Gioan's Theorem for complete bipartite graphs*

Oswin Aichholzer[1], Man-Kwun Chiu[2], Hung P. Hoang[3], Michael Hoffmann[3], Yannic Maus[1], Birgit Vogtenhuber[1], and Alexandra Weinberger[1]

1 **Institute of Software Technology, Graz University of Technology, Austria**
`[oaich,yannic.maus,bvogt,weinberger]@ist.tugraz.at`
2 **Institut für Informatik, Freie Universität Berlin, Germany**
`chiumk@zedat.fu-berlin.de`
3 **Department of Computer Science, ETH Zürich, Switzerland**
`[hung.hoang|hoffmann]@inf.ethz.ch`

Abstract

For a drawing of a labeled graph, the rotation of a vertex or crossing is the cyclic order of its incident edges, presented by the labels of their other endpoints. The extended rotation system of the drawing is the collection of the rotations of all vertices and crossings. A drawing is simple if each pair of edges has at most one common point. Gioan's Theorem states that for any two simple drawings of the complete graph K_n with the same crossing edge pairs, one drawing can be transformed into the other by a sequence of *triangle flips* (a.k.a. Reidemeister moves of Type 3). Intuitively, this operation refers to the act of moving one edge of a triangular cell formed by three pairwise crossing edges over the opposite vertex of the cell.

We investigate to what extent Gioan's Theorem generalizes to other classes of graphs. On the one hand, we show that it holds for complete bipartite graphs $K_{m,n}$, provided that the two drawings share the same extended rotation system. Note that the assumption is also implicit in Gioan's Theorem, because for simple drawings of the complete graph the crossing edge pairs uniquely determine the extended rotation system; however, this is not the case for complete bipartite graphs. Our proof uses a Carathéodory-type theorem for simple drawings of complete bipartite graphs, which may be of independent interest. On the other hand, we show that the theorem does not hold if the graph is slightly sparser: When removing two edges from $K_{m,n}$, there exist two simple drawings with the same extended rotation system that cannot be transformed into each other using triangle flips.

1 Introduction

Given a simple drawing of a graph $G = (V, E)$ on the sphere $\mathcal{S}$, an *edge fragment* is a maximal connected part of an edge that does not contain any endpoint or crossing. The *rotation of a vertex* is the clockwise circular order of incident edges. The *rotation of a crossing* χ is the clockwise cyclic order of the four vertices of the crossing edge pair which is induced by the cyclic order of edge fragments around χ. (In other words, the rotation of a crossing χ is the rotation of an additional degree-4 vertex v_χ obtained by splitting the crossing edge pair at χ and replacing χ by v_χ.) The *extended rotation system* (ERS) of a drawing is the collection of rotations of all vertices and crossings.

* Research on this work was initiated during the 5th D-A-CH Workshop on Arrangements and Drawings in March 2021. We thank all participants of this workshop for the great atmosphere and fruitful discussions. O.A. and A.W. partially supported by the Austrian Science Fund (FWF), project W1230. M.C. partially supported by ERC StG 757609. M.H. supported by the Swiss National Science Foundation within the collaborative D-A-CH project *Arrangements and Drawings* as SNSF project 200021E-171681. B.V. partially supported by the Austrian Science Fund within the collaborative D-A-CH project *Arrangements and Drawings* as FWF project I 3340-N35.

38th European Workshop on Computational Geometry, Perugia, Italy, March 14–16, 2022.
This is an extended abstract of a presentation given at EuroCG'22. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

A *crossing triangle* is a cell in the subdrawing of three pairwise crossing edges that is bounded by three edge fragments. To define the orientation of a crossing triangle, we fix an arbitrary orientation for each edge of G. The *orientation* of a crossing triangle Δ is the parity (odd or even) of the number of edges that bound Δ and where Δ lies to the left of the edge. A crossing triangle Δ is *invertible* if there exists another simple drawing of the same graph and with the same extended rotation system (ERS) in which Δ appears in the opposite orientation. A *triangle flip* is the elementary operation of changing the orientation of an unintersected crossing triangle by a local transformation of the given drawing; see Figure 1.

Figure 1 Two drawings of $K_{3,3}$ that can be transformed into each other via one triangle flip.

Two simple drawings γ and η of G are strongly isomorphic, denoted by $\gamma \cong \eta$, if there exists an orientation-preserving homeomorphism of $\mathcal{S}$ that maps γ to η, that is, $\gamma_v \mapsto \eta_v$, for all $v \in V$, and $\gamma_e \mapsto \eta_e$, for all $e \in E$. By Kynčl [5], the following combinatorial formulation is equivalent for connected drawings: (1) the same pairs of edges cross (this is called *weak isomorphism*); (2) the order of crossings along each edge is the same; and (3) the drawings have the same ERS. In this work, strongly isomorphic drawings are considered the same.

Gioan's Theorem [4] states that any two weakly isomorphic simple drawings of K_n can be transformed into each other via a sequence of triangle flips. Gioan announced his theorem in 2005 [4]. The original presentation contained a proof sketch, but a full proof was published only 10 years later by Arroyo, McQuillan, Richter, and Salazar [1], who also coined the name "Gioan's Theorem". In 2021, Schaefer generalized Gioan's Theorem to slightly sparser graphs by proving that any two weakly isomorphic simple drawings of $K_n \setminus M$, where M is a non-perfect matching, can be transformed into each other using triangle flips [7]. His work also includes an alternative proof for Gioan's Theorem.

Our main result is that an analogue of Gioan's Theorem also holds for simple drawings of much sparser graphs, namely, for complete bipartite graphs. To show this, we rephrase the statement to require both drawings to have the same ERS.

▶ **Theorem 1.1.** *Let D_1 and D_2 be two simple drawings of $K_{m,n}$, $m, n \geq 1$, on the sphere with the same ERS. Then there is a sequence of triangle flips that transforms D_1 into D_2.*

Note that triangle flips only change the order of crossings along edges. Hence, having the same ERS is a necessary requirement for any two drawings of any graph to be transformable into each other via triangle flips. For the complete graph, the requirement that the drawings have the same crossing edge pairs is equivalent to the requirement that they have the same ERS because the crossing edge pairs of a drawing uniquely determine its ERS [5, 6]. However, for complete bipartite graphs this is not the case, as two simple drawings of $K_{m,n}$ with the same crossing edge pairs might have different ERSs; see Figure 2 for an example.

We also show that both Gioan's Theorem and our Theorem 1.1 are almost tight.

▶ **Theorem 1.2.** *For any $m \geq 2$ and $n \geq 3$ and $K_{m,n}$ minus two edges, there exist two simple drawings with the same ERS that cannot be transformed into each other using triangle flips. The same holds for any $n \geq 5$ and K_n minus a four-cycle C_4.*

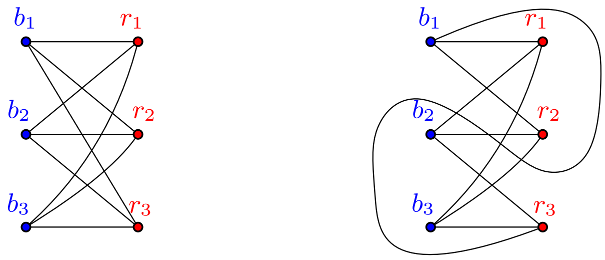

Figure 2 Two simple drawings of $K_{3,3}$ with the same crossing edge pairs but different ERSs.

In particular, the first part of Theorem 1.2 implies that an analogue to Schaefer's generalization of Gioan's Theorem for K_n minus a non-perfect matching cannot be achieved for complete bipartite graphs, as not even a generalization from $K_{m,n}$ to $K_{m,n}$ minus a matching of size two holds. Moreover, note that $K_{m,n}$ with $m \geq 4$ and $n \geq 1$ is a subgraph of $K_{n+m} \setminus C_4$. Hence the second part of Theorem 1.2 implies that—quite counterintuitively—the set of graphs for which a Gioan-type statement holds is not closed under adding edges.

To prove Theorem 1.1, we use a similar approach as Arroyo et al. [1]. In their proof, they iteratively transform one of the drawings so as to increase the parts of both drawings that are strongly isomorphic. However, several ingredients that are necessary for this transformation are known properties of drawings of complete graphs or follow directly, while it was unknown whether analogous statements hold for drawings of complete bipartite graphs. Hence, for our proof, we discover a number of useful, fundamental properties of simple drawings of complete bipartite graphs. For example, we establish an analogue to Carathéodory's Theorem for simple drawings of $K_{m,n}$.

The classic Carathéodory Theorem states that if a point $p \in \mathbb{R}^2$ lies in the convex hull of a set $A \subset \mathbb{R}^2$ of $n \geq 3$ points, then there exists a triangle spanned by points of A that contains p. In the terminology of drawings, this means that if a point p lies in a bounded cell of a straight-line drawing D of K_n in the plane, then there also exists a 3-cycle C of D so that p lies in the bounded cell of C. This statement has been generalized to simple (not necessarily straight-line) drawings of K_n [2, 3]. However, it clearly does not generalize to arbitrary (non-complete) graphs. A natural question is, for which classes of graphs this statement, or a variation of it, holds. We show that it holds for complete bipartite graphs if we replace the (non-existing) 3-cycle by a 4-cycle, which is the shortest available cycle.

▶ **Theorem 1.3** (Carathéodory's Theorem for simple drawings of $K_{m,n}$). *Let D be a simple drawing of $K_{m,n}$ in the plane, for $m, n \geq 2$, and let p be a point in some bounded cell of D. Then there exists a 4-cycle C of D such that p is contained in a bounded cell of C. This statement is tight in the sense that it does not hold for $K_{m,n}$ minus one edge.*

Outline. We prove Theorem 1.1 in Section 2. The proof relies on several lemmata, whose proofs are deferred to the upcoming full version of this paper. A sketch of the proof of Theorem 1.2 can be found in Section 3.

2 Proof of Gioan's Theorem for simple drawings of $K_{m,n}$

Denote the bipartition sets by $B = \{b_1, b_2, ..., b_m\}$ and $R = \{r_1, r_2, ..., r_n\}$. Let $D :\cong D_1$. We will do triangle flips in D, by this changing D, until we obtain $D \cong D_2$. We iteratively consider the vertices $r_1, \ldots, r_n$. For each vertex r_i, we iteratively consider the incident edges $r_ib_1 \dots, r_ib_m$. We denote by $K_{m,i}$ the subgraph of $K_{m,n}$ induced by B and the vertices $r_1, r_2, \dots, r_i$, and let $X_{i,j} = K_{m,i-1} \cup \{r_ib_k : 1 \leq k \leq j\}$ for $0 \leq k \leq m$, with $X_{i,0} = K_{m,i-1}$.

When considering an edge r_ib_j, the goal is to establish $D[X_{i,j}] \cong D_2[X_{i,j}]$, where $D[X_{i,j}]$ and $D_2[X_{i,j}]$ are the according subdrawings of D and D_2, respectively.

For the base case $i = 1$ observe that $D[K_{m,1}] \cong D_2[K_{m,1}]$ because there is only one simple drawing of $K_{m,1}$ (our graphs are labeled but the ERS is given).

For the general case $2 \leq i \leq n$ and $1 \leq j \leq m$, assume that $D[X_{i,j-1}] \cong D_2[X_{i,j-1}]$. To handle the case $j = 1$, we first argue that the position of vertex r_i is consistent between $D[K_{m,i-1}]$ and $D_2[K_{m,i-1}]$. To show this, we use the following lemma, whose proof relies on Theorem 1.3 (Carathéodory's Theorem for simple drawings of $K_{m,n}$).

▶ **Lemma 2.1.** *Let F be a simple drawing of $K_{m,n}$, $m, n \geq 1$, on the sphere. For any vertex v in F, the ERS of F uniquely determines in which cell of $F' := F \setminus \{v\}$ the vertex v lies.*

Since $D[K_{m,i-1}] \cong D_2[K_{m,i-1}]$, the two drawings topologically have the same cells. As D and D_2 have the same ERS, by Lemma 2.1 applied to $F = D[K_{m,i}]$ and to $D_2[K_{m,i}]$, both times with $v = r_i$, we conclude that r_i lies in the same cell in $D[K_{m,i-1}]$ and $D_2[K_{m,i-1}]$.

Now consider the edge r_ib_j. The aim is to use a sequence of triangle flips to transform D such that $D[X_{i,j}] \cong D_2[X_{i,j}]$. Let e_1 denote the curve that represents r_ib_j in D. We imagine to add another copy $\widetilde{e_2}$ of r_ib_j to D, which corresponds to the curve e_2 that represents the edge r_ib_j in D_2 and serves as a "target" curve which we aim to transform e_1 into.

▶ **Lemma 2.2.** *There exists a simple curve $\widetilde{e_2}$ such that $D[X_{i,j-1}] \cup \widetilde{e_2} \cong D_2[X_{i,j}]$ and e_1 and $\widetilde{e_2}$ have finitely many intersections in $D[X_{i,j}] \cup \widetilde{e_2}$.*

Now fix such a curve $\widetilde{e_2}$. Then $\Gamma = e_1 \cup \widetilde{e_2}$ forms a (not necessarily simple) closed curve. With the next lemma, we show that there is a *lens* in Γ which we can use as a starting point for transforming e_1 to $\widetilde{e_2}$. A *lens* in Γ is a cell whose boundary is formed by exactly two edge fragments of Γ, one from e_1 and one from $\widetilde{e_2}$.

▶ **Lemma 2.3.** *In Γ there is a lens that does not contain any vertex of $K_{m,i}$.*

Now consider a lens L as guaranteed by Lemma 2.3. While L does not contain any vertex of $D[X_{i,j-1}]$, it may contain crossings of $D[X_{i,j-1}]$. As a next step, we aim to transform D using triangle flips such that L does not contain any crossings of $D[X_{i,j-1}]$. Let $\chi \in L$ be a crossing of two edges a_1, a_2 in $D[X_{i,j-1}]$. As r_i and b_j are the only vertices on $e_1 \cup \widetilde{e_2}$, it follows that each of a_1, a_2 crosses ∂L twice; as both D and D_2 are simple drawings, one of these crossings is with e_1 and the other is with $\widetilde{e_2}$. Thus, a_1, a_2, and e_1 form a crossing triangle Δ_{e_1}. Moreover, the corresponding crossing triangle in D_2 has the opposite orientation, and hence Δ_{e_1} is invertible. By the following lemma, Δ_{e_1} is empty of *all* vertices of D (we already knew this for the vertices of $K_{m,i}$, but not yet for $r_{i+1}, \dots, r_n$).

▶ **Lemma 2.4** (Invertible triangles are empty). *Let D be a simple drawing of $K_{m,n}$ and Δ be an invertible crossing triangle in D. Then all vertices of D lie outside Δ.*

Since Δ_{e_1} is empty of all vertices, all edges crossing Δ_{e_1} can be "swept" outside Δ_{e_1} using a finite sequence of triangle flips (analogous to topological sweeps). None of those flips increases the number of crossings in L (while some of them might decrease this number) and after them, Δ_{e_1} is unintersected. Finally, we also flip Δ_{e_1} so that $\chi \notin L$.

Processing all remaining crossings inside L in the described fashion, we establish that in the resulting drawing, the lens L does not contain any vertex or crossing of $D[X_{i,j-1}]$. In other words, locally around L, the edge e_1 is topologically identical to $\widetilde{e_2}$ with respect to $D[X_{i,j-1}]$. Thus, we can adapt $\widetilde{e_2}$ by replacing its edge part on ∂L with a close copy of the edge part of e_1 on ∂L, effectively removing the lens L from Γ. As a result, the edges e_1 and $\widetilde{e_2}$ have fewer crossings than before in D, and the parameters D and $\Gamma = e_1 \cup \widetilde{e_2}$ again meet the conditions of Lemma 2.3. Repeatedly applying this procedure to the next cell (which exists by Lemma 2.3), we eventually obtain a drawing $D[X_{i,j}] \cup \widetilde{e_2}$ where e_1 and $\widetilde{e_2}$ do not cross, and hence Γ is a simple closed curve. By Lemma 2.3, one of the two cells bounded by Γ contains no vertices of $D[X_{i,j}]$. So after one last round of transformations as described above, we obtain a drawing $D[X_{i,j}] \cup \widetilde{e_2}$ in which all vertices and crossings lie on one side of Γ. Hence we have obtained $D[X_{i,j}] \cong D_2[X_{i,j}]$. Processing all vertices r_i, for $i = 2, \ldots, n$, and in turn handling all edges incident to r_i eventually yields a drawing $D \cong D_2$.

3 Sketch of the proof of Theorem 1.2

Figure 3 depicts the drawings we use in the proof of Theorem 1.2. The first row contains drawings of $K_{m,n}$ minus two adjacent edges, the second one drawings of $K_{m,n}$ minus two disjoint edges, and the third row is for K_{n+m} minus a C_4. In each row, the (green) edge r_1b_1 crosses b_2r_2 and b_2r_3 in a different order and these three edges do not form any crossing triangle. Thus, the drawings cannot be transformed into each other via triangle flips.

4 Conclusion & Open Questions

We have shown Gioan's Theorem for complete bipartite graphs (Theorem 1.1) and that an according statement does not hold for $K_{m,n}$ minus two edges or K_n minus a C_4 (Theorem 1.2). These results relevantly extend previous results [1, 4, 7] and show that the class of graphs for which an according statement holds is also not closed under adding edges. We believe that our result can be extended to complete k-partite graphs. But a complete characterization of graphs for which an according statement holds remains open.

▶ **Question 1.** *What is a complete characterization of all graphs for which Gioan's Theorem holds, that is, for which graphs is it true that any two drawings with the same ERS can be transformed into each other?*

Further, we have shown that an analogue of Caratheorody's Theorem holds for simple drawings of $K_{m,n}$ (Theorem 1.3). It would be interesting to know for which further graphs a similar statement is true.

Finally, in our proof of Theorem 1.1, we did not address algorithmical questions, and neither did the according proofs for Gioan's Theorem for K_n. Naturally, the minimum flip distance, that is, the minimum number of triangle flips that need to be done to transform the drawings, is of interest.

▶ **Question 2.** *What is the worst case minimum flip distance between two simple drawings of $K_{m,n}$ with the same ERS? And what is the worst case minimum flip distance between two simple drawings of K_n with the same rotation system?*

References

1 Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Drawings of K_n with the same rotation scheme are the same up to triangle-flips Gioan's theorem). *Australasian J. Combinatorics*, 67(2):131–144, 2017. URL: `https://ajc.maths.uq.edu.au/pdf/67/ajc_v67_p131.pdf`.

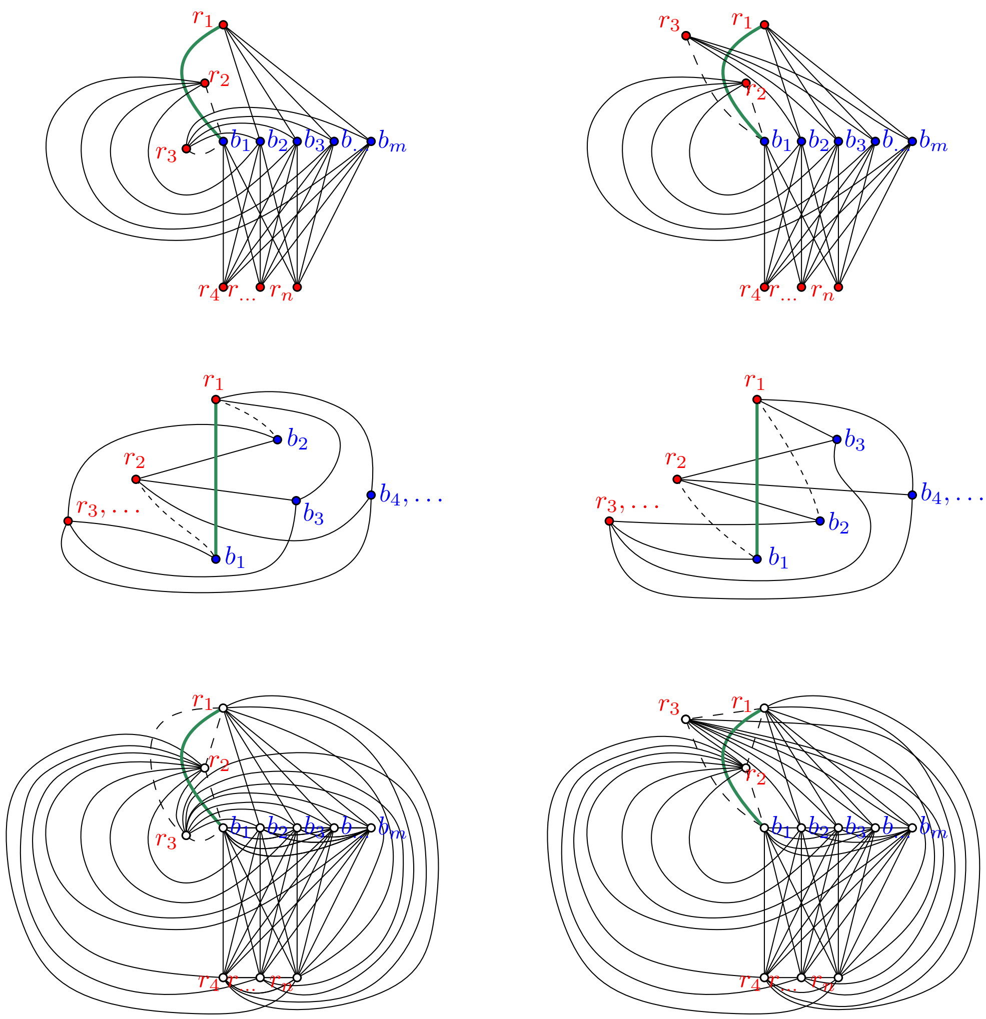

Figure 3 Constructions used in the proof of Theorem 1.2. Dashed arcs indicate omitted edges.

2 Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete Comput. Geom.*, 53:107–143, 2015. doi:10.1007/s00454-014-9644-z.

3 Helena Bergold, Stefan Felsner, Manfred Scheucher, Felix Schröder, and Raphael Steiner. Topological drawings meet classical theorems from convex geometry. In *Proc. 28th Internat. Sympos. Graph Drawing*, volume 12590 of *Lecture Notes Comput. Sci.*, pages 281–294. Springer-Verlag, 2020. doi:10.1007/978-3-030-68766-3_22.

4 Emeric Gioan. Complete graph drawings up to triangle mutations. In *Proc. 31st Internat. Workshop Graph-Theoret. Concepts Comput. Sci.*, volume 3787 of *Lecture Notes Comput. Sci.*, pages 139–150. Springer, 2005. doi:10.1007/11604686_13.

5 Jan Kynčl. Enumeration of simple complete topological graphs. *European J. Combinatorics*, 30:1676–1685, 2009. doi:10.1016/j.ejc.2009.03.005.

6 Jan Kynčl. Simple realizability of complete abstract topological graphs in P. *Discrete Comput. Geom.*, 45:383–399, 2011. doi:10.1007/s00454-010-9320-x.

7 Marcus Schaefer. Taking a detour; or, Gioan's theorem, and pseudolinear drawings of complete graphs. *Discrete & Computational Geometry*, 66:12–31, 2021. doi:10.1007/s00454-021-00296-2.