An Improved Searching Algorithm on a Line by Four Truthful Robots and Two Byzantine Robots*

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We study the following one-dimensional online search problem. A collection of n robots starts at the origin of the real line, seeking to find a treasure that is hidden at an unknown location $g \in \mathbb{R}$. The robots can move independently at unit speed and are capable to detect (and report) the treasure whenever they are at the same position g. The goal is to design an algorithm that allows to locate the treasure quickly, regardless of where it is located.

The problem here comes with an additional twist, as introduced by Czyzowicz et al. [1]. Among the n robots there are f that are *Byzantine*, which may provide false reports. That is, they may report a treasure at a position where it is not, and they may be silent at a position where the treasure is. A robot that is not Byzantine is referred to as *truthful*.

As usual in an online setting, we measure the performance of a strategy by its worst case *competitive ratio*, that is, in relation to an optimal offline algorithm that knows the target position g in advance. In this problem, the optimal offline algorithm is obvious: Directly move all robots from the origin to the goal location g. This takes $|g|$ time units. Hence, an algorithm that uses t time units to find the treasure at position g is said to have competitive ratio $t/|g|$.

Among others, the case $(n, f) = (6, 2)$ was studied by Czyzowicz et al. [1]. They claimed an algorithm with competitive ratio 4 and gave a lower bound of 3. We improve the upper bound.

Theorem 1. *There is an algorithm to find a treasure on a line with six robots, two of which are Byzantine with competitive ratio at most* $\sqrt{13} < 3.61$.

In this abstract, we only give an outline of the algorithm, and the analysis will be omitted.

*The work was done at 15th Gremo's Workshop on Open Problems, Pochtenalp, Switzerland. The authors thank the participants for an inspiring atmosphere. The work by M.H. was supported by the Swiss National Science Foundation within the collaborative DACH project *Arrangements and Drawings* as SNSF Project 200021E-171681. The work by Y.O. was partially supported by JSPS KAKENHI 15K00009 and JSPS CREST JPMJCR1402.

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The algorithm consists of up to five phases. In the first phase, the robots are split into two groups of size three arbitrarily. One group moves left and the other group moves right. The first phase ends as soon as a robot reports the treasure. Let k denote the time at which this report occurs, which is the same as the distance of both groups from the origin at the end of Phase 1. Suppose without loss of generality that a report at time k comes from the group of robots in the positive halfline. We distinguish two cases.

Case 1: two or more robots report the treasure at position k. Then, we let the two groups of robots move to exchange their position. At time $3k$, all robots have visited the location k and so we know by majority vote if this is the treasure location. If so, we are done. If not, then we know the two Byzantine robots and discard them from consideration. We continue by moving the group at position k to the right and the group at position $-k$ to the left. As soon as one robot reports the treasure, we are done.

Case 2: exactly one robot reports the treasure at position k. Then, Phase 2 of the algorithm begins. We discard the robot that issued the report from consideration so that only two robots remain in the group at position k. At the beginning of Phase 2, one of the remaining robots from each group switches back and reverses direction. We let the robots move in this way for some time $\alpha \in [0, k]$, where $\alpha = (\sqrt{13} - 3)k/2 < 0.303k$ is a good choice for this parameter. In other words, during Phase 2 there are four groups of robots moving together: one robot in $[k, 2k]$ moving right, one robot in $[0, k]$ moving left (called the *green robot*), one robot in $[-k, 0]$ moving right (the *blue robot*), and two robots in $[-2k, k]$ moving left.

At time $k + \alpha$, the second phase ends. At the beginning of the third phase, one robot (the *red robot*) from the leftmost group of two robots (at position $-k - \alpha$) switches back and reverses direction. The third phase ends when the blue robot reaches position k at time $3k$. We distinguish three cases.

Case 2.1: At some point during Phase 2 or Phase 3 another robot reports the treasure. If it is one of the three colored robots (red, blue, or green), then we can immediately conclude that it is Byzantine. As we have at most two Byzantine robots, we conclude that both reports are wrong. We simply continue to sweep the line with the two black robots in extreme position and will eventually find the treasure at a position g in optimal time $|g|$.

It remains to consider the case that one of the uncolored (*black*) robots reports the treasure at a position $k' \in (k, 3k]$. We discard the reporting robot from consideration so that only four robots remain. We let all remaining robots run their course and continue in whatever direction they are heading.

At time $3k$ the blue robot reaches k. If it confirms the treasure at k, then—one way or another—two robots lied at k. Therefore, the treasure is at either k or k' and we know where as soon as the red robot reaches k at time $3k + 2\alpha$.

Otherwise, the blue robot denies a treasure at k and we conclude that the initial report at k was wrong. We send the red robot to k' and let the other robots continue in their current direction. We consider three subcases depending on the position of k'. Note that at time $3k$, the red robot is at position $k - 2\alpha > 0$.

Case 2.1.1: $-k - \alpha \le k' < -k$. Then, the red robot has already seen k' and remained silent. Hence, when the green robot reaches k' at time $k' + 2k$ we know if the treasure is there. If so, we are done. Otherwise, we found the two Byzantine robots and either the right black robot finds the treasure in optimal time, or the green robot finds it at $g < -k$.

Case 2.1.2: $k' < -k - \alpha$. Then the red robot reaches k' at time $4k + |k'| - 2\alpha$. In addition to the reporting black robot, both the green and the red robot have visited k' at this point. Thus we know whether or not the treasure is there by a majority vote. If the treasure is at k', then we found it at time at most $\sqrt{13}k$. Otherwise, we found the two Byzantine robots and either the right black robot finds the treasure in optimal time, or the green robot finds it at a position $g < k'$.

Case 2.1.3: $k < k'$. Then, the red robot reaches k' at time $\sqrt{13}k'$. At this point, three robots (black, blue, and red) have seen k' and so we know whether or not the treasure is there by a majority vote. Therefore, if the treasure is at k', then we are done. Otherwise, we found the two Byzantine robots and either the left black robot finds the treasure in optimal time, or the red robot finds it at $g > k'$.

This completes the analysis of Case 2.1. Hence in the following we may assume that no robot reports the treasure during Phases 2 and 3. We continue our analysis at the end of Phase 3.

Case 2.2: the blue robot reports the treasure at position k, when reaching it at time $3k$ at the end of Phase 3. We discard the blue robot from consideration and wait for the red robot to arrive at k. Both extreme black robots continue in their current direction. The red robot reaches k at time $\sqrt{13}k$. If it confirms the treasure at k, then we are done. Otherwise, all remaining robots are truthful and one of the black robots finds the treasure at a position g, with $|g| > 3k$. and we know by time $2|g|$ that the report is correct.

Case 2.3: the blue robot does not report the treasure at position k at the end of Phase 3. Then, we know that the first report was wrong and only one Byzantine robot remains. We enter Phase 4, where all robots continue in their current direction except for the red robot, which switches back to the origin. Phase 4 ends when the red robot reaches the origin at time $4k - 2\alpha$.

If no robot reports the treasure during Phase 4, then Phase 5 starts where the red robot remains at the origin while the other four robots continue in their current direction. Ultimately either the blue or the green robot reports the treasure at a position g, with $|g| \ge 2(k - \alpha)$. (If one of the black robots reports it, then it is even better.) We immediately send the red robot over to check, while letting all other robots continue in their current direction.

It remains to consider the case that a robot reports the treasure at a position g during Phase 4. If the report comes from a black robot, then we can simply wait until the blue or green robots reaches g. At this point the red robot will have reached the origin and we can argue as above for Phase 5. Hence, suppose that the blue or green robot reports the treasure. There are two final cases.

Case 2.3.1: the blue robot reports the treasure during Phase 4 at a position $k' \in (k, 2(k - \alpha)]$. Then, we immediately switch around the red robot to head for k'. If the treasure is at k', then we are done. Otherwise, the remaining robots are truthful and we eventually find the treasure with a black robot in optimal time.

Case 2.3.2: the green robot reports the treasure during Phase 4 at a position $k' < -k$. If $k' \ge -k - \alpha$, we immediately know that the report at k' is wrong and continue as before with the remaining black robots, both of which are truthful.

Hence we may suppose that $k' < -k - \alpha$. The green robot reaches k' at time $|k'| + 2k$, and the red robot reaches k' at time at most $\sqrt{13}k'$.

References

[1] J. Czyzowicz, K. Georgiou, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. M. Shende. Search on a line by Byzantine robots. In *ISAAC 2016*, pp. 27:1–27:12, 2016.