On Local and Global Crossing Numbers*

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Abstract

A graph is *k-planar* if it can be drawn in $\mathbb{R}^2$ such that every edge has at most k crossings; such a drawing is called *k-plane*. The *local crossing number* $\operatorname{lcr}(G)$ of a graph G is the smallest $k \geq 0$ such that G is k-planar. The (global) *crossing number* $\operatorname{cr}(G)$ of a graph G is the minimum number of crossings over all drawings of G. Counting the number of crossings per edge in any $\operatorname{lcr}(G)$-plane drawing of G yields $2 \cdot \operatorname{cr}(G) \leq m \cdot \operatorname{lcr}(G)$, where m denotes the number of edges in G. We are interested in graphs for which this inequality holds with equality. As a main result we show that such graphs exist. More precisely, for any given $k \in \mathbb{N}$, we construct an infinite family of graphs that admit a minimum-crossing drawing in which every edge is crossed exactly k times.

1 Introduction

Are there drawings of graphs where every edge is crossed? Indeed, such drawings of K_n were first mentioned by Ringel [17] and later systematically studied by Harborth and Mengersen [11]. However, their drawings have many "unnecessary" crossings. So, what if we restrict to *minimum-crossing* drawings, that is, drawings with a minimum number $\operatorname{cr}(G)$ of edge crossings among all drawings of a graph G? Surprisingly, the answer remains the same: Ábrego, Aichholzer, Fernández-Merchant, Ramos, and Vogtenhuber [1] describe a family of drawings of K_n, for odd $n \geq 11$, where every edge is crossed and the total number of crossings is

$$H(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor .$$

According to the famous (Guy-Harary-)Hill Conjecture [10], we have $H(n) = \operatorname{cr}(K_n)$ and thus the aforementioned drawings are minimum-crossing. Despite supporting evidence (e.g., [4]), the conjecture is proven for $n \leq 14$ only [3]. Still, this leaves us with minimum-crossing drawings of K_{11} and K_{13} in which every edge is crossed. Taking any number of disjoint copies of these graphs and drawings, we obtain an infinite family of graphs with minimum-crossing drawings where every edge is crossed. In this work, we study the problem under the lens of k-planarity and give a general answer for any number k of crossings.

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▶ **Theorem 1.** *For every $k, n_0 \in \mathbb{N}$ there exists a graph G on $n \geq n_0$ vertices such that G has a drawing with $\mathrm{cr}(G)$ crossings, in which every edge is crossed exactly k times.*

In particular, in every k-plane minimum-crossing drawing of G every edge has exactly k crossings. Our motivation is to shed some light onto the relationship between the local and the (global) crossing number of graphs. To explain this relationship we introduce some standard terminology. A *drawing* of a graph $G = (V, E)$ is a map $\gamma : G \to \mathbb{R}^2$ that maps each vertex $v \in V$ to a point $\gamma(v) \in \mathbb{R}^2$ and each each edge $uv \in E$ to a simple (injective) curve $\gamma(uv)$ with endpoints $\gamma(u)$ and $\gamma(v)$, such that: (1) γ is injective on V; (2) $\gamma(uv) \cap \gamma(V) = \{\gamma(u), \gamma(v)\}$, for all $uv \in E$; and (3) for all $e_0, e_1 \in E$ with $e_0 \neq e_1$, the curves $\gamma(e_0)$ and $\gamma(e_1)$ have at most finitely many intersections, and each such intersection is either a common endpoint or a proper, transversal *crossing* between exactly two edges. A drawing is *simple* if every pair of edges has at most one common point (which is either a crossing or a common endpoint). The zoo of crossing numbers [18] is a very lively part of combinatorial geometry and graph drawing. The *crossing number* $\mathrm{cr}(D)$ of a drawing D is the number of edge crossings in D. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum of $\mathrm{cr}(D)$ over all drawings D of G. The *local crossing number* $\mathrm{lcr}(D)$ of a drawing D is the maximum number of crossings on any edge in D. The *local crossing number* $\mathrm{lcr}(G)$ of a graph G is the minimum of $\mathrm{lcr}(D)$ over all drawings D of G. A graph G is *k-planar* if $\mathrm{lcr}(G) \leq k$. The *k-planar crossing number* $\mathrm{cr}_{k\text{-pl}}(G)$ is the minimum of $\mathrm{cr}(D)$ over all drawings D of G with $\mathrm{lcr}(D) \leq k$; or $\mathrm{cr}_{k\text{-pl}}(G) = \infty$ if no such drawing exists. By definition, for every graph G we have

$$\mathrm{cr}(G) \leq \mathrm{cr}_{k\text{-pl}}(G). \tag{I1}$$

Further, as every graph G admits a drawing D with $\mathrm{lcr}(D) = \mathrm{lcr}(G)$, we have

$$\mathrm{cr}(G) \leq \mathrm{cr}(D) \leq \frac{m\mathrm{lcr}(D)}{2} = \frac{m}{2}\mathrm{lcr}(G)\,, \tag{I2}$$

where m denotes the number of edges in G. It is natural to wonder if and for what graphs these two inequalities are tight. Theorem 1 answers the if-part of this question by providing an infinite family of graphs for which both inequalities are tight. Chimani, Kindermann, Montecchiani and Valtr [7, 8] explored the opposite end of the spectrum for (I1) and constructed a family of 1-planar graphs G with $\mathrm{cr}_{1\text{-pl}}(G) \in \Omega(n)$ and $\mathrm{cr}(G) = 2$, where n denotes the number of vertices in G. Similar results have also been obtained for other beyond-planar graph classes [6, 20]. The maximum number of edges in a simple k-planar graph on n vertices is known to be at most $c_k(n-2)$, where $c_0 = 3$, $c_1 = 4$ [5], $c_2 = 5$ [15, 16], $c_3 = 5.5$ [13, 14], $c_4 = 6$ [2], and $c_k \leq 3.81\sqrt{k}$, for general $k \geq 5$ [2]. So we could plug these bounds into (I2) to obtain an upper bound on $\mathrm{cr}(G)$ in terms of n, the number of vertices in G. However, the graphs that maximize m might be quite different from the graphs for which (I2) is tight. For instance, for a 1-planar graph G it is known (see, e.g. [19, Proposition 4.4]) that it admits a drawing D such that

$$\mathrm{cr}(G) \leq \mathrm{cr}_{1\text{-pl}}(G) \leq \mathrm{cr}_{1\text{-pl}}(D) \leq n - 2\,, \tag{I3}$$

which beats the bound we get by plugging $m \leq 4n$ into (I2) by a factor of two. The bounds in (I2) and (I3) are somewhat conflicting. In fact, we show that if a graph G admits a drawing D for which both (I2) and (I3) are tight, then G is planar and thus $\mathrm{cr}(G) \ll \mathrm{cr}(D)$.

▶ **Theorem 2.** *Let D be a simple drawing of a graph G on n vertices where every edge is crossed exactly once. Then: (1) If $\mathrm{cr}(D) = n - 2$, then G is planar and disconnected; (2) If $\mathrm{cr}(D) = n - 3$, then G is planar.*

2 Preliminaries and Outline

Before we delve into the proof of Theorem 1, let us note that the statement for $k = 1$ can be obtained using edge subdivisions.

▶ **Lemma 3.** *The following statements are equivalent: (1) There exists a minimum-crossing drawing of a graph in which every edge is crossed; (2) There exists a minimum-crossing drawing of a graph in which every edge is crossed exactly once.*

Proof. The implication (2)⇒(1) is trivial. So it remains to prove (1)⇒(2). Let D be a minimum-crossing drawing of a graph G in which every edge is crossed. For every edge e in D and every pair c, c' of consecutive crossings along e in D we insert a new vertex between c and c' on e and also subdivide the corresponding edge in G. As a result, we obtain a graph G' and a drawing D' of G' such that G' is a subdivision of G and D' is a subdivision of D. By construction every edge in D' is crossed exactly once and $\mathrm{cr}(D) = \mathrm{cr}(D')$. As edge subdivisions have no effect on the crossing number of a graph, we have $\mathrm{cr}(G') = \mathrm{cr}(G) = \mathrm{cr}(D) = \mathrm{cr}(D')$ and thus D' is a minimum-crossing drawing of G'. ◀

The above subdivision argument also appears in the proof of Theorem 1. Our main challenge is to certify that a given drawing is minimum-crossing, which is very difficult in general. To get around this, we employ edge weights and correspondingly weighted crossing numbers as a framework to guarantee that the minimum-crossing drawing of the graph at hand is essentially unique. Our construction uses a highly symmetric graph based on the icosahedron graph. The symmetries of this graph help to simplify the argument. With a proper choice of weight assignment and subdivision, we can transfer the weighted minimum-crossing drawing into our target drawing where every edge is crossed exactly k times.

3 Minimum-crossing drawings in which every edge is crossed k times

We use the icosahedron graph $\mathsf{Ico} = (X, E_I)$ as a base graph, which is a plane triangulation with 12 vertices and 20 faces. It admits a combinatorially unique plane drawing, depicted by red edges in Figure 1.

Let $T = (V, E)$ be a plane triangulation. For an edge e of T, the two triangles incident to e form a 4-gon $Q(e)$. One of the diagonals of $Q(e)$ is e, we call the other diagonal of $Q(e)$ the *p-dual edge* of e. The *double-icosahedron* Dco is obtained from the icosahedron graph $\mathsf{Ico} = (X, E_I)$ by adding the p-dual edge of every edge $e \in E_I$, that is, we have $\mathsf{Dco} = (X, E_I \cup E_P)$, where E_P denotes the set of edges that are p-dual to some edge in E_I. It is interesting to note that the graph (X, E_P) is again an icosahedron graph, and Dco is a 10-regular graph on 12 vertices, isomorphic to a K_{12} with a perfect matching removed.

If we take a closer look into the drawing shown in Figure 1, we can easily count the crossing number of it since each of the 30 edges of E_I is only crossed by its p-dual edge and each edge of E_P is crossed by five other edges. Consequently, there is a total of $\frac{1}{2}(30+5\cdot30) = 90$ crossings in the drawing. We further remark that this matches the conjecture presented by Mohar [12] for K_n^t, which denotes the graph obtained from K_n by removing a matching of size t. Though an attempt to compute the exact crossing number of Dco by a computer[9] failed due to its inherent complexity, we conjecture $\mathrm{cr}(\mathsf{Dco}) = 90$ as well.

Weighted Minimum-Crossing Drawing. To derive our results, we first consider a weighted variant of the crossing number. Let $w : E \to \mathbb{R}^+$ be a weight assignment on the edges of a graph G. The *weighted crossing number* $\mathrm{cr}_w(D)$ of a drawing D of G is the sum of the

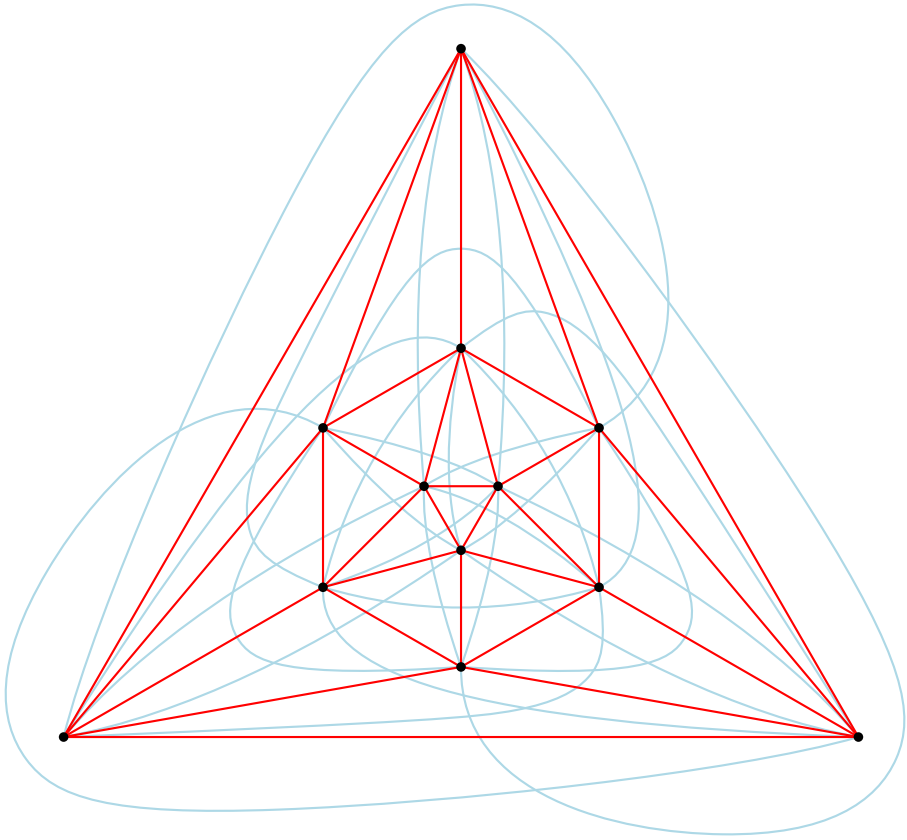

Figure 1 A drawing of Dco where every edge is crossed.

product $w(e) \cdot w(f)$ taken over all edge crossings $e \cap f$ in D. The weighted crossing number $\mathrm{cr}_w(G)$ of a graph G is the minimum of the weighted crossing numbers of its drawings.

▶ **Lemma 4.** *There is a weight assignment w on the edges of Dco such that the drawing shown in Figure 1 minimizes the weighted crossing number of this weighted graph.*

Proof. Consider any minimum-crossing drawing Γ of Dco. Recall that $\mathsf{Dco} = (X, E_I \cup E_P)$ with $|E_I| = |E_P| = 30$. If $w(e) = a$ for all $e \in E_I$ and $w(e) = 1$ for all $e \in E_P$, then for the drawing Γ_0 in Figure 1 we have $\mathrm{cr}_w(\Gamma_0) = 30a + 60$. We choose $a = 32$ s.t. $a^2 = 1024 > 30a + 60 = 1020$, and no pair of edges from E_I can cross in Γ. Hence, the drawing of (X, E_I) in Γ is a plane drawing Ico of the icosahedron. As every edge in E_P has to cross at least one edge of Ico, we have $\mathrm{cr}_w(\Gamma) \geq 30a$. At most one edge of E_P crosses two or more—and if so, exactly two—edges of Ico in Γ (else $\mathrm{cr}_w(\Gamma) \geq 30a + 2a = a^2 > \mathrm{cr}_w(\Gamma_0)$). Hence in at least 18 of the triangles of Ico, the three p-duals of the triangle edges cross inside the triangle. This yields at least $3 \cdot 18$ crossings between edges in E_P. Since $31a + 3 \cdot 18 > 30a + 60$, all the p-edges in Γ must be drawn in the union of two triangles and cross four other p-edges, just as in Γ_0. It follows that $\mathrm{cr}_w(\Gamma) = \mathrm{cr}_w(\Gamma_0)$. ◀

Let G be a graph and $w : E \to \mathbb{N}$ be an integral weight, we define the multigraph G_w by replacing each edge e of G by $w(e)$ copies of e, i.e., a set of $w(e)$ parallel edges connecting the vertices of e. Every drawing of G has a corresponding drawing of G_w by replacing the drawn edge e by a bundle of $w(e)$ edges in a small neighborhood of e. This construction shows that $\mathrm{cr}(G_w) \leq \mathrm{cr}_w(G)$. Given a drawing of G_w and an edge e of G we can look at all copies of e in the drawing and choose one, say e', involved in a minimum number of crossings. Now redraw the $w(e)$ edges of the bundle of e in a small neighborhood of e'. By the choice of e' this operation can only decrease the crossing number of the drawing. Repeating this for all edges of G we arrive at a drawing of G_w where parallel edges behave the same, i.e., which comes from a drawing of G. Therefore $\mathrm{cr}(G_w) = \mathrm{cr}_w(G)$.

▶ **Theorem 1.** *For every $k, n_0 \in \mathbb{N}$ there exists a graph G on $n \geq n_0$ vertices such that G has a drawing with $\mathrm{cr}(G)$ crossings, in which every edge is crossed exactly k times.*

Proof. We first deal with the case $k = 1$. Let Dco^+ be the multigraph obtained with multiplicity 64 for edges of E_I and multiplicity 2 for edges of E_P. The drawing Γ_0 By a similar analysis as shown in the proof of Lemma 4, this induces a minimum-crossing drawing of Dco^+ by replacing each red edge by a bundle of 64 parallel edges and each light blue edge by 2 parallel edges. For every edge e and every pair of consecutive crossings on e we place a new subdivision vertex on e; see Figure 2 (top). In this way, we obtain a subdivision Dco^+_s of Dco^+ that has a minimum-crossing drawing where every edge is crossed exactly once. To obtain arbitrarily large graphs, we take sufficiently many disjoint copies of Dco.

For $k \geq 2$ we adapt the weighting w such that if D_w is a minimum-crossing drawing of Dco^+, then in D_w the number of crossings on each edge is a multiple of k. An appropriate weighting is obtained by using the weight $64k$ for edges in E_I and $2k$ for edges in E_P. The subdivision vertices of Dco^+_s are placed on the edges of D_w such that every edge of Dco^+_s has exactly k crossings. Figure 2 (bottom) illustrates the construction in the case $k = 2$. ◀

4 On Graphs for which Both Upper Bounds are Tight

Due to space constraints we present the proof of the first statement in Theorem 2 only; the proof for the second statement can be found in a later full version.

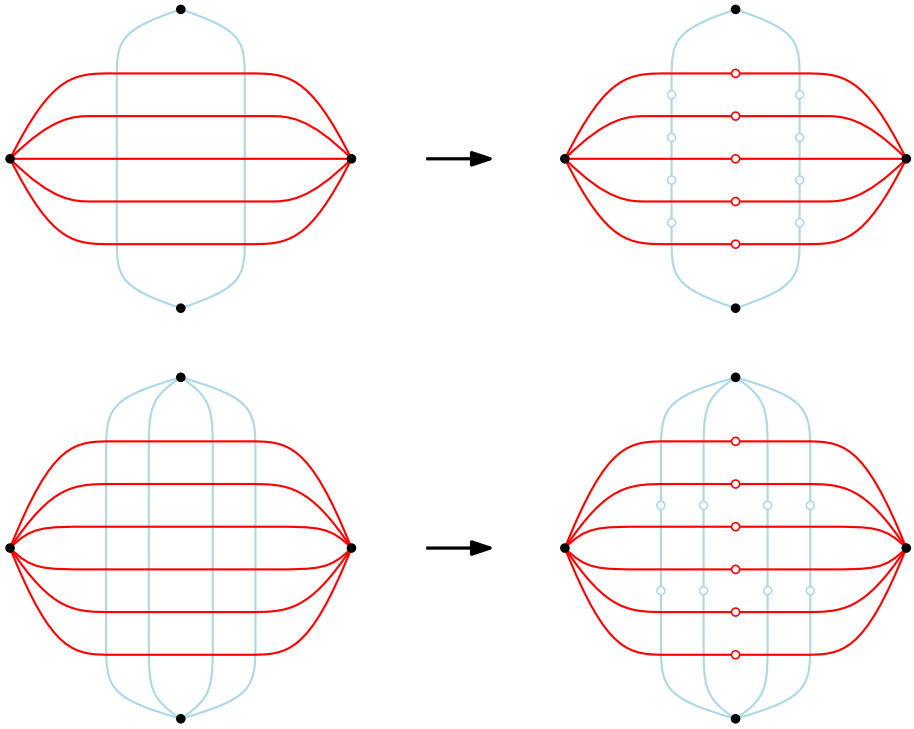

Figure 2 Subdividing edges to obtain a k-plane drawing, for $k = 1$ (top) and $k = 2$ (bottom).

▶ **Theorem 2.** *Let D be a simple drawing of a graph G on n vertices where every edge is crossed exactly once. Then: (1) If $\mathrm{cr}(D) = n - 2$, then G is planar and disconnected; (2) If $\mathrm{cr}(D) = n - 3$, then G is planar.*

Proof of (1). Let D be a drawing of G with $n - 2$ crossings such that every edge is crossed exactly once. Let D^+ be obtained from D by adding a maximal set of uncrossed edges, and let H be the plane drawing consisting of the new edges only. Each crossing pair of edges in D corresponds to a 4-face of H. Hence, H is a plane graph with n vertices and $n - 2$ many 4-faces. From Euler's formula it follows that H is a quadrangulation. In particular H is bipartite. Transferring the 2-coloring of H to D, we observe that for every pair e, e' of crossing edges the endpoints of e have the same color and both endpoints of e' have the other color. Hence, each of the two color classes induces a plane graph. Denote these two graphs by D_1 and D_2. As every edge in D is crossed, the graph D is the disjoint union of D_1 and D_2. Thus, we obtain a plane drawing of D by drawing D_1 and D_2 side by side. ◀

5 Open Problems

Our construction to prove Theorem 1 uses edge subdivisions extensively. Can a similar statement be obtained for graphs with minimum vertex degree three or higher? In Theorem 2 we show that there are no graphs for which both upper bounds (I1) and (I2) on the crossing number are (almost) tight. How far can we further relax the "almost" tightness condition?

References

1. Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Birgit Vogtenhuber. Non-shellable drawings of K_n with few crossings. In *Proc. 26th Canad. Conf. Comput. Geom.*, pages 7:1–7:6, 2014. URL: https://www.cccg.ca/proceedings/2014/papers/paper07.pdf.
2. Eyal Ackerman. On topological graphs with at most four crossings per edge. *Comput. Geom.*, 85:101574, 2019. doi:10.1016/J.COMGEO.2019.101574.
3. Oswin Aichholzer. Another small but long step for crossing numbers: cr(13) = 225 and cr(14) = 315. In *Proc. 33rd Canad. Conf. Comput. Geom.*, pages 72–77, 2021. URL: https://projects.cs.dal.ca/cccg2021/wordpress/wp-content/uploads/2021/08/CCCG2021.pdf.
4. József Balogh, Bernard Lidický, and Gelasio Salazar. Closing in on Hill's Conjecture. *SIAM J. Discrete Math.*, 33(3):1261–1276, 2019. doi:10.1137/17M1158859.
5. Rainer Bodendiek, Heinz Schumacher, and Klaus Wagner. Bemerkungen zu einem Sechsfarbenproblem von G. Ringel. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 53:41–52, 1983. doi:10.1007/BF02941309.
6. Markus Chimani, Torben Donzelmann, Nick Kloster, Melissa Koch, Jan-Jakob Völlering, and Mirko H. Wagner. Crossing numbers of beyond planar graphs re-revisited: A framework approach. In *Proc. 32nd Internat. Sympos. Graph Drawing*, volume 320 of *Leibniz Internat. Proc. Informatics (LIPIcs)*, pages 33:1–33:17. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2024. URL: https://doi.org/10.4230/LIPIcs.GD.2024.33.
7. Markus Chimani, Philipp Kindermann, Fabrizio Montecchiani, and Pavel Valtr. Crossing numbers of beyond-planar graphs. In *Proc. 27th Internat. Sympos. Graph Drawing*, volume 11904 of *Lecture Notes in Computer Science*, pages 78–86, 2019. doi:10.1007/978-3-030-35802-0_6.
8. Markus Chimani, Philipp Kindermann, Fabrizio Montecchiani, and Pavel Valtr. Crossing numbers of beyond-planar graphs. *Theor. Comput. Sci.*, 898:44–49, 2022. doi:10.1016/J.TCS.2021.10.016.
9. Markus Chimani and Tilo Wiedera. An ILP-based proof system for the crossing number problem. In *24th Annual European Symposium on Algorithms*, volume 57 of *LIPIcs*, pages 29:1–29:13, 2016. doi:10.4230/LIPICS.ESA.2016.29.
10. Frank Harary and Anthony Hill. On the number of crossings in a complete graph. *Proc. Edinburgh Math. Soc.*, 13(4):333–338, 1963. doi:10.1017/S0013091500025645.
11. Heiko Harborth and Ingrid Mengersen. Edges without crossings in drawings of complete graphs. *J. Combin. Theory Ser. B*, 17(3):229–311, 1974. doi:10.1016/0095-8956(74)90035-5.
12. Bojan Mohar. On a conjecture by Anthony Hill, 2020. arXiv 2009.03418. URL: https://arxiv.org/abs/2009.03418, arXiv:2009.03418.
13. János Pach, Radoš Radoičić, Gábor Tardos, and Géza Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. In *Proc. 20th Annu. Sympos. Comput. Geom.*, pages 68–75, 2004. doi:10.1145/997817.997831.
14. János Pach, Radoš Radoičić, Gábor Tardos, and Géza Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete Comput. Geom.*, 36(4):527–552, 2006. doi:10.1007/s00454-006-1264-9.
15. János Pach and Géza Tóth. Graphs drawn with few crossings per edge. In *Proc. 4th Internat. Sympos. Graph Drawing*, volume 1190 of *Lecture Notes in Computer Science*, pages 345–354, 1996. doi:10.1007/3-540-62495-3_59.
16. János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997. doi:10.1007/BF01215922.

17 Gerhard Ringel. Extremal problems in the theory of graphs. In *Theory of Graphs and Its Applications, Proc. Sympos. Smolenice*, pages 85–90, 1963.

18 Marcus Schaefer. The graph crossing number and its variants: A survey. *Electron. J. Comb.*, Dynamic Surveys, DS21(8):1–166, 2024. doi:10.37236/2713.

19 Yusuke Suzuki. 1-planar graphs. In *Beyond Planar Graphs, Communications of NII Shonan Meetings*, pages 47–68. Springer, 2020. doi:10.1007/978-981-15-6533-5_4.

20 Nathan van Beusekom, Irene Parada, and Bettina Speckmann. Crossing numbers of beyond-planar graphs revisited. *J. Graph Algorithms Appl.*, 26(1):149–170, 2022. doi:10.7155/JGAA.00586.