Crossing Number of 3-Plane Drawings*

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Abstract

We study 3-plane drawings, that is, drawings of graphs in which every edge has at most three crossings. We show how the recently developed Density Formula for topological drawings of graphs [9] can be used to count the crossings in terms of the number n of vertices. As a main result, we show that every 3-plane drawing has at most $5.5(n-2)$ crossings, which is tight. In particular, it follows that every 3-planar graph on n vertices has crossing number at most $5.5n$, which improves upon a recent bound [3] of $6.6n$. To apply the Density Formula, we carefully analyze the interplay between certain configurations of cells in a 3-plane drawing. As a by-product, we also obtain an alternative proof for the known statement that every 3-planar graph has at most $5.5(n-2)$ edges.

1 Introduction

One of the most basic combinatorial questions one can ask for a class of graphs is: How many edges can a graph from this class have as a function of the number n of vertices? Prominent examples include upper bounds of $\binom{n}{2}$ for the class of all graphs and $\frac{n^2}{4}$ for bipartite graphs. These bounds are immediate consequences of the definition of these graph classes, and they are tight, that is, there exist graphs in the class with exactly this many edges. But for several other graph classes good upper bounds on the number of edges are much more challenging to obtain. Notably this holds for classes that relate to the existence of certain geometric representations. One the most fundamental questions one can ask about a class of geometrically represented graphs is: What is the minimum number of edge crossings required in such a representation, as a function of the number n of vertices? We study both of these fundamental questions in combination, for the class of 3-planar graphs. A graph is *k-planar* if it can be drawn in the plane such that every edge has at most k crossings. The study of k-planar graphs goes back to Ringel [16] and has been a major focus in graph drawing over the past two decades [8], as a natural generalization of planar graphs ($k = 0$).

The maximum number of edges in a simple k-planar graph on n vertices is known to be at most $c_k(n-2)$, where $c_0 = 3$, $c_1 = 4$ [5], $c_2 = 5$ [14, 15], $c_3 = 5.5$ [10, 11], $c_4 = 6$ [1], and $c_k \leq 3.81\sqrt{k}$, for general $k \geq 5$ [1]. The bounds for $k \leq 2$ are tight and those for $k \leq 4$ are tight up to an additive constant [1, 4]. The bounds for $k \leq 4$ also generalize to *non-homotopic* drawings of multigraphs [12, 13], that is, where every continuous transformation

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that transforms one copy of an edge to another passes over a vertex. Interestingly, the upper bound for 3-planar graphs is tight in this more general setting only [4, 6].

The *crossing number* of a drawing Γ is the number of edge crossings in Γ. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum crossing number over all drawings of G. By definition every k-planar graph G admits a k-plane drawing and thus

$$\mathrm{cr}(G) \leq \frac{km}{2}, \tag{S}$$

where m denotes the number of edges in G. For a k-planar graph, this simple inequality connects upper bounds on the number of edges with lower bounds on the crossing number. Both of these come together in the well-known Crossing Lemma [2, Chapter 45], as the best constants in the Crossing Lemma are obtained by analyzing k-plane drawings [1, 6, 10, 11]. Conversely, combining the lower bound on $\mathrm{cr}(G)$ from the Crossing Lemma with an upper bound on $\mathrm{cr}(G)$ we obtain an upper bound on the number of edges in G. While (S) would work here, it is probably not an ideal choice because the graphs for which (S) is tight might be very different from those graphs that have a maximum number of edges, for any fixed n. For instance, for a 1-planar graph G we have $\mathrm{cr}(G) \leq n - 2$ [17, Proposition 4.4], which beats the bound we get by plugging $m \leq 4n - 8$ into (S) by a factor of two. Can we obtain similar improvements by bounding $\mathrm{cr}(G)$ in terms of n, rather than m, for $k \geq 2$?

Indeed, very recently it has been shown that $\mathrm{cr}(G) \leq 3.\overline{3}n$ if G is 2-planar and $\mathrm{cr}(G) \leq 6.6n$ if G is 3-planar [3]. There is some indication that the bound for 2-planar graphs could be tight up to an additive constant, as it is achieved by the standard drawings of optimal 2-planar graphs (Figure 1). But the crossing number of these graphs is not known.

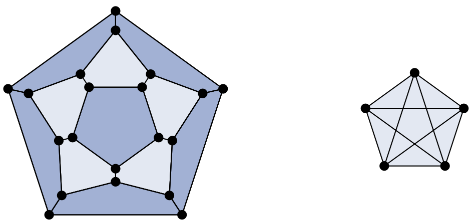

Figure 1 Construction by Pach and Tóth [15, Figure 3]. Left: A planar drawing with pentagonal faces. Right: To each pentagonal face all diagonals are added.

In contrast, there exists a family of simple 3-planar graphs with $5.5n - 15$ edges whose standard drawings have $5.5n - 21$ crossings (Figure 2). Thus, there is a gap of $1.1n$ between the lower and the upper bound for the crossing number of 3-plane drawings.

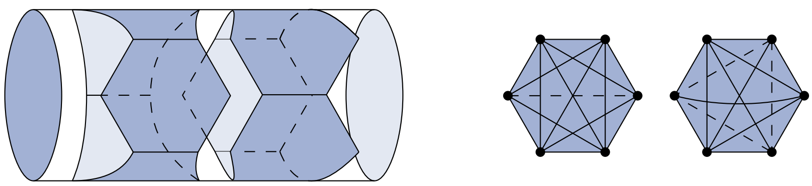

Figure 2 Construction from [11, Figure 8]. Left: A cylinder with two layers, each consisting of three hexagonal faces. Right: To each face of a layer all but one diagonal is added. To the top and bottom face six diagonals are added. Missing diagonals are represented by dashed lines.

Results. We close the gap and present an upper bound on the crossing number of 3-plane drawings that is tight up to an additive constant. Using the same approach we also obtain an alternative proof to show that a 3-planar n-vertex graph has at most $5.5(n-2)$ edges.

▶ **Theorem 1.** *Every non-homotopic 3-plane drawing of a graph on n vertices, $n \geq 3$, contains at most $5.5(n-2)$ edges and at most $5.5(n-2)$ crossings.*

Our proof relies on the recently developed Density Formula (cf. Theorem 2 below) for topological drawings of graphs [9]. It relates the number of vertices, edges, and cells of various sizes in a drawing, in a way similar to the Euler Formula in the case of plane graphs. Previously, the Density Formula has been used to derive upper bounds on the number of edges in k-plane drawings, for $k \leq 2$ [9]. In order to apply it to 3-plane drawings, to bound the number of crossings, and to obtain tight bounds, we study cells not only in isolation but also as part of what we call *configurations*, which consist of several connected cells. We then develop a number of new constraints that relate the number of cells and/or configurations of a certain type in any 3-plane drawing. The combination of all these constraints with the Density Formula yields a linear program that we can solve in two different ways—maximizing either the number of edges or the number of crossings—to prove Theorem 1.

Using Theorem 1 we can derive better upper bounds on the number of edges in k-planar graphs without short cycles. Plugging our bound of at most $5.5n$ crossings into the proofs from [3] we obtain that

- C_3-free 3-planar graphs on n vertices have at most $\sqrt[3]{891/8}n < 4.812n$ edges (down from $\approx 5.113n$ [3, Theorem 18]),
- C_4-free 3-planar graphs on n vertices have at most $\sqrt[3]{1'254'825/12'544}n < 4.643n$ edges (down from $\approx 4.933n$ [3, Theorem 20]), and
- 3-planar graphs of girth 5 on n vertices have at most $\sqrt[3]{122,793/1600}n < 4.25n$ edges (down from $\approx 4.516n$ [3, Theorem 21]).

2 Preliminaries

We consider *drawings* of graphs on the sphere with vertices as points, edges as Jordan arcs, and the usual assumption that any two edges share only finitely many points, each being a common endpoint or a proper crossing, and that no three edges cross in the same point. We also assume that no edge crosses itself and that no two adjacent edges cross. As is customary, we do not distinguish between the points and curves in Γ and the vertices and edges of G they represent, respectively. The graphs we consider may contain parallel edges, but no loops. In order to avoid an arbitrary number of parallel edges within a small corridor, a drawing Γ is called *non-homotopic* if every region that is bounded by exactly two parts of edges, called a *lens*, contains a crossing or a vertex in its interior; see Figure 3.

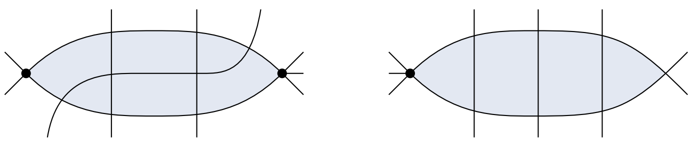

Figure 3 *Left:* A lens (blue) with two crossings in its interior. *Right:* An empty lens (blue).

Let Γ be a drawing of a graph $G = (V, E)$. If every edge is crossed at most three times, we say that Γ is 3-*plane*. We denote the set of crossings by X. For $i \in \{0, 1, 2, 3\}$, let $E_i \subseteq E$ be the set of all edges with exactly i crossings, and let $E_\times = E_1 \cup E_2 \cup E_3$.

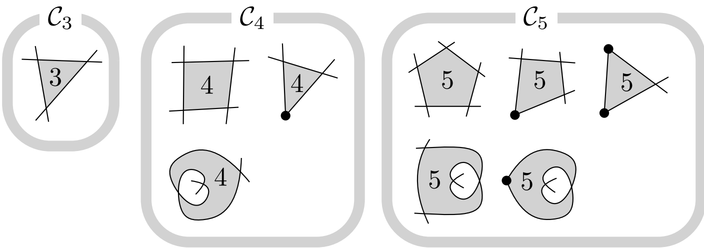

Figure 4 Taken from [9, Figure 2]. All types of cells c of size $\|c\| \leq 5$ in a non-homotopic connected drawing on at least three vertices. The bottom row shows the degenerate cells.

Edge-Segments and Cells. An edge with i crossings is split into $i+1$ parts, called *edge-segments*. An edge-segment is *inner* if both its endpoints are crossings, and *outer* otherwise. The *planarization* of Γ is the graph obtained by replacing every crossing x with a vertex of degree 4 that is incident to the four edge-segments of x. We say that the drawing Γ is *connected*, if its planarization is a connected graph, and shall henceforth only consider connected drawings. Removing all edges and vertices of Γ splits the sphere into several components, called *cells*. We denote the set of all cells by $\mathcal{C}$. Since Γ is connected, the *boundary* ∂c of a cell c corresponds to a cyclic sequence alternating between edge-segments and elements in $V \cup X$ (i.e., vertices and crossings). If a crossing or a vertex appears multiple times on the boundary of the same cell c, then c is *degenerate*. The *size* of a cell c, denoted by $\|c\|$, is the number of vertex incidences plus the number of edge-segment incidences of c. Note that incidences with crossings are not taken into account, see Figure 4 for examples. For $a \in \mathbb{N}$, we denote by $\mathcal{C}_a = \{c \in \mathcal{C} \colon \|c\| = a\}$ the set of all cells of size a.

▶ **Theorem 2** (Density Formula [9]). *If Γ is a connected drawing with at least one edge, and t is a real number, then*

$$|E| = t(|V| - 2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} \|c\| - t \right) - |X|$$

To apply the Density Formula, we count the cells of different sizes. We distinguish several types of cells based on their size and boundary and denote these by small pictograms, such as or . We call a cell *large* if it has size at least 6 and write for this type of cells. By abuse of notation, we denote the number of cells of a certain type by their pictogram.

Configurations are connected labeled embedded subgraphs of the planarization of a drawing Γ. We denote configuration types by pictograms such as and (see Figure 5).

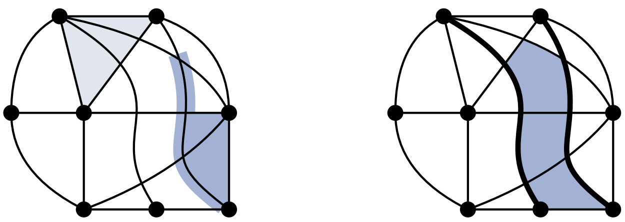

Figure 5 Left: A -configuration (light blue) and a -configuration (dark blue). Right: A --trail (dark blue) and its bounding edges (thick).

A configuration is an *A-B-trail* if its dual is a path P whose endpoints are cells of type $A \neq$ and $B \neq$, respectively, whose edges correspond to inner segments, and whose interior vertices are $\boxed{4}$-cells whose two edge-segments on P are opposite along their boundary, see Figure 5. We denote by $(A \leftrightarrow B)$ the number of A-B-trails in Γ.

▶ **Observation 3.** *Every inner edge-segment of a drawing is interior to exactly one trail.*

A drawing is *filled* if any two vertices $u \neq v$ on the boundary of a cell c are joined by an uncrossed edge along ∂c. A 3-plane, non-homotopic, connected, filled drawing of a graph on at least three vertices is 3-*saturated.*

3 Crossing-Number and Edge-Density via Density Formula

To obtain our upper bounds we prove a number of (in)equalities, each relating the number of certain cells, configurations, edges and crossings. The Density Formula is one such equality. In total, we obtain a system of linear inequalities where each quantity (such as $|E|$, $(|V|-2)$, $|X|$, $|\mathcal{C}_2|$, $|E_1|$, ▽, ○, etc.) can be considered as a variable. Setting the "variable" $(|V|-2)$ to 1, we can maximize the value of $|X|$ by solving the obtained linear program (LP). The resulting maximum represents the number of crossings per vertex; more precisely, per $(|V|-2)$. We want to prove that the number of crossings in any 3-plane drawing on n vertices is at most $5.5(n-2)$. It thus suffices to show that the maximum value of $|X|$ in the LP is 5.5 if we set the variable representing the number of vertices to 1. Our LP comprises 21 constraints, which are summarized in Figure 6. The validity of two constraints (namely (3.C) and (5.A)) is proven in Section 4. Constraints that are only proven in the full version are marked with $(\star)$. Summing up all constraints with the coefficients in Figure 6, we obtain $|X| \leq 5.5(|V|-2)$.

If we maximize $|E|$ instead, we obtain $|E| \leq 5.5(|V|-2)$ from the same constraints (with different coefficients; also in Figure 6). Hence, by verifying that all 21 constraints hold for every connected, non-homotopic 3-plane drawing on $n \geq 3$ vertices, we obtain our result.

▶ **Theorem 1.** *Every non-homotopic 3-plane drawing of a graph on n vertices, $n \geq 3$, contains at most $5.5(n-2)$ edges and at most $5.5(n-2)$ crossings.*

4 Relating Crossing, Edge, Cell, Trail, and Configuration Counts

In this section, we present a number of (in)equalities, each relating the number of certain cells, configurations, edges, or crossings. Due to space constraints we discuss only two of these inequalities, the rest can be found in the full version. Our proof relies on the Density Formula for $t = 5$. For this value of t, ○-cells contribute negatively in the formula. Intuitively, large cells account for many crossings: If many trails end in large cells, we obtain a lower bound on the sum $\sum_{a\geq 6} a|\mathcal{C}_a|$ of sizes of large cells. This yields a lower bound on the sum $\sum_{c\in\mathcal{C}_{\geq 6}}(\|c\|-5)$ in the Density Formula, where $\mathcal{C}_{\geq 6}$ denotes the set of large cells. If there are few such trails, we obtain configurations that contain many crossed edges.

▶ **Lemma 4.** *If Γ is a 3-saturated drawing, then*

$$\sum_{a\geq 6} a|\mathcal{C}_a| \geq (\triangle \leftrightarrow ○) + (\boxed{5} \leftrightarrow ○) + (▽ \leftrightarrow ○) + (\langle 5\rangle \leftrightarrow ○) + 5\,\boxed{6}. \qquad (5.A)$$

Proof. As we want to obtain a lower bound on the sum $\sum_{a\geq 6} a|\mathcal{C}_a|$, it suffices to count the number of vertex and edge-segment incidences of large cells. Each trail that ends in a large cell enters this cell via an inner edge-segment. As no two trails share such an inner edge-segment, we obtain one edge-segment incidence for each such trail.

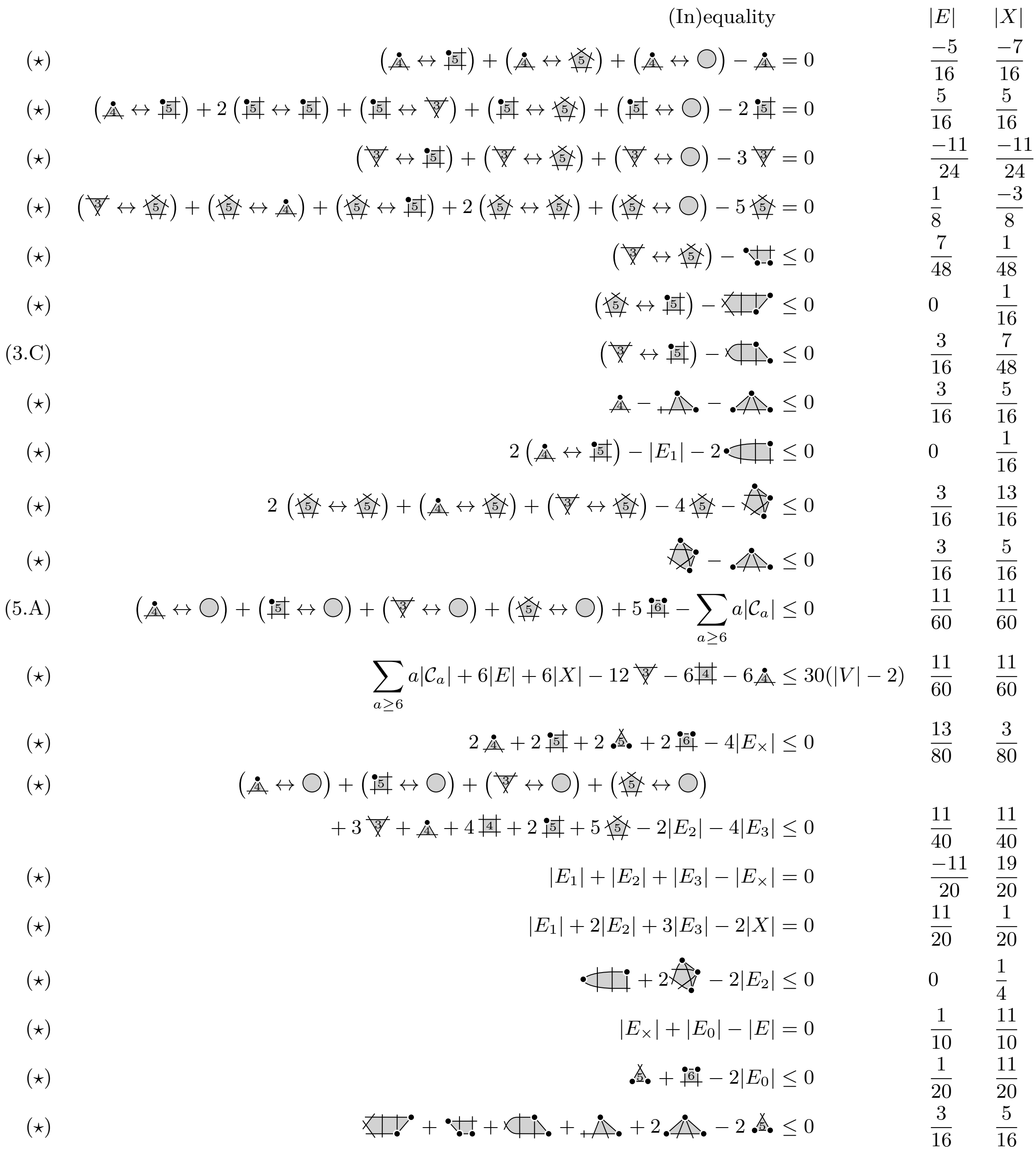

Figure 6 Certificates for the upper bound on the number of edges and crossings in 3-saturated drawings in terms of the number of vertices. Each row corresponds to one inequality. In order to obtain the upper bound on the number of edges, we multiply each inequality with the third entry in the corresponding row and sum up all the inequalities. To obtain the upper bound on the number of crossings we proceed likewise using the fourth entry of each row as a coefficient.

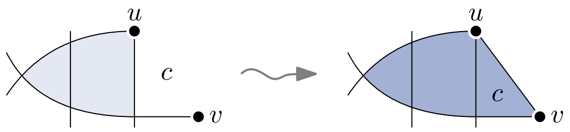

Figure 7 A [3]-[5]-trail (light blue). It forms a [configuration]-configuration (dark) with an adjacent cell.

A [6]-cell is in particular large. As it is incident to only one inner-segment, it is the endpoint of only one trail. We have not counted the remaining three edge-segment incidences and the two vertex incidences when considering trails. Therefore, each [6]-cell yields at least five more edge-segment and vertex incidences. ◀

▶ **Lemma 5.** *If* Γ *is a* 3*-saturated drawing, then*

$$([3] \leftrightarrow [5]) \leq [\text{configuration}]. \tag{3.C}$$

Proof. Consider a [3]-[5]-trail. As every edge is crossed at most three times, the trail contains no [4]-cell and we are in the situation represented in Figure 7. The vertices u and v lie on the boundary of a cell c. As the drawing is 3-saturated, the edge uv is contained in G and the cell c is a [5]-cell. The trail together with c forms a [configuration]-configuration. As every [3]-[5]-trail is only part of one such configuration, the statement follows. ◀

5 Discussion

The *k-planar crossing number* $\mathrm{cr}_k(G)$ is similar to the crossing number, except that the minimum is taken over all k-plane drawings of G. Clearly, $\mathrm{cr}(G) \leq \mathrm{cr}_k(G)$ for all k and G. But there are k-planar n-vertex graphs G with $\mathrm{cr}(G) \in \mathcal{O}(k)$ and $\mathrm{cr}_k(G) \in \Omega(kn)$ [7, Theorem 2]. By Theorem 1, every 3-plane drawing of an n-vertex graph G has $|X| \leq 5.5(n-2)$ crossings, and hence $\mathrm{cr}(G) \leq \mathrm{cr}_3(G) \leq 5.5(n-2)$. Although Theorem 1 is tight, we could have $\mathrm{cr}(G), \mathrm{cr}_3(G) < 5.5(n-2)$, and a similar question arises for 2-planar graphs.

▶ **Question 6.**

- *Are there* 3*-planar* n*-vertex graphs* G *with* $\mathrm{cr}_3(G) = 5.5(n-2)$ *or* $\mathrm{cr}(G) = 5.5(n-2)$*?*
- *Are there* 2*-planar* n*-vertex graphs* G *with* $\mathrm{cr}_2(G) = 3.\overline{3}(n-2)$ *or* $\mathrm{cr}(G) = 3.\overline{3}(n-2)$*?*

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