

# Convex hull alignment through translation

Michael Hoffmann\* Vincent Kusters\* Günter Rote† Maria Saumell‡ Rodrigo I. Silveira§

## Abstract

Given  $k$  finite point sets  $A_1, \dots, A_k$  in  $\mathbb{R}^2$ , we are interested in finding one translation for each point set such that the union of the translated point sets is in convex position. We show that if  $k$  is part of the input, then it is NP-hard to determine if such translations exist, even when each point set has at most three points.

The original motivation of this problem comes from the question of whether a given triangulation  $T$  of a point set is the *empty shape triangulation* with respect to some (strictly convex) shape  $S$ . In other words, we want to find a shape  $S$  such that the triangles of  $T$  are precisely those triangles about which we can circumscribe a homothetic copy of  $S$  that does not contain any other vertices of  $T$ . This is the Delaunay criterion with respect to  $S$ ; for the usual Delaunay triangulation,  $S$  is the circle.

## 1 Introduction

We study the following problem: given  $k$  finite point sets  $A_1, \dots, A_k$  in  $\mathbb{R}^2$ , are there translations  $t_1, \dots, t_k$  such that the union of all  $t_i(A_i)$  is in convex position? For  $k = 1$  the problem is simply convexity testing. For  $k = 2$ , the problem can be solved using linear programming, under the additional assumption that the order of points along the convex hull is fixed. Even without this assumption, the case  $k = 2$  is solvable in polynomial time: if there is a solution for a given instance, then

there is also a solution where a point  $p$  from the first set is collinear with two points  $q$  and  $r$  from the second set. For each such triple, it remains only to determine where  $p$  should be placed along the line  $qr$ , which can easily be done in polynomial time.

For general  $k$  (being part of the input), the straightforward formulation does not yield a linear program, because triples of points may come from different polygons and, thus, involve several translations, which makes the constraints quadratic. Similarly, the problem is easy if the size of the point sets is at most two: sort the line segments  $s_1, \dots, s_n$  by increasing slope and translate such that the right endpoint of each  $s_i$  is identified with the left endpoint of  $s_{i+1}$ . In contrast we prove that the general problem is NP-hard by reduction from 3-SAT:

**Theorem 1** *Given  $k$  finite point sets  $A_1, \dots, A_k$  in  $\mathbb{R}^2$ , it is NP-hard to decide if there are translations  $t_1, \dots, t_k$  such that the union of all  $t_i(A_i)$  is in convex position.*

The reduction uses point sets of size three, as well as a regular polygon with size linear in the size of the 3-SAT formula. We also show that this regular polygon can be replaced by a set of triangles:

**Theorem 2** *Given  $k$  finite point sets  $A_1, \dots, A_k$  in  $\mathbb{R}^2$ , it is NP-hard to decide if there are translations  $t_1, \dots, t_k$  such that the union of all  $t_i(A_i)$  is in convex position, even if each  $A_i$  has size at most three.*

**Motivation.** The original motivation of this problem comes from the question of whether a given triangulation  $T$  of a point set is the *empty shape triangulation* [10, 11] with respect to some (strictly convex) shape  $S$  (Problem A). In other words, we want to find a shape  $S$  such that the triangles of  $T$  are precisely those triangles about which we can circumscribe a homothetic copy of  $S$  that does not contain any other vertices of  $T$ . This is the Delaunay criterion with respect to  $S$ . For the usual Delaunay triangulation,  $S$  is the circle.

An abstraction of this question is the following more general problem (Problem B). We are given families  $(P_i^+, P_i^=, P_i^-)_{i=1,2,\dots}$  of point sets. We look for a (strictly) convex shape  $S$  with the following property: for each  $i$ , we can scale and translate  $S$  so that the three sets  $P_i^+, P_i^=, P_i^-$  lie inside  $S$ , on the boundary, and outside  $S$ , respectively.

In Problem A, each  $P_i^=$  is a triplet corresponding to a triangle, and  $P_i^-$  is the complementary set of points.  $P_i^+$

\*Institute of Theoretical Computer Science, ETH Zürich, Switzerland. [hoffmann|vincent.kusters]@inf.ethz.ch. Partially supported by the ESF EUROCORES programme EuroGIGA, CRP GraDR and the Swiss National Science Foundation, SNF Project 20GG21-134306.

†Institut für Informatik, Freie Universität Berlin, Germany. rote@inf.fu-berlin.de. Supported by the ESF EUROCORES programme EuroGIGA-VORONOI, Deutsche Forschungsgemeinschaft (DFG): grant RO 2338/5-1.

‡Département d'Informatique, Université Libre de Bruxelles, Belgium. maria.saumell.m@gmail.com. Supported by ESF EuroGIGA project ComPoSe as F.R.S.-FNRS - EUROGIGA NR 13604 and ESF EuroGIGA project GraDR as GAČR GIG/11/E023.

§Dept. Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Spain. rodrigo.silveira@upc.edu. Partially supported by projects MINECO MTM2012-30951, Gen. Cat. DGR2009SGR1040, ESF EUROCORES programme EuroGIGA, CRP ComPoSe: MICINN Project EU1-EURC-2011-4306, and by FP7 Marie Curie Actions Individual Fellowship PIEF-GA-2009-251235.

is always empty. Equivalently, we may form quadruples with three triangle points  $P_i^-$  and each remaining point as a singleton set  $P_i^-$ .

We obtain a variation of this problem (Problem B') by allowing only translation of  $S$ , but no scaling. Let Problem C be the special case of Problem B where  $P_i^+ = P_i^- = \emptyset$  and let Problem C' be the same special case of Problem B'. The problem that we consider in this paper is problem C'. Thus, our answer to problem C' does not imply an answer to our original question, since it is *more specialized* in one respect (allowing translations only) and *more general* in another respect (considering arbitrary point sets  $P_i^-$ ).

**Related work.** We are not aware of previous work on the related problems mentioned above. Regarding our main motivation, Problem A, quite some work has been devoted to studying properties of Voronoi diagrams and their duals—often triangulations—based on a particular convex distance function (see e.g. [6, 12]), but not, to the best of our knowledge, to tackling the inverse question in which we are interested here.

The problem of finding a set of translations to place a set of points in convex position is related to certain matching and polygon placement problems. In some matching problems the goal is to find a rigid motion of one shape to make it as *similar* as possible to another shape. Here, similarity can be measured in a variety of ways, such as using Hausdorff distance [2], Fréchet distance [3] or maximizing their area of overlap [7], among others. Polygon placement problems are usually concerned with finding a transformation of a polygon to place it inside another polygon or to contain/avoid certain objects (such as points or other polygons). Multiple variants have been studied in the past, depending on the type of polygon (e.g. convex [14] or simple [4]), the type of transformation (e.g. translation versus translation and rotation [4]), and the final goal (e.g. to fit the largest possible copy of a polygon inside another one [1], or to cover as many points as possible [8]). We remark that, besides having clearly different goals, these problems are usually concerned with two single shapes or objects, whereas we are interested in translating  $k > 2$  points sets altogether.

**2 Reduction from 3-SAT**

This section is devoted to the proof of Theorem 1. We prove the theorem by reduction from 3-SAT. Given a 3-SAT formula  $F$  with variables  $x_0, \dots, x_{n-1}$  and clauses  $C_0, \dots, C_{m-1}$ , construct point sets as follows.

Let  $R = r_0, r'_0, \dots, r_{t-1}, r'_{t-1}$  be a regular<sup>1</sup> convex

<sup>1</sup>A sufficiently fine rational approximation [5] of  $R$  suffices. The main property of  $R$  that is important for our reduction is that opposite sides are parallel.

polygon on  $2t = 8n + 16m$  points centered at the origin. Given an edge  $r_i r'_i$  of  $R$ , we define the *pocket*  $P_i$  of  $r_i r'_i$  to be the compact set bounded by the line segment  $r_i r'_i$  and the lines  $\ell^-$  through  $r'_{i-1} r_i$  and  $\ell^+$  through  $r'_i r_{i+1}$ . The intersection of  $\ell^-$  and  $\ell^+$  is the *apex* of pocket  $P_i$ . Our reduction relies on the following fact: if a point  $p$  is placed on an open line segment  $r_i r'_i$ , then placing a point  $q$  in  $P_i \setminus r_i r'_i$  destroys convexity of  $R \cup \{p, q\}$ . We say that  $p$  *blocks* the pocket  $P_i$ . Unblocked pockets are called *free*. Note that placing a point in  $P_i$  does not prevent us from placing a point in any other pocket. We say that two convex sets are *compatible* if their union is convex.

**Variable gadget.** In order to encode the variables of our formula, we will first define triangles  $Q_i^d = (q_i, q_{i+d}, q_{i+t/2})$  for  $0 \leq i < t$  and  $0 \leq d < t$  as follows (note that in this section all indices are taken modulo  $t$ , and that  $t$  is even). Consider the diametrically opposed segments  $r_i r'_i$  and  $r_{i+t/2} r'_{i+t/2}$ . Place  $q_{i+t/2}$  on  $r_{i+t/2}$  and place  $q_{i+d}$  on the apex of  $P_{i+d}$  (Figure 1). Translate  $q_{i+t/2}$  and  $q_{i+d}$  rigidly, parallel along  $r_{i+t/2} r'_{i+t/2}$  until  $q_{i+d}$  hits the line segment  $r_{i+d} r'_{i+d}$ . Now place  $q_i$  on  $r_i$ . Note that  $Q_i^d$  can slide freely along  $r_i r'_i$  until:

- (1)  $q_i$  hits  $r_i$  and  $q_{i+d}$  hits the line segment  $r_{i+d} r'_{i+d}$ : moving further would move  $q_{i+d}$  inside the convex hull of  $R$ ; or
- (2)  $q_{i+t/2}$  hits  $r_{i+t/2}$  and  $q_{i+d}$  hits the apex of  $P_{i+d}$ : moving further would push  $q_{i+d}$  outside the pocket, removing  $r_{i+d}$  and  $r'_{i+d}$  from the joint convex hull.

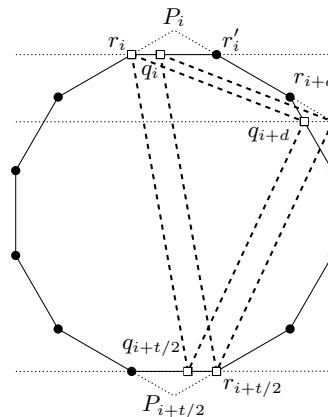


Figure 1: The construction of variable gadget  $Q_i^d$ . The figure shows the two extreme positions of the gadget.

Since  $r_i r'_i$  and  $r_{i+t/2} r'_{i+t/2}$  are diametrically opposed segments, moving  $Q_i^d$  in any other direction would place either  $q_i$  or  $q_{i+t/2}$  inside the convex hull of  $R$ . For most of the compatible translations, the pockets  $P_i$  and

$P_{i+t/2}$  are both blocked. However, for the two extreme positions (shown in Figure 1), where  $q_{i+d}$  touches the apex or the line segment, exactly one of the pockets is free.

We say that the *state* of  $Q_i^d$  is *true* if  $Q_i^d$  does not block  $P_i$ , *false* if it does not block  $P_{i+t/2}$  and *undefined* if  $P_i$  and  $P_{i+t/2}$  are both blocked. After defining the other two gadgets, we will associate some  $Q_j^d$  with each variable  $x_i$ , with the interpretation that  $x_i$  is true if and only if the state of  $Q_j^d$  is true and  $\bar{x}_i$  is true if and only if the state of  $Q_j^d$  is false. If the state of  $Q_j^d$  is undefined, then both  $x_i$  and  $\bar{x}_i$  are false. Note that if  $R$  and a translation of  $Q_i^d$  are compatible, then  $x_i$  and  $\bar{x}_i$  are not both true.

**Copy gadget.** Given a variable gadget  $Q_i^d$ , we can copy the state of  $Q_i^d$  with the gadget  $Q_{i+k}^{d-k}$ , provided  $k \notin \{0, t/2, d\}$ . This can be seen as follows. Assume for the moment that  $Q_i^d$  is in one of its two extreme positions. The gadget  $Q_{i+k}^{d-k}$  shares the pocket  $P_{i+d}$  with  $Q_i^d$  (see Figure 2). If  $Q_i^d$  touches the apex of  $P_{i+d}$  (as in Figure 2), then so must  $Q_{i+k}^{d-k}$  in order to be compatible with  $R$  and the translation of  $Q_i^d$ . Similarly, if  $Q_i^d$  touches the open line segment  $r_{i+d}r'_{i+d}$ , then so must  $Q_{i+k}^{d-k}$ . Hence, the state of  $Q_{i+k}^{d-k}$  is completely determined by the state of  $Q_i^d$ . Specifically, the state of  $P_i$  is copied to  $P_{i+k}$  and the state of  $P_{i+t/2}$  is copied to  $P_{i+k+t/2}$ . If  $Q_i^d$  is not in an extreme position, then neither is  $Q_{i+k}^{d-k}$  and hence both states are undefined.

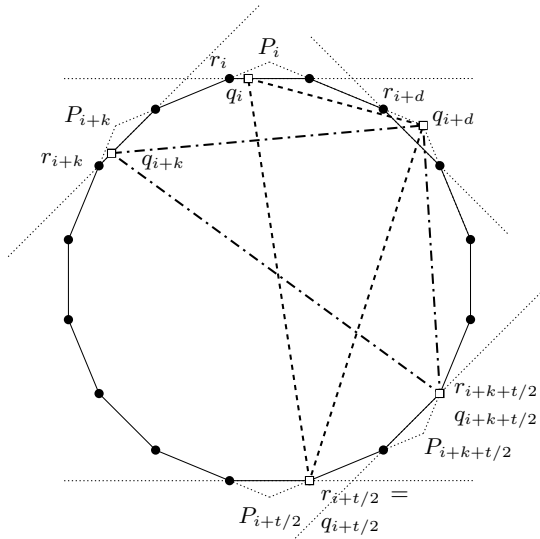


Figure 2: The construction of copy gadget  $Q_{i+k}^{d-k}$  with  $k = -1$ .

**Clause gadget.** Given  $i, j$  and  $k$ , let  $T_{ijk}$  be the triangle whose vertices  $t_i, t_j, t_k$  are the midpoints of the seg-

ments  $r_i r'_i, r_j r'_j$  and  $r_k r'_k$ , respectively. For each clause  $C$  on variables  $x, y, z$ , we will use three copy gadgets to copy the states of the literals  $x$  (or  $\bar{x}$ ),  $y$  (or  $\bar{y}$ ) and  $z$  (or  $\bar{z}$ ) of  $C$  to pockets  $P_i, P_j, P_k$  such that  $T_{ijk}$  contains the origin. Some care must be taken to ensure that this clause does not interfere with existing gadgets, but we will cover this issue in the subsection below. Figure 3 shows an example for a clause  $C = x \vee \bar{y} \vee z$ . Only the relevant corner of each copy gadget is shown.

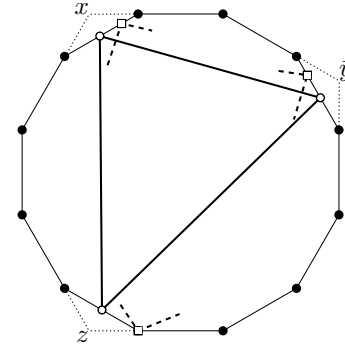


Figure 3: The construction of a clause gadget for a clause  $C = x \vee \bar{y} \vee z$ .

Note that  $x$  and  $\bar{y}$  are both in the false (or undefined) state, whereas  $z$  is in the true state. We now scale  $T_{ijk}$  by a factor of  $1 + \varepsilon$  for  $\varepsilon$  sufficiently small. Since triangle  $T_{ijk}$  contains the origin, it no longer fits inside  $R$ . If it did not contain the origin, a small translation towards the origin would potentially bring it back in convex position with  $R$ . If one of the pockets  $P_i, P_j$  or  $P_k$  is free, e.g.  $P_i$ , then we can translate  $T_{ijk}$  such that  $t_j$  is again on  $r_j r'_j$  and  $t_k$  is again on  $r_k r'_k$  and  $t_i$  is inside  $P_i$ . This translation of  $T_{ijk}$  is compatible with  $R$ : hence, the clause  $C$  is satisfied. If all three pockets are blocked, then there is no translation for which  $T_{ijk}$  is compatible with  $R$  and hence  $C$  is not satisfied.

**Selecting the gadgets.** We are now ready to explicitly define the gadgets to be used for the given 3-SAT formula  $F$  with variables  $x_0, \dots, x_{n-1}$  and clauses  $C_0, \dots, C_{m-1}$ . We partition the polygon into paths  $A, B'_1, B_2, Q, B'_3, X, A', B_1, B'_2, Q', B_3, X'$  as shown in Figure 4. The paths  $A, Q, A'$  and  $Q'$  all have length  $2n$ . The other paths have length  $2m$ . Hence  $R$  has size  $2t = 8n + 16m$ . We will not use the paths  $X, Q'$  and  $X'$ . Associate with each variable  $x_i$  the variable gadget  $Q_i^{2m+n}$ . This places the variable gadgets in paths  $A, Q$  and  $A'$ . For each clause  $C_i = \{\ell_a, \ell_b, \ell_c\}$  add copy gadgets as follows. If  $\ell_a = \bar{x}_a$  (i.e. the literal is negative), then add  $Q_{i+n}^{a+2m-i}$  to copy the value of  $x_a$  to the  $i$ th pocket in  $B'_1$ . This copies the value of  $\bar{x}_a$  into the  $i$ th pocket in  $B_1$ . Alternatively, if  $\ell_a = x_a$  (i.e. the literal is positive), then add  $Q_{i+3n+4m}^{a-2n-2m-i}$  to copy the value of  $x_a$  to the  $i$ th pocket in  $B_1$ . Copy  $\ell_b$  and  $\ell_c$  analogously to

$B_2$  and  $B_3$  and add the clause gadget corresponding to the selected pocket. Note that any triangle of vertices selected from  $B_1, B_2, B_3$  contains the origin.

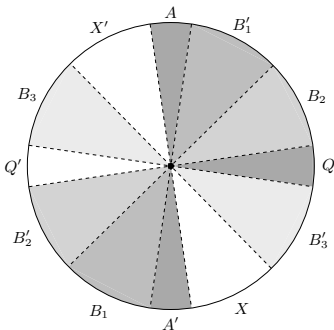


Figure 4: Placing the gadgets on the polygon.

**Correctness.** If our 3-SAT formula  $F$  is satisfiable, then set the state of the initial variable gadgets according to a satisfying assignment of  $F$ . The copy gadgets preserve the state of these initial variable gadgets, so each clause gadget has at least one free pocket. Hence, the union of  $R$  with all gadgets is in convex position. Conversely, if  $F$  is unsatisfiable, then suppose for the sake of obtaining a contradiction that the union of  $R$  with all gadgets is in convex position. Consider the assignment  $\alpha$  of  $F$  corresponding to the state of the variable gadgets (this may set some  $x_i$  and  $\bar{x}_i$  both to false). Since  $F$  is unsatisfiable, there must be a clause  $C$  that is not satisfied by assignment  $\alpha$ . Since the copy gadgets preserve the state of the original variable gadgets, all pockets of  $C$  are blocked. Hence,  $C$  is not compatible with the other gadgets and the union of  $R$  with all gadgets cannot be in convex position, which yields a contradiction. Theorem 1 follows.

### 3 Replacing a regular polygon by triangles

In this section we show that we can replace the regular polygon from our reduction above by a set of triangles. This will prove Theorem 2.

**Proposition 3** *Let  $p_1, p_2, \dots, p_n$  be the vertices of a regular  $n$ -gon in counterclockwise order. Let us consider the family of triangles*

$$\mathcal{T} = \{ \Delta p_1 p_2 p_3, \Delta p_2 p_3 p_4, \dots, \Delta p_n p_1 p_2, \Delta p_1 p_2 p_{n-2}, \Delta p_2 p_3 p_{n-1}, \dots, \Delta p_n p_1 p_{n-3} \}.$$

*Let  $S$  be a point set obtained by translating each triangle in  $\mathcal{T}$  independently. If  $S$  is in convex position, then  $S$  consists of the vertices of a regular  $n$ -gon.*

The rest of this section is devoted to the proof of Proposition 3. For a given set  $S$  of points in the plane, let  $CH(S)$  denote the *vertices* of the convex hull of  $S$ .

Let  $T \subseteq \mathcal{T}$ . Suppose that every triangle  $T_i \in T$  is translated according to some vector  $\lambda_i$ , and let  $S_{T_\lambda}$  be the resulting set of points. We call  $S_{T_\lambda}$  a *geometric placement* of  $T$ . A placement of  $T$  is called *combinatorial* if the order of the vertices of  $T$  around the convex hull is known, but not the exact position of the triangles. We say that  $S_{T_\lambda}$  satisfies the *convex condition* if all points of  $S_{T_\lambda}$  belong to  $CH(S_{T_\lambda})$ . In this case we might also say that the placement of the triangles is *valid*. We say that  $p \in S_{T_\lambda}$  *violates* the convex condition if  $p \notin CH(S_{T_\lambda})$ .

Every vertex of the regular  $n$ -gon belongs to several triangles of  $\mathcal{T}$ . To distinguish the distinct copies of the vertex, we use superscripts, so for example  $p_1^i$  will denote vertex  $p_1$  in some particular triangle of  $\mathcal{T}$ . We set  $T_1 = \Delta p_1^1 p_2^1 p_3^1$ ,  $T_2 = \Delta p_2^2 p_3^2 p_4^2, \dots$ , and  $T_n = \Delta p_n^n p_1^n p_2^n$ . We also set  $T_{n+1} = \Delta p_1^{n+1} p_2^{n+1} p_{n-2}^{n+1}$ ,  $T_{n+2} = \Delta p_2^{n+2} p_3^{n+2} p_{n-1}^{n+2}, \dots$ , and  $T_{2n} = \Delta p_n^{2n} p_1^{2n} p_{n-3}^{2n}$ . We assume that  $n$  is a multiple of 4, that the regular  $n$ -gon is oriented so that two of its sides are horizontal, and that  $p_2 p_3$  is the bottom horizontal side.

**Observation 1** *Let  $p \in S \subseteq S'$ . If  $p \notin CH(S)$ , then  $p \notin CH(S')$ .*

**Lemma 4** *Let  $T = \{T_1, T_2\}$ . Suppose that  $S_{T_\lambda}$  satisfies the convex condition, and  $p_2^1 p_3^1$  is not collinear with  $p_2^2 p_3^2$ . Then the counterclockwise order of  $S_{T_\lambda}$  in  $CH(S_{T_\lambda})$  is either  $p_1^1 p_2^1 p_2^2 p_3^2 p_4^2 p_3^1$  or  $p_1^1 p_2^1 p_3^1 p_3^2 p_4^2 p_2^2$ .*

**Proof.** Suppose that we fix the position of  $T_1$ . Then if we extend the line segments of  $T_1$  to lines, the plane is decomposed into four open regions  $F_1, \dots, F_4$  and three closed regions  $R_1, \dots, R_3$ , as shown in Figure 5. It is easy to verify that in order to satisfy the convex condition, no point of  $T_2$  can lie in any of the regions  $F_1, F_2, F_3$ , and no vertex of  $T_2$  can lie in  $F_4$ . In particular,  $p_2^2, p_3^2$ , and  $p_4^2$  can lie only in  $R_1 \cup R_2 \cup R_3$ . In principle, there are 27 cases to consider, based on all possible combinations. Fortunately, we only need to distinguish a few situations.

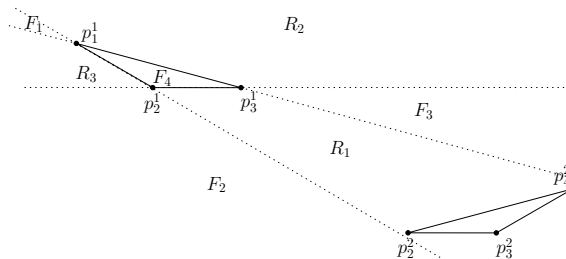


Figure 5: A valid combinatorial placement of  $T_1$  and  $T_2$ .

We first suppose that  $p_2^2 \in R_1$ . Notice that in this case it is not possible that  $p_3^2 \in R_2$  or  $p_3^2 \in R_3$ . So we can assume that  $p_3^2 \in R_1$ . Now we can easily rule

out the case  $p_4^2 \in R_3$ . Point  $p_4^2$  cannot lie in  $R_2$  either, because in this case the side  $p_3^2 p_4^2$  would intersect  $F_3$ . Therefore  $p_4^2$  can lie only in  $R_1$ . This gives a valid combinatorial placement of  $T_1$  and  $T_2$  where the counterclockwise order of  $S_{T_\lambda}$  in  $CH(S_{T_\lambda})$  is  $p_1^1 p_2^1 p_3^2 p_4^2 p_3^1 p_4^1$  (see Figure 5).

Next suppose that  $p_2^2 \in R_2$ . Then,  $p_3^2 \in R_2$  and  $p_4^2 \in R_2$ . This gives another valid combinatorial placement of  $T_1$  and  $T_2$  where the counterclockwise order of  $S_{T_\lambda}$  in  $CH(S_{T_\lambda})$  is  $p_1^1 p_2^1 p_3^2 p_4^2 p_2^2 p_3^1$  (see Figure 6). Note that there are no other possible combinatorial placements since we assume that  $p_2^1 p_3^1$  and  $p_2^2 p_3^2$  are not collinear.

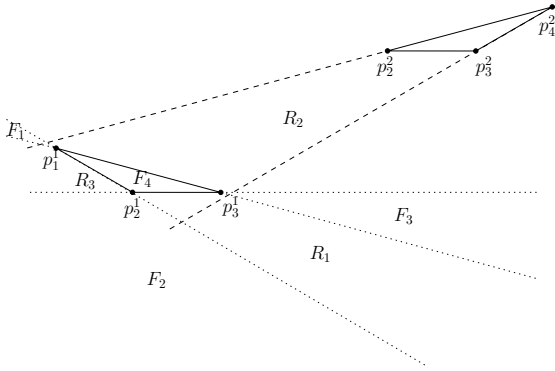


Figure 6: A valid combinatorial placement of  $T_1$  and  $T_2$ .

Finally suppose that  $p_2^2 \in R_3$ . Then,  $p_3^2 \notin R_1$  and  $p_4^2 \notin R_1$ . Recall that  $p_2^2$  is on the same horizontal line as  $p_3^2$  and to its left, and  $p_4^2$  is above and to the right of  $p_3^2$ . If  $p_3^2 \in R_2$  and  $p_4^2 \in R_2$ , then we have that  $p_3^1$  is below and to the right of  $p_2^2$ . Consequently,  $p_3^2$  lies inside the triangle  $\Delta p_2^2 p_4^2 p_3^1$  and violates the convex condition. The combination  $p_3^2 \in R_2$  and  $p_4^2 \in R_3$  is not possible. If  $p_3^2 \in R_3$ , then we have that  $p_2^2$  is below and to the right of  $p_3^2$  (since we are assuming that  $p_2^1 p_3^1$  is not collinear with  $p_2^2 p_3^2$ ). Consequently,  $p_3^2$  lies inside the triangle  $\Delta p_2^2 p_4^2 p_2^1$  and violates the convex condition.  $\square$

**Lemma 5** *Let  $T = \{T_1, T_2, T_{\frac{n}{2}+1}, T_{\frac{n}{2}+2}\}$ . If  $S_{T_\lambda}$  satisfies the convex condition, then  $p_2^1 p_3^1$  is collinear with  $p_2^2 p_3^2$ .*

**Proof.** [Sketch] Due to space limitations, we only sketch the proof here. It suffices to prove that the combinatorial placements for  $\Delta p_1^1 p_2^1 p_3^1$  and  $\Delta p_2^2 p_3^2 p_4^2$  of Lemma 4 are no longer valid. To show that the combinatorial placement in which the counterclockwise order around the convex hull is  $p_1^1 p_2^1 p_3^2 p_4^2 p_3^1 p_4^1$  is no longer valid, we try to add  $T_{\frac{n}{2}+1}$  to the placement, and prove that this is not possible. The other combinatorial placement from Lemma 4 can be ruled out analogously.  $\square$

By symmetry, we have the following corollary:

**Corollary 6** *If  $S_{T_\lambda}$  satisfies the convex condition, then  $p_2^1 p_3^1$  is collinear with  $p_2^2 p_3^2$ ,  $p_3^2 p_4^2$  is collinear with  $p_3^3 p_4^3, \dots$ , and  $p_1^n p_2^n$  is collinear with  $p_1^1 p_2^1$ .*

**Lemma 7** *If  $S_{T_\lambda}$  satisfies the convex condition, then  $p_2^1 p_3^1$  is collinear with  $p_2^2 p_3^2$ , and  $p_2^1$  is not right of  $p_2^2$ .*

We omit the proof due to space considerations.

Suppose that, in some placement of  $T_{i-1}$  and  $T_i$ ,  $p_i^{i-1} p_{i+1}^{i-1}$  is collinear with  $p_i^i p_{i+1}^i$ . Suppose that we rotate the triangles so that  $T_{i-1}$  has the same orientation as  $T_1$  and  $T_i$  has the same orientation as  $T_2$ . We say that  $T_{i-1}$  and  $T_i$  cross if, after this rotation,  $p_i^{i-1}$  is to the right of  $p_i^i$ . By symmetry, we have the following corollary, which subsumes Corollary 6 and Lemma 7:

**Corollary 8** *Suppose that  $S_{T_\lambda}$  satisfies the convex condition. Then  $p_2^1 p_3^1$  is collinear with  $p_2^2 p_3^2$ ,  $p_3^2 p_4^2$  is collinear with  $p_3^3 p_4^3, \dots$ , and  $p_1^n p_2^n$  is collinear with  $p_1^1 p_2^1$ . Furthermore, none of the pairs  $\{T_i, T_{i+1}\}$  cross.*

**Lemma 9** *Suppose that  $S_{T_\lambda}$  satisfies the convex condition. Then  $p_2^1 p_3^1$  is collinear with  $p_2^2 p_3^2$ , and  $p_2^1$  is not to the left of  $p_2^2$ .*

**Proof.** We proceed by contradiction. Consider the regular  $n$ -gon having one side at  $p_2^2 p_3^2$ . Since none of the pairs  $\{T_i, T_{i+1}\}$  cross, this polygon lies inside the polygon formed by the placement of  $T_1, \dots, T_n$  (see Figure 7 for an example). Since we are assuming that  $p_2^1$  is to the left of  $p_2^2$ , we also have that  $p_n$  of the polygon lies strictly inside (that is, not on the boundary) the polygon formed by the placement of  $T_1, \dots, T_n$ . We now try to place  $T_{n+3}$ . In order to maintain the convex condition, the vertices of  $T_{n+3}$  must be placed on the boundary of the polygon formed by the placement of  $T_1, \dots, T_n$ , or on the triangles bounded by an edge of the polygon and two dotted segments shown in Figure 7. Such a placement of  $T_{n+3}$  is not possible.  $\square$

By symmetry, we obtain the following corollary, which subsumes Lemma 9 and Corollary 8:

**Corollary 10** *Suppose that  $S_{T_\lambda}$  satisfies the convex condition. Then  $p_2^1$  is on the same position as  $p_2^2$ ,  $p_3^2$  is on the same position as  $p_3^3, \dots$ , and  $p_1^n$  is on the same position as  $p_1^1$ . Equivalently, the vertices of  $\Delta p_1^1 p_2^1 p_3^1, \Delta p_2^2 p_3^2 p_4^2, \dots, \Delta p_n^n p_1^n p_2^n$  form a regular  $n$ -gon.*

To complete the proof of Proposition 3, it only remains to see that the triangles  $T_{n+1}, T_{n+2}, \dots, T_{2n}$  are also placed in the “natural” way. We prove it in the next lemma for  $T_{n+2}$  and, by symmetry, the result also holds for the other triangles.

**Lemma 11** *Suppose that  $S_{T_\lambda}$  satisfies the convex condition. Then,  $\Delta p_2^{n+2} p_3^{n+2} p_{n-1}^{n+2}$  is placed so that  $p_2^{n+2}$  is on the same position as  $p_2^2$ .*

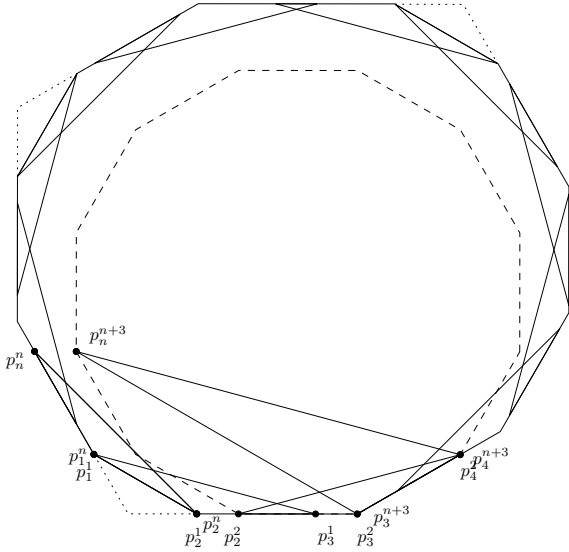


Figure 7: The vertex  $p_n^{n+3}$  cannot be placed in Lemma 9 if  $p_2^1$  is to the left of  $p_2^2$  (case 1).

**Proof.** We know that the vertices of  $T_1, T_2, \dots, T_n$  form a regular  $n$ -gon. In order to maintain the convex condition, the vertices of  $T_{n+2}$  must be placed on the boundary of the regular  $n$ -gon or on the triangles bounded by an edge of the polygon and two dotted segments shown in Figure 8. It is clear that the only way to do it consists in placing  $p_2^{n+2}$  on the same position as  $p_2^2$ .  $\square$

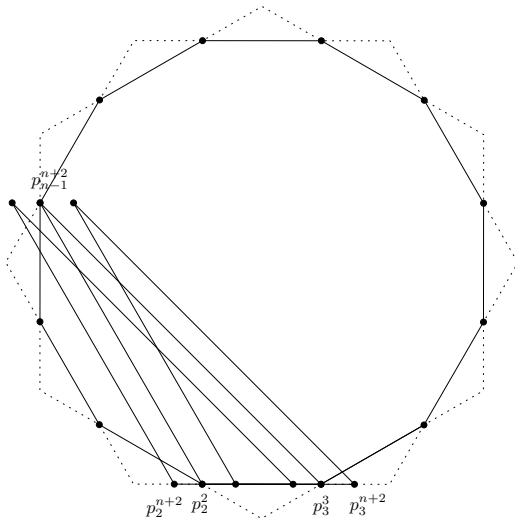


Figure 8: Points with the same superscript must again coincide after the translations.

#### 4 Concluding remarks

If we allow scaling in addition to translation (Problem C from the introduction), then it is not known if the prob-

lem is still NP-hard. In addition, it is not clear if our problem is in NP. The obvious certificate would be the sequence of translations, but one must argue that the representation of these translations is not too large in terms of the input representation. In fact, our problem has some similarities to the order type realizability problem, which is known to be complete for the existential theory of the reals [13], and the coordinate representation of some order types requires exponential storage [9].

**Acknowledgments.** This research was initiated during a EuroGIGA workshop in Assisi (Italy) in March 2012. We thank Jan Kratochvíl, Tillmann Miltzow, Alexander Pilz, and Vera Sacristán for helpful discussions.

#### References

- [1] P. K. Agarwal, N. Amenta, and M. Sharir. Largest placement of one convex polygon inside another. *Discrete Comput. Geom.*, 19(1):95–104, 1998.
- [2] H. Alt, B. Behrends, and J. Blömer. Approximate matching of polygonal shapes. *Ann. Math. Artif. Intell.*, 13(3-4):251–265, 1995.
- [3] H. Alt and M. Godau. Measuring the resemblance of polygonal curves. *Proc. SCG '92*, pp. 102–109, 1992.
- [4] F. Avnaim and J.-D. Boissonnat. Polygon placement under translation and rotation. *ITA*, 23(1):5–28, 1989.
- [5] J. Canny, B. R. Donald, and E. K. Ressler. A rational rotation method for robust geometric algorithms. *Proc. SCG '92*, pp. 251–260, 1992.
- [6] L. P. Chew and R. L. S. Drysdale, III. Voronoi diagrams based on convex distance functions. *Proc. SCG '85*, pp. 235–244, 1985.
- [7] M. de Berg, O. Cheong, O. Devillers, M. van Kreveld, and M. Teillaud. Computing the maximum overlap of two convex polygons under translations. *Theory Comput. Syst.*, 31(5):613–628, 1998.
- [8] M. Dickerson and D. Scharstein. Optimal placement of convex polygons to maximize point containment. *Comput. Geom.*, 11(1):1–16, 1998.
- [9] J. E. Goodman, R. Pollack, and B. Sturmfels. Coordinate representation of order types requires exponential storage. *Proc. STOC '89*, pp. 405–410, 1989.
- [10] T. Lambert. Systematic local flip rules are generalized delaunay rules. *Proc. CCCG '93*, pp. 352–357, 1993.
- [11] T. Lambert. *Empty-Shape Triangulation Algorithms*. PhD thesis, University of Manitoba, 1994.
- [12] L. Ma. Bisectors and Voronoi diagrams for convex distance functions. Master’s thesis, FernUniversität Hagen, 2000.
- [13] M. Schaefer. Complexity of some geometric and topological problems. In *Graph Drawing*, volume 5849 of *LNCS*, pp. 334–344. Springer, 2010.
- [14] M. Sharir and S. Toledo. External polygon containment problems. *Comput. Geom.*, 4:99–118, 1994.