Column Planarity and Partial Simultaneous Geometric Embedding for Outerplanar Graphs*

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Abstract

Given a graph $G = (V, E)$, a set $R \subseteq V$ is *column planar* in G if we can assign x-coordinates to the vertices in R such that every assignment of y-coordinates to R gives a partial embedding of G that can be completed to a plane straight-line embedding of the whole graph. This notion is strongly related to unlabeled level planarity. We prove that every outerplanar graph on n vertices contains a column planar set of size at least $n/2$.

We use this result to show that every pair of outerplanar graphs G_1 and G_2 on the same set V of n vertices admit an $(n/4)$-*partial simultaneous geometric embedding* (PSGE): a plane straight-line embedding of G_1 and a plane straight-line embedding of G_2 such that $n/4$ vertices are mapped to the same point in the two drawings. This is a relaxation of the well-studied notion of *simultaneous geometric embedding*, which is equivalent to n-PSGE.

1 Introduction

The notion of *column planarity* was originally introduced by Evans et al. [6]. Informally, given a graph $G = (V, E)$, a set $R \subseteq V$ is *column planar* in G if we can assign x-coordinates to the vertices in R such that any assignment of y-coordinates to R gives a partial embedding of G that can be completed to a plane straight-line embedding of the whole graph. More formally, R is column planar in G if there exists an injection $\rho : R \to \mathbb{R}$ such that for all ρ-*compatible* injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G where each $v \in R$ is embedded at $(\rho(v), \gamma(v))$. Injection γ is ρ-compatible if the combination of ρ and γ does not embed three vertices on a line. See Figure 1.

Column planarity is both a generalization and a strengthening of *unlabeled level planarity (ULP)*. A graph $G = (V, E)$ is ULP if for all injections $\gamma : V \to \mathbb{R}$, there exists an injection $\rho : V \to \mathbb{R}$, so that embedding each $v \in V$ at $(\rho(v), \gamma(v))$ results in a plane straight-line embedding of G. If V is column planar in G, then G is ULP. Estrella-Balderrama, Fowler and Kobourov [5] introduced ULP graphs and characterized ULP trees in terms of forbidden subgraphs. Fowler and Kobourov [7] extended this characterization to general ULP graphs. ULP graphs are exactly the graphs that admit a simultaneous geometric embedding with a monotone path: this was the original motivation for studying them.

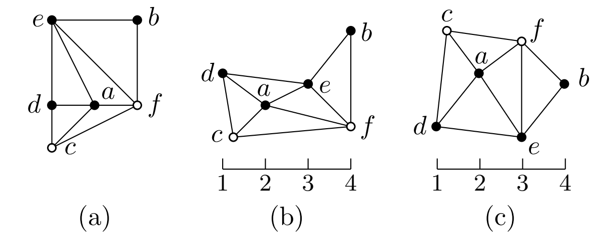

Figure 1: A graph with column planar set $R = \{a, b, d, e\}$ and $\rho = \{a \mapsto 2, b \mapsto 4, d \mapsto 1, e \mapsto 3\}$. (b-c) depict completed embeddings for two different assignments $\gamma : R \to \mathbb{R}$.

Following the characterization of ULP graphs, Di Giacomo et al. [4] introduce a family of graphs called *fat caterpillars* and prove that they are exactly the graphs $G = (V, E)$ where V is column planar in G (they call such graphs EAP graphs). Evans et al. [6] prove near-tight bounds for column planar subsets of trees: any tree on n vertices contains a column planar set of size at least $14n/17$ and for any $\epsilon > 0$ and any sufficiently large n, there exists an n-vertex tree in which every column planar subset has size at most $(5/6 + \epsilon)n$. Furthermore, they show that outerpaths (outerplanar graphs whose weak dual is a path) always contain a column planar subset of size at least $n/2$. In this paper, we prove that this bound holds for general outerplanar graphs.

Evans et al. [6] apply their results on column planarity to give bounds for *k-partial simultaneous geometric embedding (k-PSGE)*. This problem is a relaxation of *simultaneous geometric embedding (SGE)*, which was introduced by Brass et al. [3]. Given graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of n vertices, an SGE of G_1 and G_2 is a pair of plane straight-line embeddings $\varphi_1 : V \to \mathbb{R}^2$ of G_1 and

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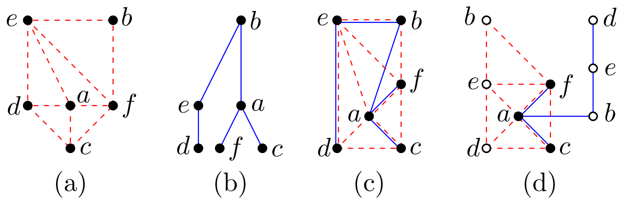

Figure 2: (a-b) Graphs G_1 and G_2 on the same vertex set. (c) An SGE of G_1 and G_2. (d) A 3-PSGE of G_1 and G_2.

$\varphi_2 : V \to \mathbb{R}^2$ of G_2 such that $\varphi_1 = \varphi_2$. See Figure 2c. Conversely, in a k-PSGE of G_1 and G_2, we require $\varphi_1(v) = \varphi_2(v)$ only for some k vertices in V. More formally, a k-PSGE of G_1 and G_2 is a pair of injections $\varphi_1 : V \to \mathbb{R}^2$ and $\varphi_2 : V \to \mathbb{R}^2$ such that (i) the straight-line drawings $\varphi_1(G_1)$ and $\varphi_2(G_2)$ are both plane; (ii) if $\varphi_1(v_1) = \varphi_2(v_2)$ then $v_1 = v_2$; and (iii) $\varphi_1(v) = \varphi_2(v)$ for at least k vertices $v \in V$ [6]. See Figure 2d. An n-PSGE is simply an SGE.

Brass et al. [3] show that two paths, cycles, or caterpillars always admit an SGE. On the negative side, they prove that two outerplanar graphs or three paths sometimes do not admit an SGE. Bläsius et al. [2] give an excellent survey of the subsequent papers on simultaneous embeddings. We highlight the negative result by Geyer et al. [8] that there exist two trees that do not admit an SGE and the result by Angelini et al. [1] that there exist a tree and a path that do not admit an SGE. These negative results motivated the study of PSGE. Evans et al. [6] show that if a set R is column planar in both G_1 and G_2, then G_1 and G_2 admit a $|R|$-PSGE. Di Giacomo et al. [4] independently prove this for $R = V$. Combining their lower bounds on the size of column planar sets with a pigeonhole argument, Evans et al. show that every two trees admit a (11/17)-PSGE.

A result from Goaoc et al. [9] on the untangling of outerplanar graphs, implies that any two outerplanar graphs G_1 and G_2 on n vertices admit a $\sqrt{n/2}$-PSGE.

In this paper, we prove that every outerplanar graph contains a column planar set of size at least $n/2$. We then use this result to show that every two outerplanar graphs on n vertices admit an $(n/4)$-PSGE.

1.1 Outline

We first give an outline of our approach. Consider an outerplanar graph G on n vertices. We first define the *chord graph* of G, which contains only the "long" chords of the graph. We show that the chord graph has an independent set $\mathcal{I}$ of size at least $\frac{n+2}{2}$. We show that $\mathcal{I}$ is almost column planar in G: it suffices to remove at most one vertex. This gives a column planar set of size at least $n/2$ in G.

For our second result, consider two outerplanar graphs G_1 and G_2 on the same set of n vertices. It suffices to compute a set R with $|R| \geq n/4$ that is column planar in both G_1 and G_2. The result of Evans et al. [6] implies then that G_1 and G_2 admit a $|R|$-PSGE. We first compute a column planar set R_1 in G_1. Next, we compute a column planar set R in $G_2[R_1]$ with $|R| \geq n/4$. Since $R \subseteq R_1$, the set R is column planar in both G_1 and G_2, and hence the statement follows.

2 Column Planarity in Outerplanar Graphs

In this section we show that every outerplanar graph has a column planar subset containing at least half of its vertices. Let $G = (V, E)$ be an outerplanar graph with n vertices. Assume without loss of generality that G is maximal outerplanar.

Let $v_0, v_1, \ldots, v_{n-1}$ be the sequence of vertices of V along the unique Hamiltonian cycle of G. Consider the following *removal procedure*: Choose an arbitrary vertex of G of degree two different from v_0 and v_{n-1}, remove it from the graph and repeat recursively. Since every maximal outerplanar graph has two nonadjacent vertices of degree 2, and since removing such a vertex maintains maximal outerplanarity, such a vertex always exists. The *removal order* of the the vertices $V \setminus \{v_0, v_{n-1}\}$ is the order in which they are removed by this procedure. For $0 \leq i < n$, let

$$V(v_i) = \{v_j \in V : v_j \text{ was removed before } v_i\}.$$

Let $N^+(v_i)$ be the closed neighborhood of v_i. For $0 < i < n-1$, the *left index* ℓ_i of v_i is the smallest index such that $v_{\ell_i} \in N^+(v_i)$. Similarly, the *right index* r_i of v_i is the largest index with $v_{r_i} \in N^+(v_i)$. Naturally, $v_{\ell_i} \leq v_i \leq v_{r_i}$.

Lemma 1 *Let v_i be a vertex with $0 < i < n-1$ and suppose that there is a vertex v_j with $i \neq j$ and $\ell_i < j < r_i$. Then all neighbors of v_j are in $V(v_i)$.*

Proof. Let $\ell = \ell_i$ and $r = r_i$ and assume without loss of generality that $i < j$. Since $i < r - 1$, the edge $v_i v_r$ is a chord of G. See Figure 3. Hence, the removal of v_i and v_r splits G into two connected components H_1 and H_2 such that $v_j \in H_1$ and $v_0 \in H_2$. Note that v_j neighbors no vertex in H_2. We claim that all the vertices in $V \setminus V(v_i)$ lie in H_2. If this claim is true, then v_j neighbors no point in $V \setminus V(v_i)$, which proves the statement.

Assume for a contradiction that there is a vertex $v \in V \setminus V(v_i)$ that belongs to H_1. Therefore, v lies after v_i in the removal order. Since (i) there is no edge between a vertex of H_1 and a vertex of H_2, (ii) H_1 contains a vertex after removing v_i (namely v), and (iii) H_2 contains a vertex after removing v_i (namely v_0), the graph $G[V \setminus V(v_i)]$ induced by $V \setminus V(v_i)$ is disconnected. However, the removal procedure described above only removes ears of the graph and cannot disconnect it—a contradiction that comes from assuming that v belongs to H_1. ☐

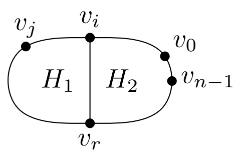

Figure 3: Division into connected components in the proof of Lemma 1.

Let $E_C \subset E$ be the set of all chords of G having endpoints whose removal splits G into components with at least 2 vertices. That is, the chords adjacent to ears of G are not part of E_C; see Figure 4. Let $C = (V, E_C)$ be the *chord graph* of G.

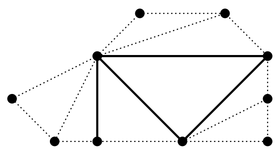

Figure 4: An outerplanar graph $G = (V, E)$. The edge set E_C is drawn solid; the other edges are dotted.

Lemma 2 *Let $\mathcal{I} \subset V$ be an independent set of C such that there is an edge of the Hamiltonian cycle of G whose endpoints are both not in $\mathcal{I}$. Then $\mathcal{I}$ is column planar in G.*

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the sequence of vertices of V along the unique Hamiltonian cycle of G such that v_0 and v_{n-1} are not in $\mathcal{I}$. To set the x-coordinate of the vertices in $\mathcal{I}$, we define the injection $\rho : \mathcal{I} \to \mathbb{R}$ such that $\rho(v_i) = i$.

For any ρ-compatible injection $\gamma : \mathcal{I} \to \mathbb{R}$, we need to show that there exists a plane straight-line embedding of G where each $v_i \in \mathcal{I}$ is embedded at $\varphi(v_i) = (\rho(v_i), \gamma(v_i))$.

We first show that φ is a plane straight-line embedding of the graph $G[\mathcal{I}]$. Since $\mathcal{I}$ is an independent set of C, we know that if two vertices $v_i, v_j \in \mathcal{I}$ are adjacent in G such that $i < j$, then either $j = i + 1$ or $j = i + 2$. Otherwise, the removal of v_i and v_j splits G into two graphs, each with at least two vertices, which implies that the edge v_iv_j belongs to C: a contradiction. Furthermore, if $\{v_i, v_{i+2}\} \in E$ then the neighbours of v_{i+1} are exactly v_i and v_{i+2}. Therefore, φ is a plane straight-line embedding of $G[\mathcal{I}]$.

We now describe an algorithm that places the remaining vertices of V to obtain a plane straight-line embedding of G. The algorithm is incremental and adds one vertex at the time in the order given by the removal order.

Let X_i be the set of vertices that have already been placed, starting with, $X_0 = \mathcal{I}$. We never embed two vertices at the same x-coordinate. We say that the *visibility invariant* holds if each vertex of X_i that neighbors a vertex of $V \setminus X_i$ in G is *visible from below*, i.e., the ray shooting downwards from this vertex intersects no edge of the embedding of $G[X_i]$. We can see that the visibility invariant holds for X_0 as follows. Suppose that there is a vertex v_k that is not visible from below. Then the ray from v_k downward intersects some edge $\{v_i, v_j\}$. Since v_i and v_j are independent in C and since $i < k < j$, we must have $k = i + 1$ and $j = i + 2$. But then the only neighbors of v_k are v_i and v_j, and hence v_k does not neighbor a vertex of $V \setminus X_0$, as required.

For any $i \geq 0$, let v_j be the first vertex in $V \setminus X_i$ according to the removal order and let $X_{i+1} = X_i \cup \{v_j\}$. Let $\ell = \ell_j$ and $r = r_j$ be the left and right indices of v_j, respectively.

We place v_j at coordinates (j, y_j), where y_j is a sufficiently small number such that all neighbors of v_j in X_i are visible from v_j. This number always exist by the visibility invariant and since we never embed two vertices with the same x-coordinate. Because $\ell \leq j \leq r$ by Lemma 1, only vertices strictly between v_ℓ and v_r in the x-order can become not visible from below. However, since $V(v_i) \subset X_{i+1}$, Lemma 1 implies that for every $\ell < k < r$, all neighbors of v_k are in X_{i+1}. Therefore, the visibility invariant is preserved for X_{i+1}.

After this process completes, the only remaining vertices to embed are v_0 and v_{n-1}. Embed v_0 at $x = 0$ and v_{n-1} at $x = n - 1$. Move both down sufficiently far so that the edge $\{v_0, v_{n-1}\}$ does not intersect the drawing so far and so that v_0 and v_{n-1} can both see their neighbors from below. This completes the plane straight-line embedding of G. $\square$

Lemma 3 *The graph C has an independent set of size at least $\frac{n+2}{2}$.*

Proof. Let $\overline{G}$ be the weak dual graph of the complete outerplanar graph G. Let x_i be the number of vertices of degree i in $\overline{G}$. Notice that $\overline{G}$ is a binary tree whose leaves correspond to ears of G. Since the the degree two vertex of an ear in G becomes an isolated vertex in C, we know that C has at least x_1 isolated vertices. Since $\overline{G}$ is a binary tree, we know that $x_1 = x_3 + 2$.

We describe a greedy procedure to construct an independent set $\mathcal{I}$ of C. The algorithm chooses the vertex of smallest degree in the current graph (initially C), adds it to $\mathcal{I}$, and removes its neighbors from the graph. Clearly this procedure generates an independent set. We claim that that $|\mathcal{I}| \geq \frac{n+2}{2}$.

Because C is outerplanar, it is 2-degenerate. Therefore, whenever we add a vertex to $\mathcal{I}$, it has degree 0, 1, or 2. Let n_i be the number of vertices in $\mathcal{I}$ that had degree i at the moment they were chosen. Thus, $|\mathcal{I}| = n_0 + n_1 + n_2$. Moreover, we know that $n_0 \geq x_1$ as isolated vertices of C will be added to $\mathcal{I}$ before any other vertex of C. Thus, $n_0 \geq x_1 = x_3 + 2$.

Let m be the number of bounded faces of C. Since $m \leq x_3$, we conclude that $m + 2 \leq n_0$.

Since removing vertices of degree zero or one does not change the number of bounded faces, we remove a bounded face of the current graph exactly when we add a vertex of degree 2 to $\mathcal{I}$. Thus, $m = n_2$. Therefore, $n_2 \leq n_0 - 2$.

Since every time our algorithm chooses a vertex of degree i we remove its i neighbors from the graph, and since only vertices of degree 0, 1 or 2 are chosen, we conclude that $n = n_0 + 2n_1 + 3n_2$. Because $|\mathcal{I}| = n_0 + n_1 + n_2$, we infer that

$$n = n_0 + 2n_1 + 3n_2 \leq 2(n_0 + n_1 + n_2) - 2 = 2|\mathcal{I}| - 2.$$

Consequently $|\mathcal{I}| \geq \frac{n+2}{2}$. □

If the independent set $\mathcal{I}$ guaranteed by Lemma 3 does not satisfy the condition of Lemma 2, for instance when n is even and $\mathcal{I}$ is the set of vertices with an even index, then take any $v_i \in V \setminus \mathcal{I}$ and remove v_{i+1} from $\mathcal{I}$. Since the modified $\mathcal{I}$ satisfies Lemma 2, we have

Theorem 4 *Every outerplanar graph on n vertices contains a column planar set of size at least $n/2$.*

3 Application to Partial Simultaneous Geometric Embedding

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, both on the same set V of n vertices. Let $R_1 \subseteq V$ be column planar in G_1 and let $R_2 \subseteq V$ be column planar in G_2. Evans et al. [6] proved that then G_1 and G_2 admit an $|R|$-PSGE where $R = R_1 \cap R_2$.

For outerplanar graphs G_1 and G_2, let C_1 and C_2 be their chord graphs, respectively. First use Lemma 3 compute an independent set $\mathcal{I}_1$ of size at least $n/2+1$ in C_1. Remove at most one vertex from $\mathcal{I}_1$ to obtain a set R_1 of size at least $n/2$ that is column planar in G_1 by Lemma 2. Next, use Lemma 3 to compute an independent set $\mathcal{I}_2$ of size at least $n/4 + 1$ in the chord graph of $G_2[R_1]$ (after adding edges to make $G_2[R_1]$ maximal outerplanar). Note that $\mathcal{I}_2$ is also independent in C_2, and hence we can remove at most one vertex from $\mathcal{I}_2$ to obtain a set $R \subseteq R_1 \subseteq V$ of size at least $n/4$ that is column planar in G_2 using Lemma 2. Note that R is also column planar in G_1 since $R \subseteq R_1$. Combining this with the aforementioned result of Evans et al. [6] gives our second result.

Theorem 5 *Every two outerplanar graphs on a set of n vertices admit an $(n/4)$-PSGE.*

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