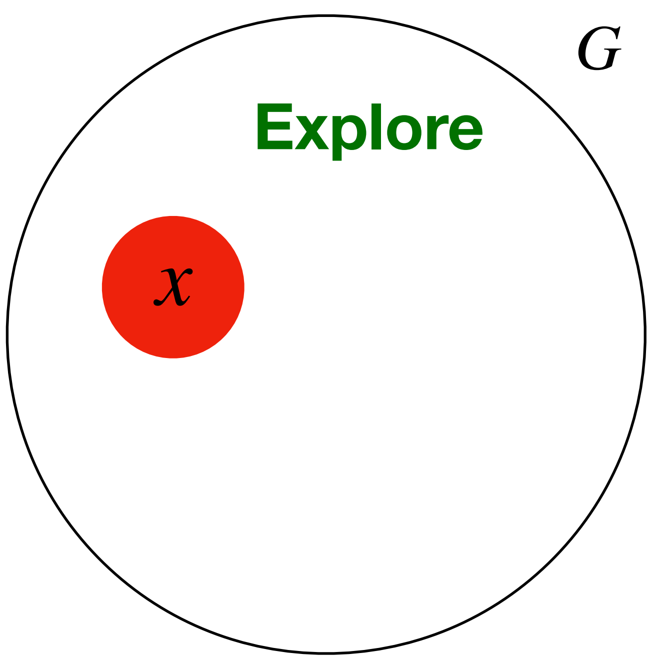

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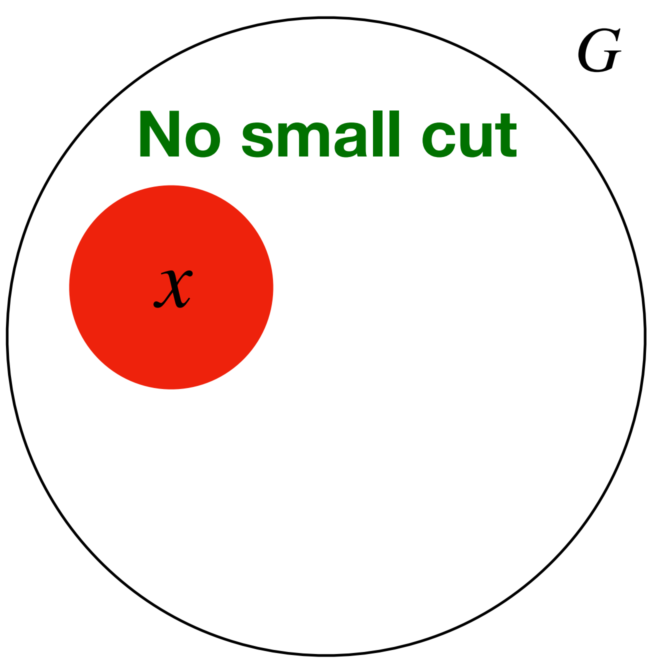

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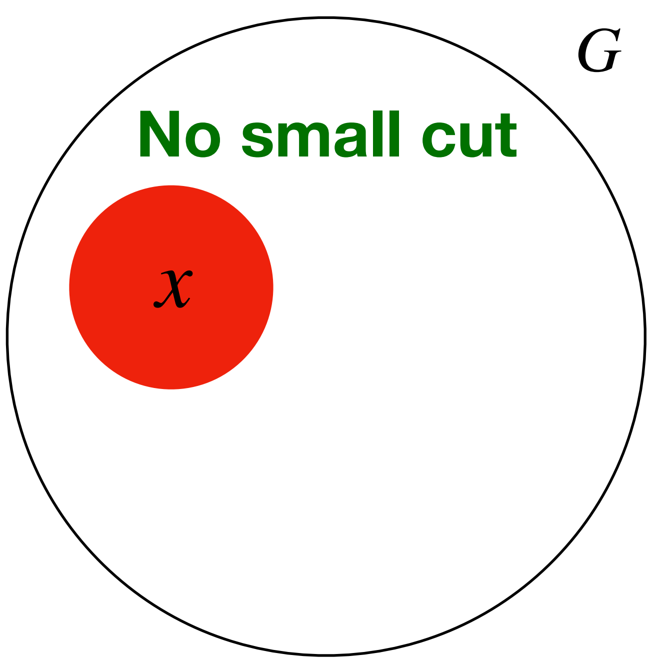

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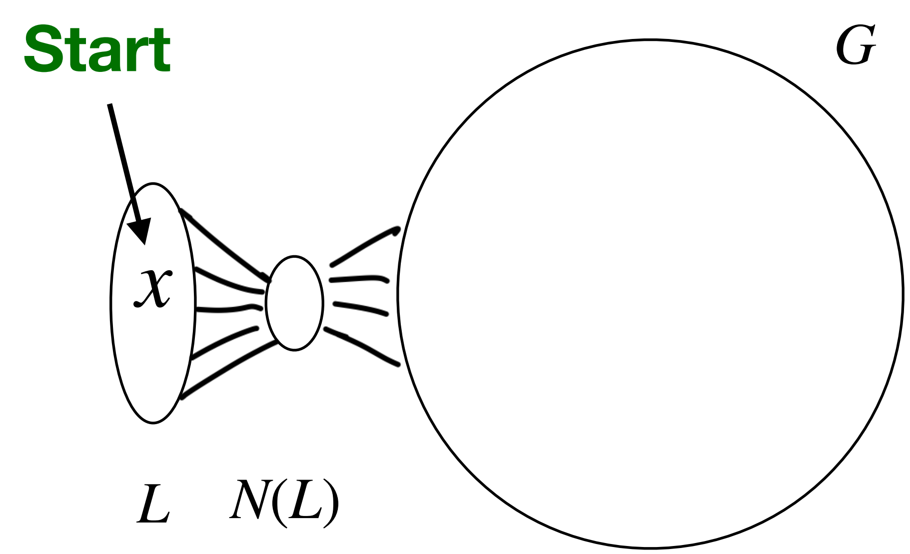

Local Vertex Connectivity

Input: LocalVC$(x \in V, \nu, k)$ **where** $\nu < \dfrac{m}{k}$

Output: Either

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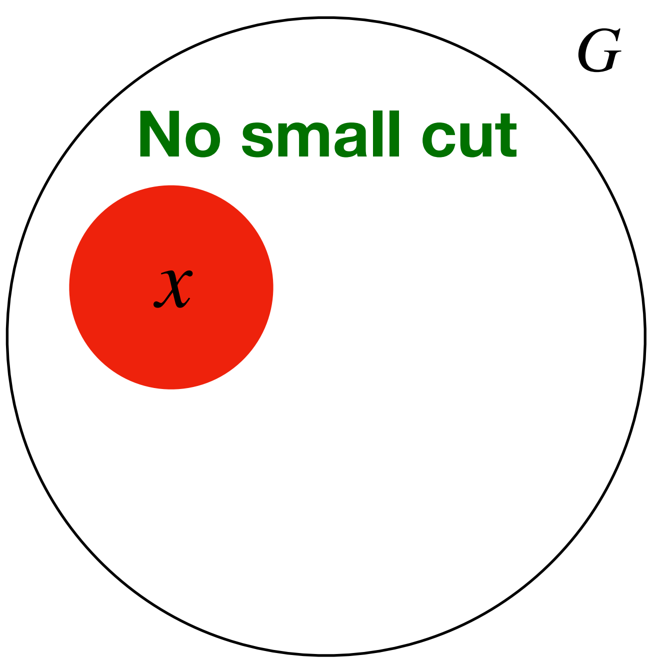

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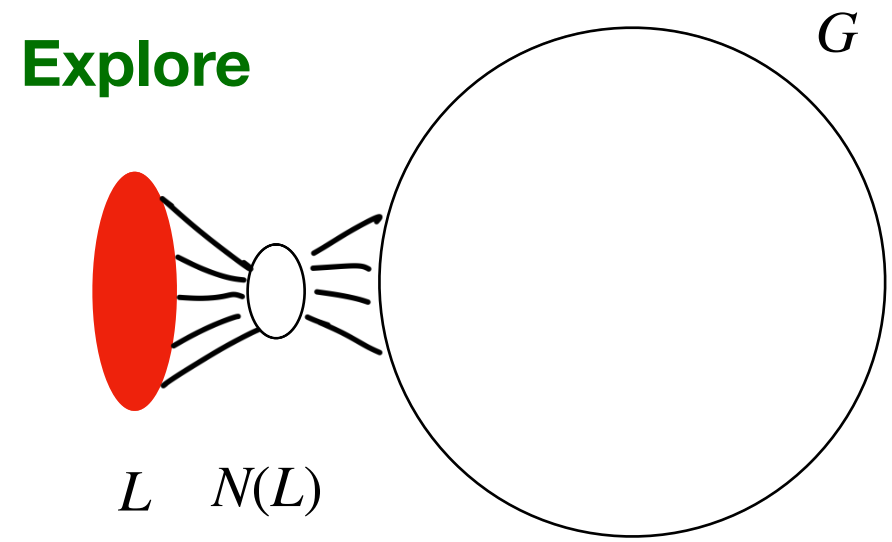

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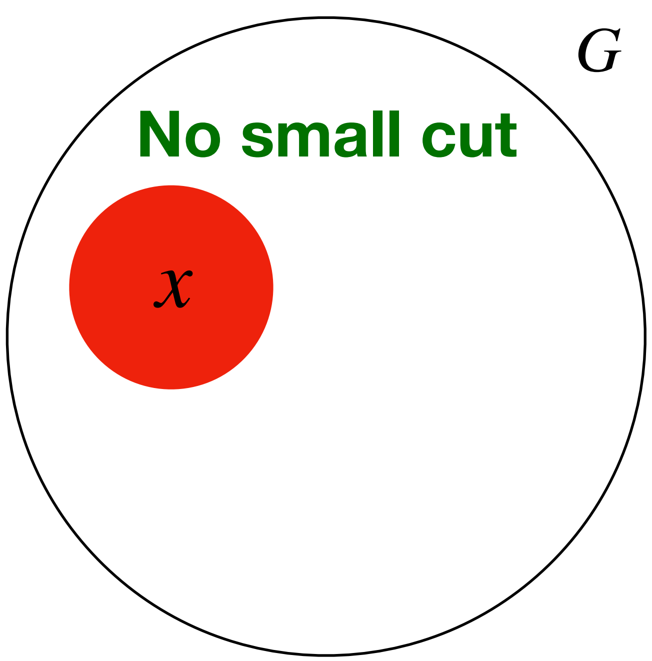

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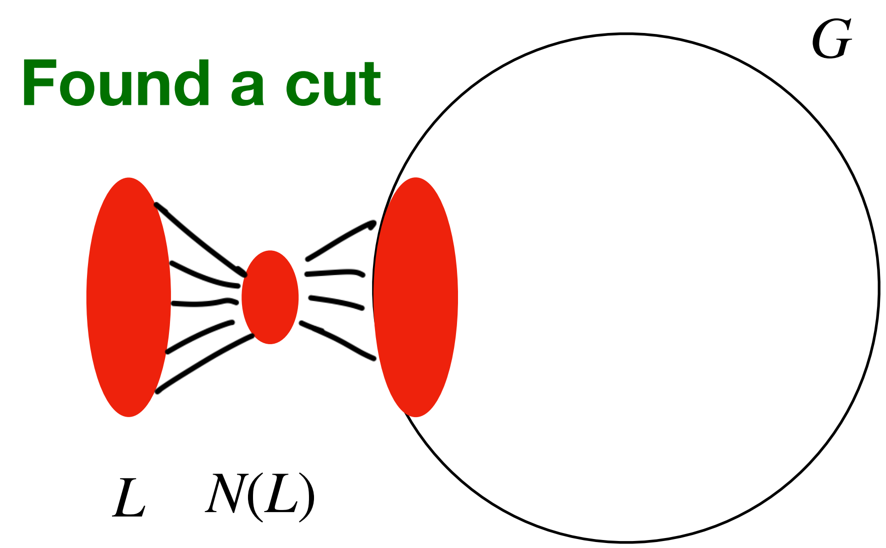


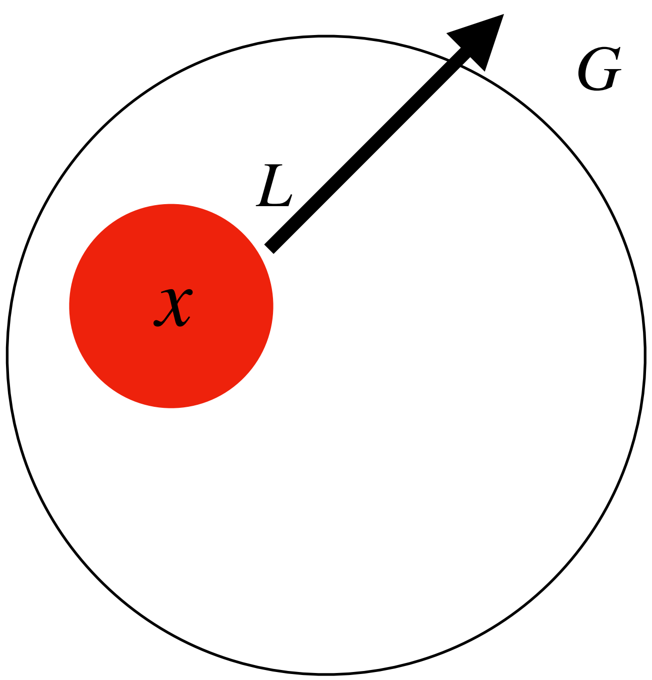
Local V… …nnectivity

Input: LocalVC(…

Output: Either

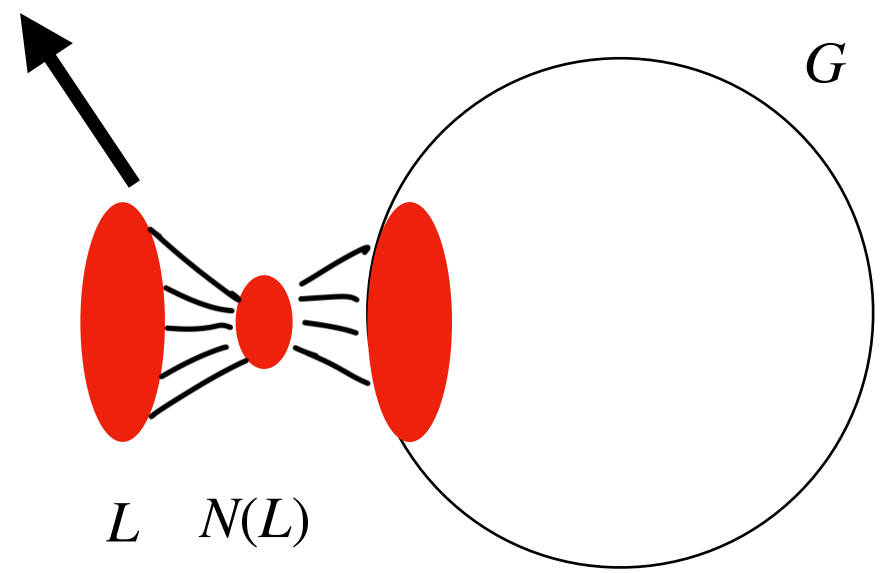
(1) declare that n…

where $L \ni x, |L| \le$ …



…2) a vertex-set L

where $|N(L)| < k$



Reading

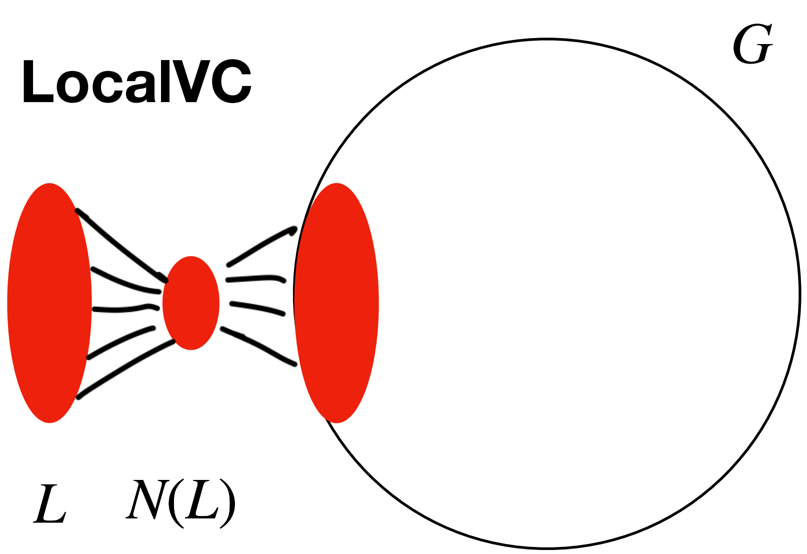
$O(\nu k)$

edges

possible

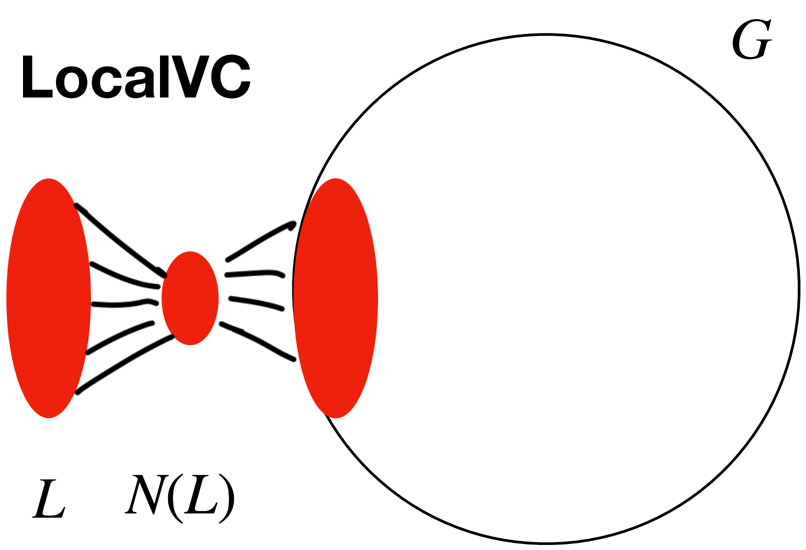
History

LocalVC



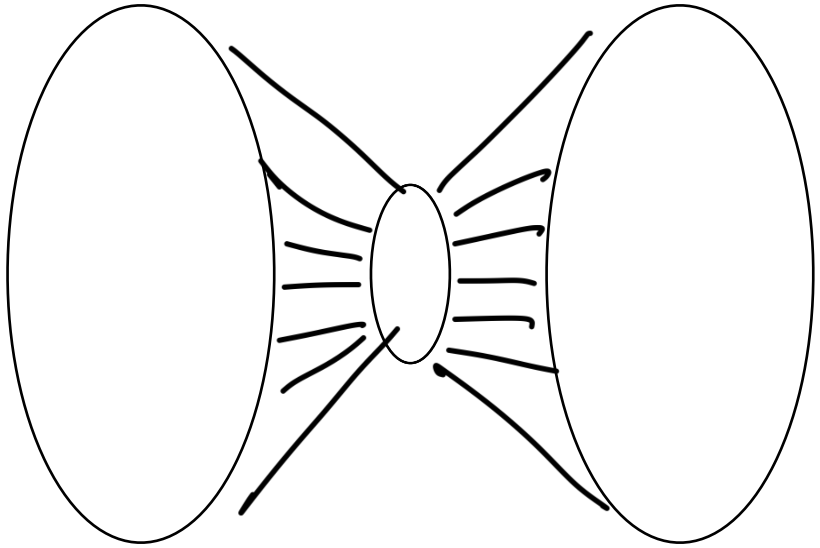
	LocalVC	Approximate LocalVC	Deterministic
This work (STOC '19)	$\nu^{1.5}k$	$\frac{\nu^{1.5}}{\epsilon^{1.5}\sqrt{k}}$	Yes
Adaptation of (Chechik, Hansen, Italiano, Loitzenbauer, Parotsidis SODA'17)	νk^k	-	Yes
Forster Young'19	νk^2	-	No
Our follow-up'19	νk^2	$\frac{\nu k}{\epsilon}$	No

Part 2.2:
Vertex Connectivity
via LocalVC



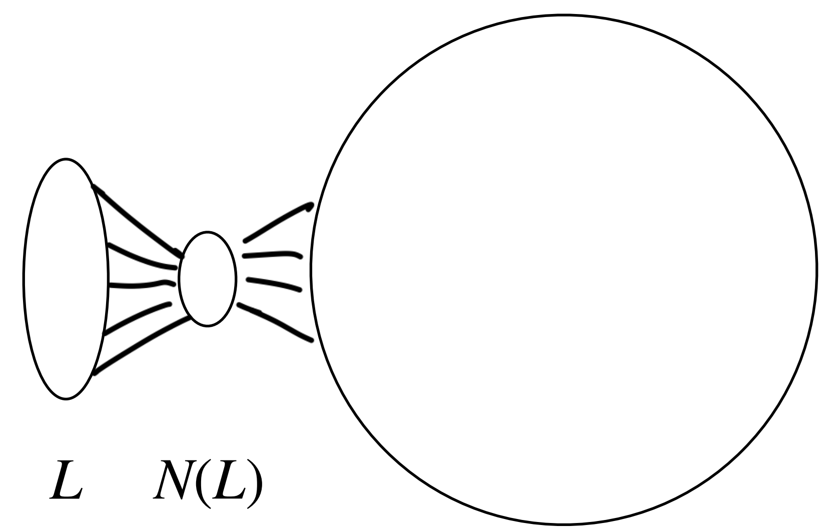
Suppose *G* has a vertex cut of size less than *k*

Highly balanced



$|L| \geq n/10$ $|R| \geq n/10$

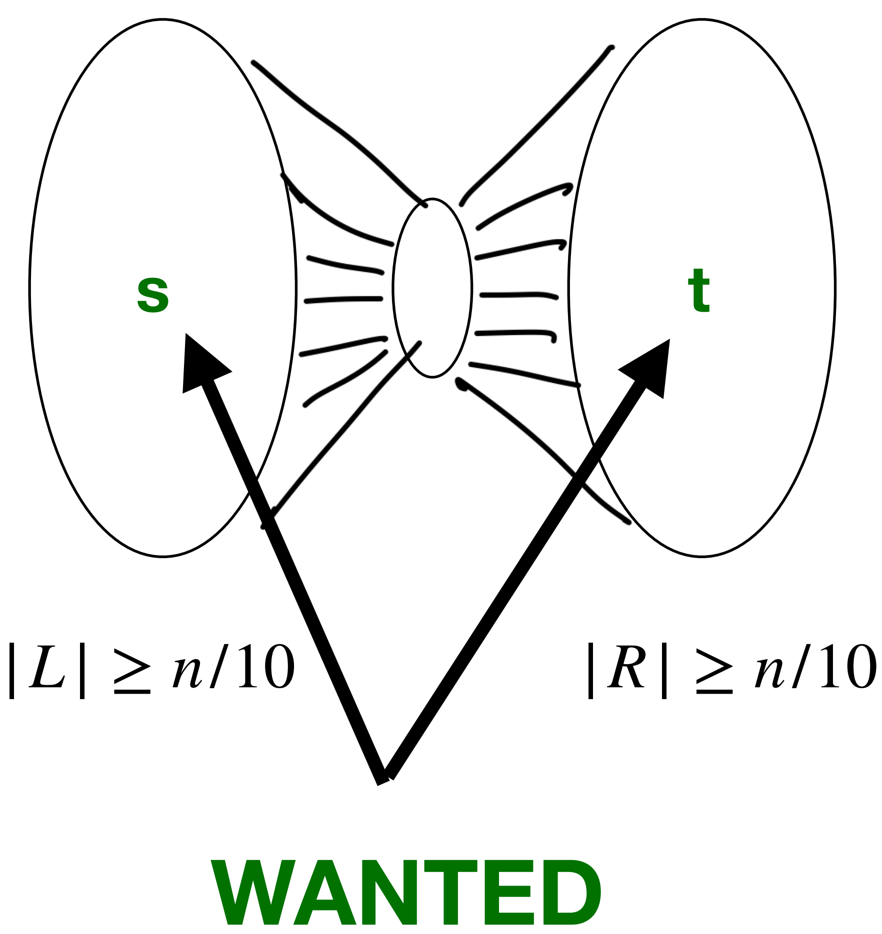
Highly lopsided



$|L| = \sqrt{n}$

Suppose *G* has a vertex cut of size less than *k*

Highly balanced



Fact: Compute s, t-connectivity by s, t-Max-Flow in $O(mk) = O(n)$

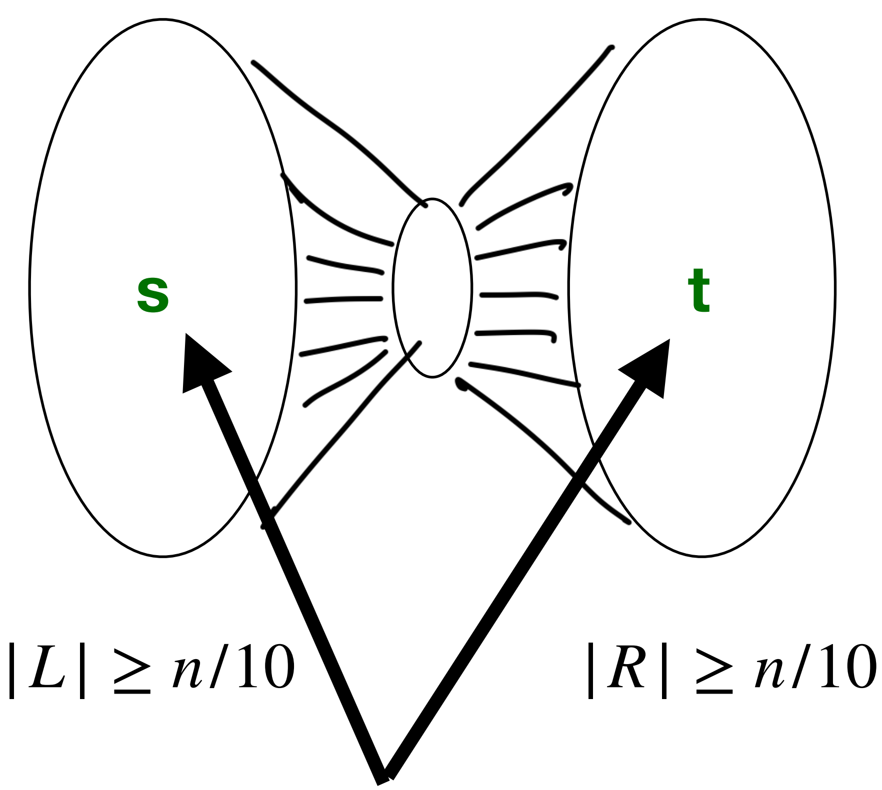
(e.g. Ford Fulkerson)

Suppose $|L| \geq n/10, |R| \geq n/10$

Compute s, t-connectivity

Suppose G has a vertex cut of size less than k

Highly balanced



WANTED

Fact: Compute s, t-connectivity by s, t-Max-Flow in $O(mk) = O(n)$

Suppose $|L| \geq n/10, |R| \geq n/10$

Sample $O(1)$ **pairs of nodes**

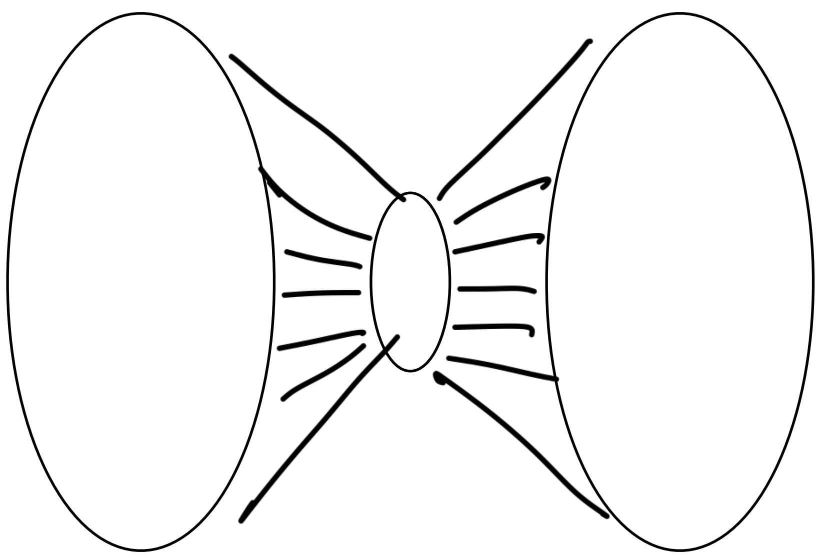
For each pair,

Compute s, t-connectivity

Total Time $= O(1) \times O(n) = O(n)$

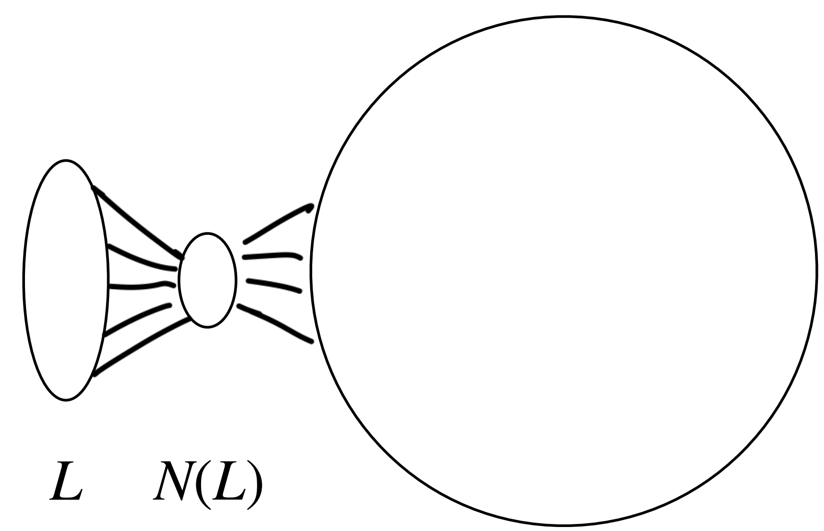
Suppose G has a vertex cut of size less than k

Highly balanced



$|L| \geq n/10$ $|R| \geq n/10$

Highly lopsided



$|L| = \sqrt{n}$

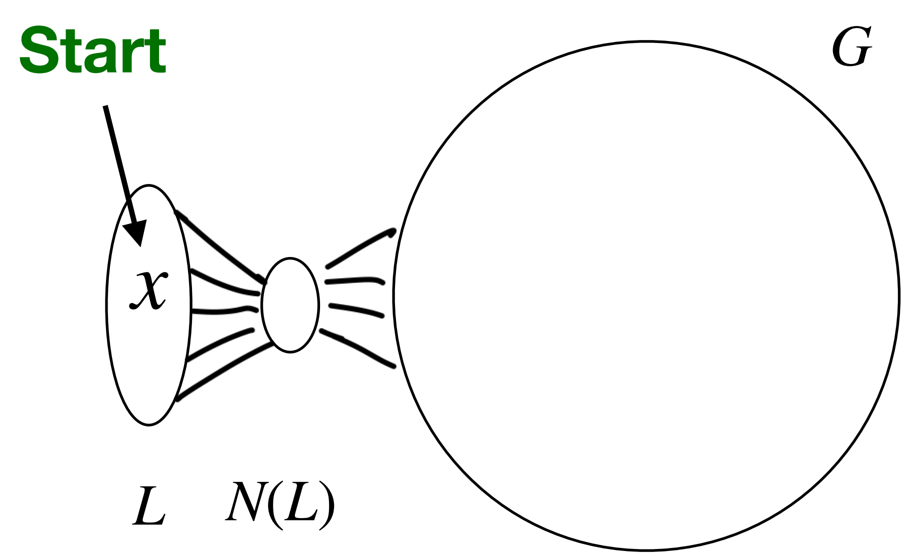
Suppose G has a vertex cut of size less than k

New

Theorem: Local Vertex Connectivity

$|L| \leq \nu, |N(L)| < k$ **in** $O(\nu k^2)$

Highly lopsided



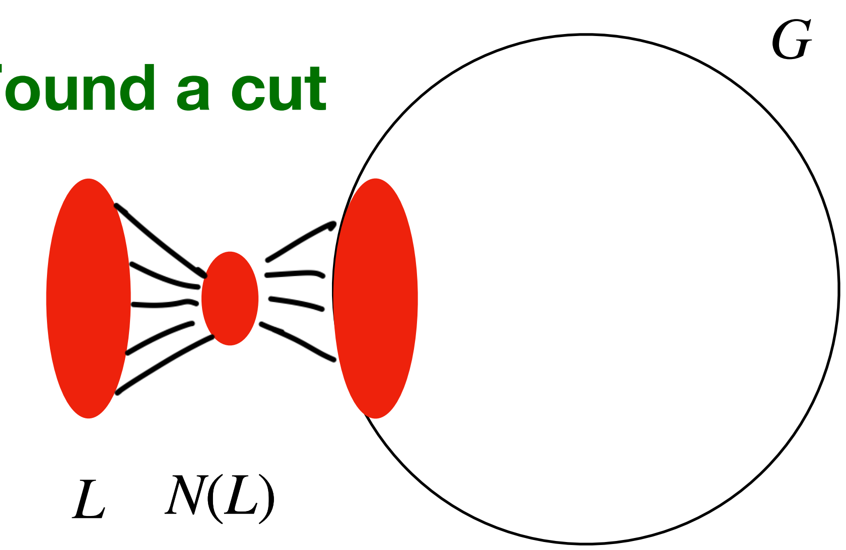
Suppose $|L| = \sqrt{n}$

Compute LocalVC$(x, \nu := \sqrt{n}, k)$

Suppose G has a vertex cut of size less than k

Highly lopsided

Found a cut



New

> **Theorem: Local Vertex Connectivity**
>
> $|L| \leq \nu, |N(L)| < k$ **in** $O(\nu k^2)$

Suppose $|L| = \sqrt{n}$

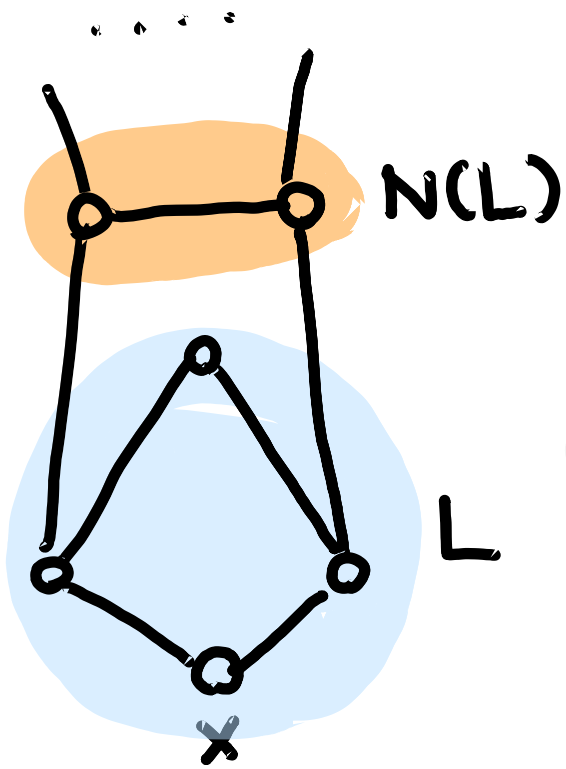
Sample $O(\sqrt{n})$ **nodes**

For each node,

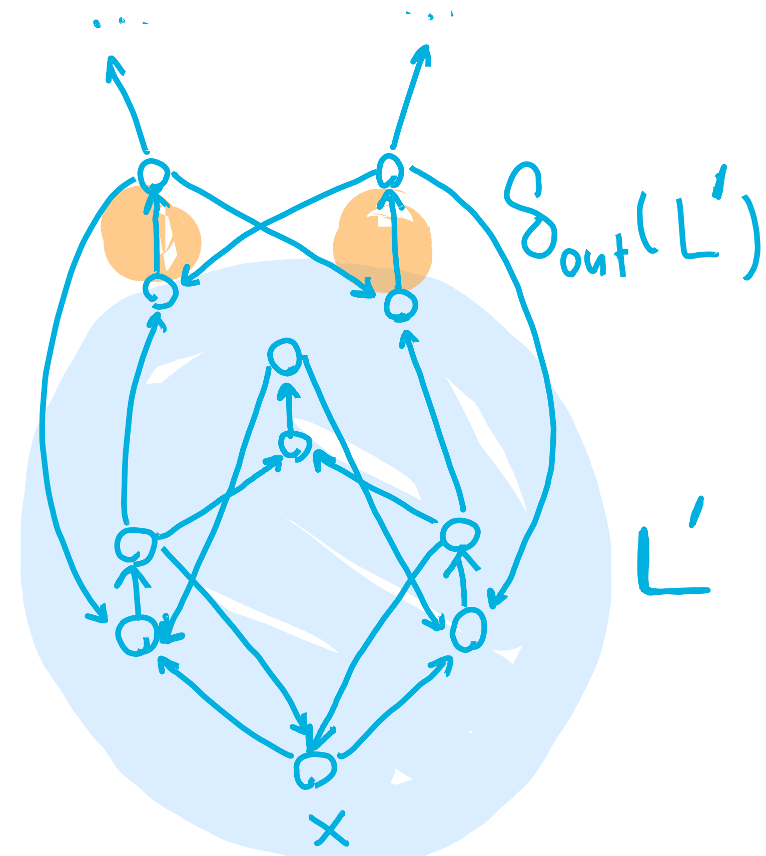
Compute LocalVC$(x, \nu := \sqrt{n}, k)$

Total Time $= O(\sqrt{n}) \times O(\sqrt{n}k^2) = O(n)$

Part 3: Local Vertex Connectivity in $O(\nu k^2)$



To

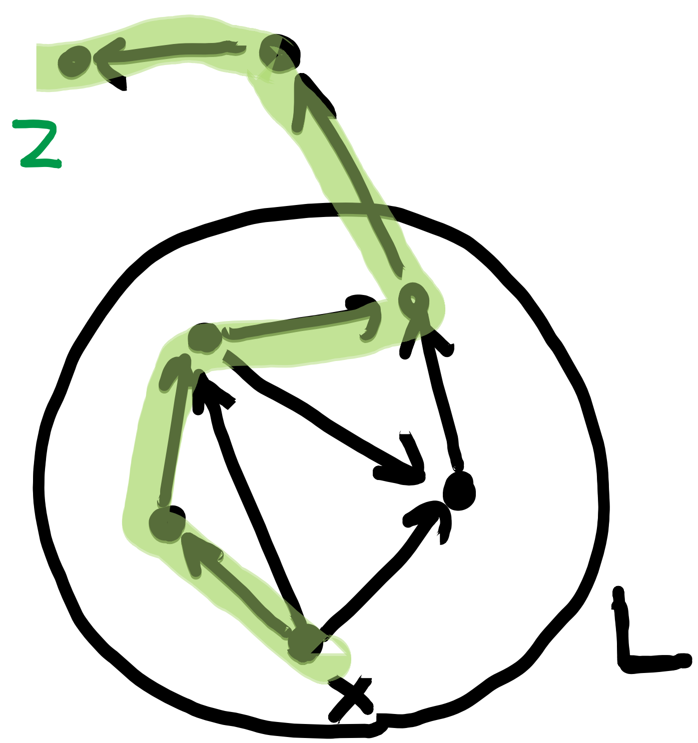


Vertex Connectivity

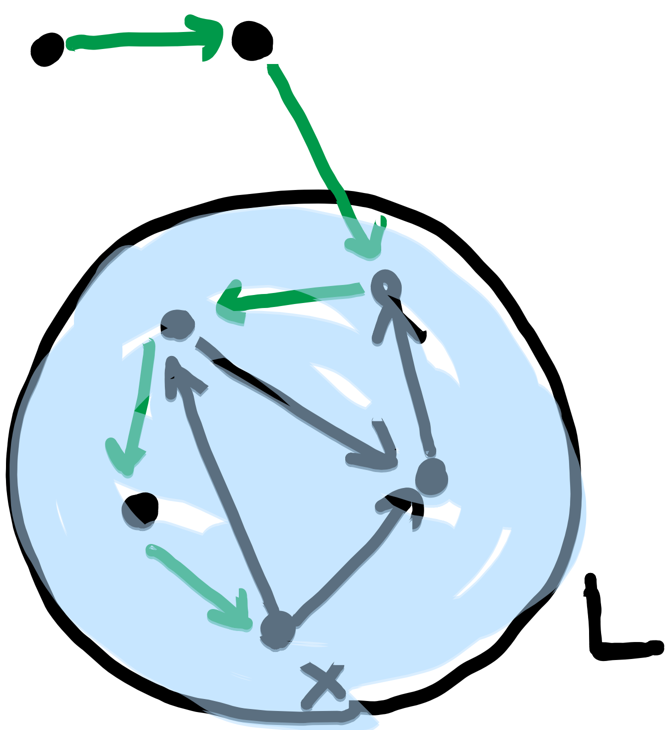
Directed Edge Connectivity

Local Edge Connectivity

Suppose *L* has 1 leaving edge



DFS exactly
$\nu + 1$
edges



1. Reverse x,z-path
2. DFS will get stuck
and we get *L*

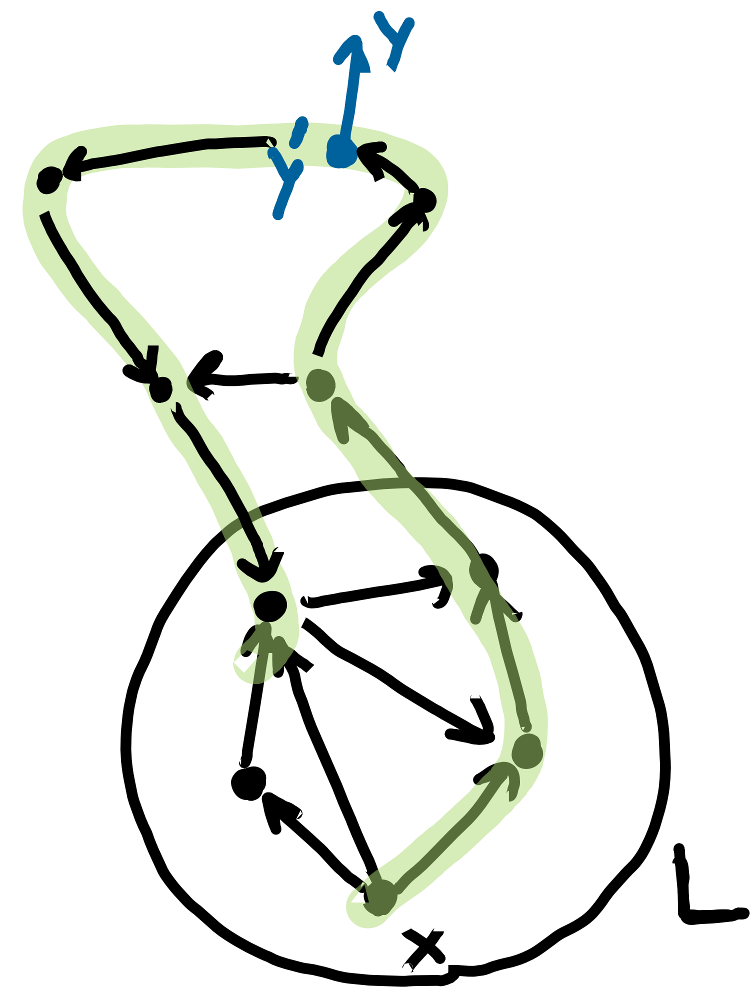
Local Edge Connectivity

Complete pseudocode:

Repeat *k* times

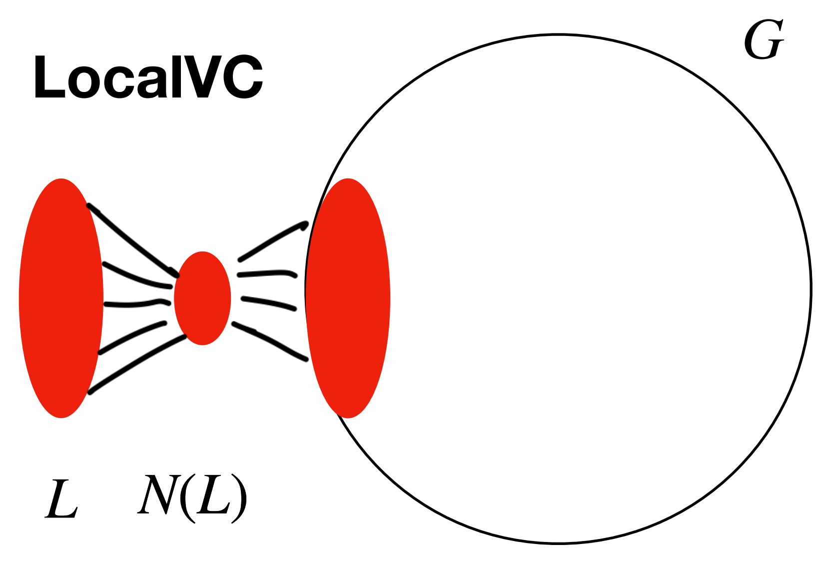
1. Grow DFS tree from *x* using exactly νk edges
2. **If** stuck, **return** a cut from DFS
3. Sample an explore edge (y’,y)
4. Reverse the x,y’-path

Terminate with no cut.



Take Home Message:

"Local algorithms are useful for vertex connectivity"



Open problems

1. **Deterministic** in near-linear time when $m = O(n)$?
   $O(n^{1.8})$-time by [LNSY'19] using expander and page rank.

2. $o(n^3)$ time when $k = \Omega(n)$ or even in **weighted** graph?
   $O(mn)$-time by [Henzinger Rao Gabow'00]

3. $O(m)$ time better **approx**. algorithm?
   $O(\log n)$ approx. by [Censor-Hillel,Ghaffari, and Khun'14]

4. **Local vertex/edge connectivity**
   $O(\nu k)$ time ? More applications?

Questions?

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