# Coloring Dynamic Point Sets on a Line

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## Abstract

We consider a coloring problem on dynamic, one-dimensional point sets: points appearing and disappearing on a line at given times. We wish to color them with $k$ colors so that at any time, any sequence of $p(k)$ consecutive points, for some function $p$, contains at least one point of each color.

We prove that no such function $p(k)$ exists in general. However, in the restricted case in which points appear gradually, but never disappear, we give a coloring algorithm guaranteeing the property at any time with $p(k) = 8k - 5$.

This can be interpreted as coloring point sets in $\mathbb{R}^2$ with $k$ colors such that any bottomless rectangle containing at least $8k - 5$ points contains at least one point of each color. Chen *et al.* (2009) proved that such colorings do not always exist in the case of general axis-aligned rectangles. Our result also complements recent results from Keszegh and Pálvölgyi (2011).

## 1 Introduction

It is straightforward to notice that $n$ points lying on a line can be colored with $k$ colors in such a way that any set of $k$ consecutive points receive different colors: it is sufficient to color them cyclically with the colors $1, 2, \ldots, k, 1, \ldots$. What can we do if points can appear and disappear on the line, and we wish a similar property to hold at any time? More precisely, we fix the number $k$ of colors, and wish to maintain the property that at any given time, any sequence of $p(k)$ consecutive points, for some function $p$, contains at least one point of each color.

We show that in general, such a function does not exist: there are dynamic point sets on a line that are impossible to color with two colors so that monochromatic subsequences have bounded length. This holds even if the whole schedule of appearances and disappearances is known in advance. This family of point sets is described in Section 2.

We prove, however, that there exists a linear function $p$ in the case where points can appear on the line at any time, but *never disappear*. Furthermore, this is achieved in a constructive, *semi-online* fashion: the coloring decision for a point can be delayed, but at any time the currently colored points yield a suitable coloring of the set. The algorithm is described in Section 3.

In Section 4, we restate the result in terms of a coloring problem in $\mathbb{R}^2$: there exists a constant $c$ such that for any integer $k \geq 1$, every point set in $\mathbb{R}^2$ can be colored with $k$ colors so that any *bottomless* rectangle containing at least $ck$ points contains one point of each color. Here, an axis-aligned rectangle is said to be bottomless whenever the $y$-coordinate of its bottom edge is $-\infty$.

**Motivations and previous works.** The problem is motivated by previous intriguing results in the field of geometric hypergraph coloring. Here, a geometric hypergraph is a set system defined by a set of points and a set of geometric ranges, typically polygons, disks, or pseudodisks. Every hyperedge of the hypergraph is the intersection of the point set with a range.

It was shown recently [12, 1, 4] that for every convex polygon $P$, there exists a constant $c$, such that any point set in $\mathbb{R}^2$ can be colored with $k$ colors in such a way that any translation of $P$ containing at least $p(k) = ck$ points contains at least one point of each color.

For the range spaces defined by translates of a given convex polygon, this corresponds to partitioning a given point set into $k$ subsets, each subset being an $\varepsilon$-net for $\varepsilon = ck/n$. More on the relation between this coloring problem and $\varepsilon$-nets can be found in the recent papers of Varadarajan [13], and Pach and Tardos [9].

The problem for translates of polygons can be cast in its dual form as a covering decomposition problem: given a set of translates of a polygon $P$, we wish to color them with $k$ colors so that any point covered by at least $p(k)$ of them is covered by at least one of each color. The two problems can be seen to be equivalent by replacing the points by translates of a symmetric image of $P$ centered on these points. The covering decomposition problem has a long history that dates back to conjectures by János Pach in the early 80s (see for instance [7, 2], and references therein). The decomposability of coverings by unit disks was con-

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sidered in a seemingly lost unpublished manuscript by Mani and Pach in 1986. Up to recently, however, surprisingly little was known about this problem.

For other classes of ranges, such as axis-aligned rectangles, disks, or translates of some concave polygons [3, 8, 10, 11], such a coloring does not always exists, even when we restrict ourselves to two colors. For instance, the following result holds: for any integer $p \geq 2$, there exists a set of points in $\mathbb{R}^2$, every 2-coloring of which is such that there exists an open disk containing $p$ monochromatic points.

Keszegh [5] showed in 2007 that every point set could be 2-colored so that any bottomless rectangle containing at least 4 points contains both colors. More recently, Keszegh and Pálvölgyi [6] proved the cover-decomposability of octants in $\mathbb{R}^3$ : every collection of translates of the first octant can be 2-colored so that any point of $\mathbb{R}^3$ that is covered by at least 12 octants is covered by at least one of each color. This result generalizes the previous one (with a loser constant), as incidence systems of bottomless rectangles in the plane can be produced by restricted systems of octants in $\mathbb{R}^3$. It also implies similar covering decomposition results for homotopic copies of triangles. The generalization to $k$-colorings, however, still seems elusive.

Our positive result on bottomless rectangles (Corollary 4) is a generalization of Keszegh's results [5] to $k$-colorings. To our knowledge, this is the first example of a $k$-coloring achieving a linear bound $p(k)$ for ranges that are not translates of a given convex body. We see this as a first step towards similar results for more general range spaces, such as homothetic copies of convex polygons.

## 2 Coloring dynamic point sets

A *dynamic point set* $S$ in $\mathbb{R}$ is a collection of triples $(v_i, a_i, d_i) \in \mathbb{R}^3$, with $d_i \geq a_i$, that is interpreted as follows: the point $v_i \in \mathbb{R}$ appears on the real line at time $a_i$ and disappears at time $d_i$. Hence the set $S(t)$ of points that are present at time $t$ are the points $v_i$ with $t \in [a_i, d_i]$. A $k$-coloring of a dynamic point set assigns one of $k$ colors to each such triple.

We now show that it is not possible to find a 2-coloring of such a point set while avoiding long monochromatic subsequences at any time.

**Theorem 1** *For every $p \in \mathbb{N}$, there exists a dynamic point set $S$ with the following property: for every 2-coloring of $S$, there exists a time $t$ such that $S(t)$ contains $p$ consecutive points of the same color.*

**Proof.** In order to prove this result, we work on an equivalent two-dimensional version of the problem. From a dynamic point set, we can build $n$ horizontal segments in the plane, where the $i$th segment goes from $(a_i, v_i)$ to $(d_i, v_i)$. At any time $t$ the visible points $S(t)$ correspond to the intervals that intersect the line $x = t$. It is therefore equivalent, in order to obtain our result, to build a collection of horizontal segments in the plane that cannot be 2-colored in such a way that any set of $p$ segments intersecting some vertical segment contains one element of each color.

Our construction borrows a technique from Pach, Tardos, and Tóth [10]. In this paper, the authors provide an example of a set system whose ground set cannot be 2-colored without leaving some set monochromatic. This set system $\mathcal{S}$ is built on top of the $1 + p + \cdots + p^{p-1} = \frac{1-p^p}{1-p}$ vertices of a $p$-regular tree $\mathcal{T}^p$ of depth $p$, and contains two kinds of sets :

- the $1 + p + \cdots + p^{p-2}$ sets of *siblings*: the sets of $p$ vertices having the same father,
- the $p^{p-1}$ sets of $p$ vertices corresponding to a path from the root vertex to one of the leaves in $\mathcal{T}^p$.

It is not difficult to realize that this set system is not 2-colorable: by contradiction, if every set of siblings is non-monochromatic, we can greedily construct a monochromatic path from the root to a leaf.

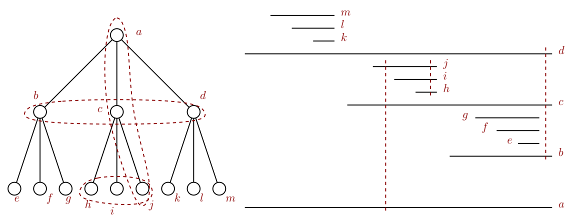
We now build a collection of horizontal segments corresponding to the vertices of $\mathcal{T}^p$, in such a way that for any set $E \in \mathcal{S}$ there exists a time $t$ at which the elements of $E$ are consecutive among those that intersect the line $x = t$. For any $p$ (see Fig. 1), the construction starts with a building block $B_p^1$ of $p$ horizontal segments, the $i$th segment going from $(-\frac{i}{p}, i)$ to $(0, i)$. Because these $p$ segments represent *siblings* in $\mathcal{T}^p$, they are consecutive on the vertical line that goes through their rightmost endpoint, and hence cannot all receive the same color.

Block $B_p^{j+1}$ is built from a copy of $B_p^1$ to which are added $p$ resized and translated copies of $B_p^j$ : the $i$th copy lies in the rectangle with top-right corner $(-\frac{i-1}{p}, i+1)$ and bottom-left corner $(-\frac{i}{p}, i)$. With this construction, the ancestors of a segment are precisely those that are below it on the vertical line that goes through its leftmost point. When such sets of ancestors are of cardinality $p$, which only happens when one considers the set of ancestors of a leaf, the set is required to be non-monochromatic.

By adding to $B_p^{p-1}$ a last horizontal segment below all others, corresponding to the root of $\mathcal{T}^p$, we ensure that a feasible 2-coloring of the segments would yield a proper 2-coloring of $\mathcal{S}$, which we know does not exist. □

The above result implies that no function $p(k)$ exists for any $k$ that answers the original question. If it were the case, then we could simply merge color classes of a $k$-coloring into two groups and contradict the above statement.

Theorem 1 can also be interpreted as the indecomposability of coverings by a specific class of unbounded



(a) The tree $\mathcal{T}^3$.

(b) The corresponding set of horizontal segments $B_3^2$, with a root segment $a$.

Figure 1: The recursive construction of theorem 1, for $p = 3$.

polytopes in $\mathbb{R}^3$. We omit the details of this interpretation.

## 3 Coloring point sets under insertion

Since we cannot bound the function $p(k)$ in the general case, we now consider a simple restriction on our dynamic point sets: we let the deletion times $d_i$ be infinite for every $i$. Hence points appear on the line, but never disappear.

A natural idea to tackle this problem is to consider an online coloring strategy, that would assign a color to each point in order of their arrival times $a_i$, without any knowledge of the points appearing later. However, we cannot guarantee any bound on $p(k)$ unless we delay some of the coloring decisions. To see this, consider the case $k = 2$, and call the two colors red and blue. An online algorithm must color each new point in red or blue as soon as it is presented. We can design an adversary such that the following invariant holds: at any time, the set of points is composed of a sequence of consecutive red points, followed by a sequence of consecutive blue points. The adversary simply chooses the new point to lie exactly between the two sequences at each step.

Our computation model will be *semi-online*: The algorithm considers the points in their order of the arrival time $a_i$. At any time, a point in the sequence either has one of the $k$ colors, or is uncolored. Uncolored points can be colored later, but once a point is colored, it keeps its color for the rest of the procedure. At any time, the colors that are already assigned suffice to satisfy the property that any subsequence of $p(k)$ points have one point of each color, where $p(k) = O(k)$.

**Theorem 2** *Every dynamic point set without disappearing points can be $k$-colored in the semi-online model such that at any time, every subsequence of at least $8k - 5$ consecutive points contains at least one point of each color.*

**Proof.** We define a *gap* for color $i$, or an $i$-gap, as a maximal interval containing no point of color $i$, that is, either between two successive occurrences of color $i$, or before the first occurrence, or after the last occurrence. At every step, we decide to assign a color to some of the uncolored points, in order to maintain the property that every gap contains at most $p(k) - 1$ points. This implies that any subsequence of $p(k)$ consecutive points contains at least one point of each color. We show that $p(k)$ can be linear in $k$. For this purpose, the algorithm will maintain another invariant:

*(a) every gap contains at most $B = 4k - 2$ uncolored points.*

At every step, we consider the new point, and leave it uncolored. This may cause some $i$-gap to contain $B+1$ uncolored points, and therefore violate invariant *(a)*. In that case, we pick one such gap and color its $(B/2 + 1)$th uncolored point with color $i$. This splits the $i$-gap into two parts, each containing $B/2$ uncolored points. If another gap was violating the invariant, say for color $j$, two cases can occur. Either the $j$-gap also contains the $(B/2 + 1)$th uncolored point of the $i$-gap, in which case the invariant is not violated anymore, or we can color with color $j$ the $(B/2 + 1)$th uncolored point of the $j$-gap. In the latter case, note that the newly colored point cannot belong to the $i$-gap, as otherwise the first case would apply.

We now show that this algorithm also maintains the following invariant:

*(b) every gap contains at least $k$ uncolored points.*

From the above steps, we know that when a point is colored with color $i$, its is separated from the two occurrences of color $i$ on its left and right by at least $B/2$ uncolored points. Later, however, some of these uncolored points can be colored with a color distinct from $i$. Hence the number of uncolored points can decrease below $B/2$. However, from the time it is less than $B/2$, there cannot be more than one point of each color distinct from $i$ appearing, since any new pair of colored points is separated by at least $B/2$ uncolored points. Hence there must still be at least $B/2 - (k - 1) = k$ uncolored points between two points of color $i$.

Now it remains to check that the two invariants *(a)* and *(b)* imply that any two successive points of the same color $i$ cannot be separated by more than $O(k)$ other points. First, from invariant *(a)*, there can be at most $B$ uncolored point in this $i$-gap. Then, every pair of occurrences of a color distinct from $i$ is separated by at least $k$ uncolored points. This implies that there cannot be more than $1 + \lfloor \frac{B}{k} \rfloor$ occurrences of each of the $k-1$ other colors in the $i$-gap. Hence the total number of points cannot exceed

$$\begin{aligned} p(k) - 1 &= B + (k-1)\left(1 + \left\lfloor \frac{B}{k} \right\rfloor\right) \\ &= 4k - 2 + (k-1)\left(1 + \left\lfloor \frac{4k-2}{k} \right\rfloor\right) \\ &= 8k - 6. \end{aligned}$$

□

The function is not tight for every $k$, as witnessed by Keszegh's result for $k = 2$ [5]. For $k = 3$, we have the following result, the proof of which is omitted.

**Theorem 3** *Every dynamic point set without disappearing points can be 3-colored in the semi-online model such that at any time, every subsequence of at least 8 consecutive points contains at least one point of each color.*

## 4 Coloring points with respect to bottomless rectangles

A *bottomless rectangle* is a set of the form $\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, y \leq c\}$, for a triple of real numbers $(a,b,c)$ with $a \leq b$. We consider the following geometric coloring problem: given a set of points in the plane, we wish to color them with $k$ colors so that any bottomless rectangle containing at least $p(k)$ points contains at least one point of each color. It is not difficult to realize that the problem is equivalent to that of the previous section.

**Corollary 4** *Every point set $S \subset \mathbb{R}^2$ can be colored with $k$ colors so that any bottomless rectangle containing at least $8k-5$ points of $S$ contains at least one point of each color.*

**Proof.** The algorithm proceeds by sweeping $S$ vertically in increasing $y$-coordinate order. This defines a dynamic point set $S'$ that contains at time $t$ the $x$-coordinates of the points below the horizontal line of equation $y = t$. The set of points of $S$ that are contained in a bottomless rectangle $\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, y \leq t\}$ correspond to the points in the interval $[a,b]$ in $S'(t)$. Hence the two coloring problems are equivalent, and Theorem 2 applies. □

The following is a corollary of Theorem 3.

**Corollary 5** *Every point set $S \subset \mathbb{R}^2$ can be colored with 3 colors so that any bottomless rectangle containing at least 8 points of $S$ contains at least one point of each color.*

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