Connecting Points in the Presence of Obstacles in the Plane

Michael Hoffmann[1] Csaba D. Tóth[1,2]

Abstract
Given a point set P and a set B of polygonal obstacles in the plane, we consider planar geometric graphs connecting the points of P and crossing few obstacles from B. We describe two finite constructions and an algorithm. The constructions show that it is not possible to draw a spanning tree or Hamiltonian circuit without crossing an obstacle a certain number of times. The algorithm shows that the vertices of axis-parallel rectangles – in contrast to general rectangles – can be connected by a Hamiltonian circuit without crossing obstacles.

1 Introduction

Let B be a set of disjoint closed convex objects in the plane and let P be a point set disjoint from the relative interior of every object. A *geometric planar graph* G_P is a graph drawn in the plane on the vertex set P such that (1) the edges are straight line segments between the points in the plane and (2) the relative interiors of distinct edges are disjoint.

The *crossing number* $\mathrm{cr}(G_P, b)$ of a geometric graph G_P with respect to an object $b \in B$ is the number of edges that *cross* b. For an object $b \in B$ with non-empty interior, an edge e crosses b if it intersects $\mathrm{int}(b)$. For a line or line segment b, edge e crosses b if e intersects the relative interior of b but e is not collinear with b. The crossing number $\mathrm{cr}(G_P, B)$ of the set B of objects with respect to G_P is simply the sum $\sum_{b\in B} \mathrm{cr}(G_P, b)$.

We are interested in geometric planar *spanning trees* (for short, spanning tree) and geometric planar *Hamiltonian circuits* (for short Hamiltonian circuit) with low or zero crossing number w.r.t. certain types of families of polygonal objects. Our setting generalizes two classes of problems described below.

The first class includes problems where the point set is independent of the obstacles. Matoušek [3] proved that if B contains one line only, then there is a spanning tree T_P with $\mathrm{cr}(T_P, B) \leq \sqrt{|P|}$, and this bound is optimal. Asano et al. [1] studied the crossing number of spanning trees where every point of P is disjoint from the objects in B. They showed that there exists a spanning tree T_P such that $\mathrm{cr}(T_P, B) = O(|P| + |B|)$. Moreover, if S is a set of disjoint line segments, then there exists a spanning tree T_P such that $\mathrm{cr}(T_P, s) \leq 4$ for every $s \in S$, and therefore $\mathrm{cr}(T_P, S) \leq 4|S|$.

The best know lower bound for this last result relies on a construction where S consists of n disjoint sides of a convex $2n$-gon and P consists of n points along those sides outside the $2n$-gon: In this example, , the crossing number is $\mathrm{cr}(T_P, S) \geq 2|S| - 2 = 2n - 2$ for any spanning tree T_P and there is at least one segment $s \in S$ for which $\mathrm{cr}(T_P, s) \geq 2$. No linear upper bound is known for $\mathrm{cr}(H_P, S)$ where H_P is a Hamiltonian circuit.

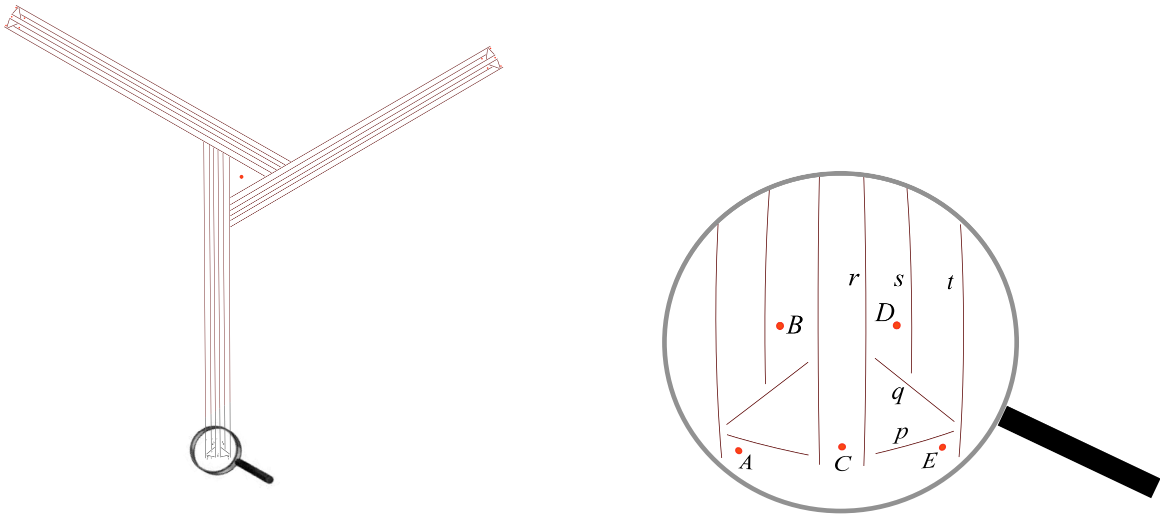
In Section 2, we give a construction for a set S of disjoint line segments and a point set P where for any spanning tree T_P there is at least one segment $s \in S$ such that $\mathrm{cr}(T_P, s) \geq 3$.

The second class of problems are discussed in Section 3 and Section 4. Here the point set P is predetermined by B: The convex objects in B are polygonal and the point set P is the set of all their vertices $V(B)$. If $P = V(B)$ then a spanning tree $T_{V(B)}$ with $\mathrm{cr}(T_{V(B)}, B) = 0$ always exists, therefore we turn our attention to Hamiltonian circuits.

Answering a question of Mirzaian [4], we have shown recently [2] that for any set S of n disjoint line segments (not all on a line) and the set $V(S)$ of $2n$ segment endpoints there exists a Hamiltonian circuit $H_{V(B)}$ with $\mathrm{cr}(H_{V(S)}, S) = 0$. This implies that the segment endpoint visibility graph of S is Hamiltonian. Earlier, Urabe and Watanabe [6] gave a construction where S does not always have a Hamiltonian circuit $H_{V(S)}$ circumscribing all segments. Rappaport [5] showed that it is NP-complete to decide if S admits a Hamiltonian circuit $H_{V(S)}$ containing every segment $s \in S$ as an edge of $H_{V(S)}$.

A natural generalization of the problem asks the following: For a set B of n disjoint k-gons in the plane and the set $V(B)$ of kn vertices of the k-gons, does there always exists a Hamiltonian circuit $H_{V(B)}$ with $\mathrm{cr}(H_{V(B)}, B) = 0$? We give negative answer to this question for all $k \geq 4$. Our construction for $k = 4$ can be realized by rectangles. We give a positive answer for $k = 4$ if all quadrilaterals in B are axis-parallel rectangles. The question remains open for $k = 3$.

[1]Institute for Theoretical Computer Science, ETH Zürich, CH-8092, Switzerland, {hoffmann,toth}@inf.ethz.ch.
[2]Supported by the joint Berlin-Zürich graduate program "Combinatorics, Geometry, and Computation".



(a) Global view.

(b) Points in one bundle.

Figure 1: Construction for Theorem 1, consisting of 16 points and 30 segments.

2 Segments and points

Theorem 1 *There is a set S of 30 line segments and a set P of 16 points in the plane, such that for any geometric spanning tree T_P there is a segment from S which is crossed by T_P at least three times.*

Proof. Consider the construction in Figure 1, and let T be an arbitrary (not necessarily planar) geometric spanning tree for P. Note that P contains a point M at the center, and five points in each each of the outer parts of the three rotational symmetric segment bundles. Point M has to be connected to at least one other point, but the bundled segments have been chosen to be sufficiently long, such that any segment connecting M to another point crosses one of the three bundles completely. Let us restrict our focus to this particular bundle, whose outer part is shown in Figure 1(b).

Look at point C: obviously there has to be a path in T from C to the points lying in the other bundles. The first option is, that C is directly connected to M or to another bundle, w.l.o.g. by an edge to the right. This edge crosses the two vertical segments next to D. But since all vertical segments in this bundle are already crossed once by assumption, and since any edge incident to D (there must be one such edge in T) crosses at least one of r and s, there are three crossings on one of these two edges.

Next possibility is that edge CD or BC, w.l.o.g. CD, is in T, and there is an edge from D to M or to another bundle to the right (to the left would intersect r the third time). But then there is no way to connect E in T: edge DE generates a third crossing of s, and any other edge would cross either r or t for a third time.

Another option is to connect C via edge CE or CA, w.l.o.g. CE, and to connect E to M or to another bundle to the right. But similarly to the previous case, there remains no way to connect D without crossing one of r, t, and p three times.

Finally, there remain two possibilities, again w.l.o.g. we restrict our attention to the part to the right of C: either CE and ED are edges of T, and D is connected to the outside world; or CD and DE are edges of T, and E is connected to the outside world. In the first case one of r and s is crossed three times, in the latter case q is crossed three times. □

3 Rectangles and all vertices

Lemma 2 *There is a set R of 13 disjoint rectangles such that for any Hamiltonian circuit $H_{V(R)}$, the crossing number $\mathrm{cr}(H_{V(R)}, R)$ is at least one.*

Proof. Our construction is depicted on Figure 2. Six rectangles are labeled by numbers at their vertices, seven rectangles are labeled by capital letters. Suppose that there is a Hamiltonian circuit $H^0_{V(R)}$ with $\mathrm{cr}(H^0_{V(R)}, R) = 0$. Our first observation is that $H^0_{V(R)}$ contains the vertices of the convex hull of R in the same (cyclic) order as they appear along the convex hull. The convex hull of R contains the ver-

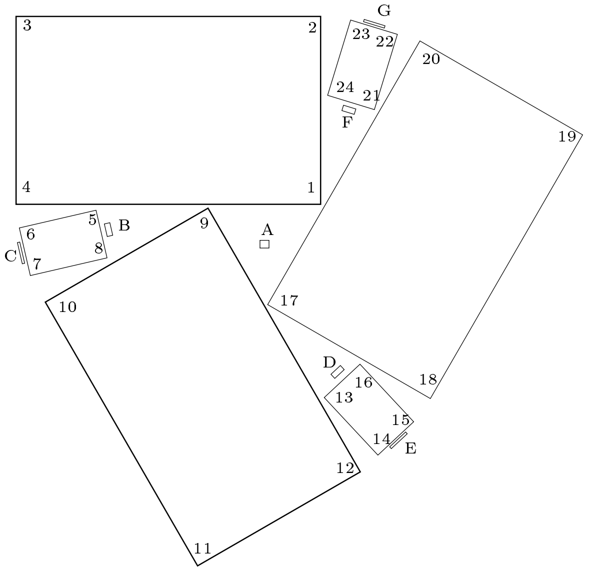


Figure 2: Construction with 13 rectangles.

tices 2, 3, 4, 6, 7, 10, 11, 12, 14, 15, 18, 19, 20, 22, 23 in this order and also two-two vertices of the rectangles C, E, and G.

Next we observe that the segments $(1, 5)$, $(9, 13)$, and $(17, 21)$ crosses the rectangles $[9, 10, 11, 12]$, $[17, 18, 19, 20]$ and $[1, 2, 3, 4]$ respectively. Therefore, in order to access the vertices of A, the circuit $H^0_{V(R)}$ contains one of the subsequences $(4, 1, 9, 10)$, $(12, 9, 17, 18)$, and $(20, 17, 1, 2)$ – in counterclockwise direction with possibly other vertices between them.

Assume w.l.o.g. that $H^0_{V(R)}$ contains the subsequence $(4, 1, 9, 10)$. We argue that $H^0_{V(R)}$ cannot visit the vertices of F. Indeed: the vertex 1 is visited by the subsequence $(4, 6, C, 7, 10)$ along the convex hull, while vertex 20 is in the subsequence $(20, 22, G, 23, 2)$. Neither subsequence can visit any vertex of F. □

For every $k \geq 5$, we can generate similar constructions with already 10 disjoint k-gons. We consider the construction depicted on Figure 2 and add new vertices close to one of the vertices of each rectangle. We can remove the polygons C, E, and G, if we add the fifth and further vertices of the polygons $[5, 6, 7, 8]$, $[13, 14, 15, 16]$, and $[21, 22, 23, 24]$ on the convex hull between $(6, 7)$, $(14, 15)$, and $(22, 23)$ respectively. These vertices can assure that a Hamiltonian circuit $H_{V(B)}$ with zero crossing number traverses these three intervals in this order. Therefore:

Theorem 3 *For any k, there is a set B of n disjoint k-gons such that for any Hamiltonian circuit $H_{V(B)}$, the crossing number $\mathrm{cr}(H_{V(B)}, B)$ is at least one.*

4 Axis-parallel rectangles

In the rest of the paper, we consider disjoint axis-parallel rectangles in the plane and outline the proof of the following theorem.

Theorem 4 *For any set R of disjoint axis-parallel rectangles, there is a Hamiltonian circuit $H_{V(R)}$ with $\mathrm{cr}(H_{V(R)}, R) = 0$.*

Given a set R of axis-parallel rectangles, we build a Hamiltonian circuit recursively. We say that a vertex v of H_i is *convex* (*reflex*) if the simple polygon enclosed by H_i has a convex (reflex) angle at v.

We initialize our algorithm by $H_1 = r_1$, the circuit around the rectangle whose top side has maximal y-coordinate. For every $i \geq 1$, the graph H_i is a Hamiltonian circuit on a subset of $V(R)$ with $\mathrm{cr}(H_i, R) = 0$. In each step, H_{i+1} contains strictly more vertices than H_i, therefore we generate a graph $H_{V(A)}$ with no crossing in a finite number of steps.

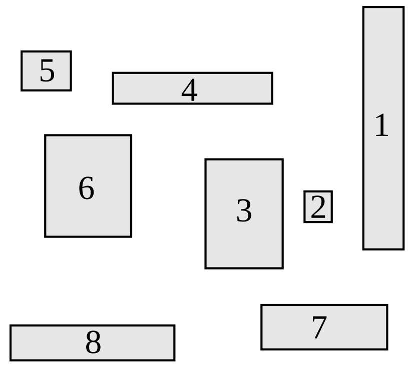


Figure 3: An example with twelve rectangles and a possible processing order.

During the algorithm, we determine recursively a linear (*processing*) order on the rectangles. The first rectangle in our order is r_1. Furthermore, let $R_i = \{r_j : 1 \leq j \leq i\}$, for $j = 1, 2, \ldots, |R|$, and let $\overline{R_i} = R \setminus R_i$. We maintain the following three properties on the circuits H_i.

(1) the vertices of H_i are exactly the $4i$ vertices of the rectangles in R_i,
(2) H_i circumscribes the rectangles in R_i,
(3) Every rectangle in $\overline{R_i}$ is exterior to H_i,
(4) $\mathrm{cr}(H_i, R) = 0$.

Suppose we are given a circuit H_i, $i \in \mathbb{N}$, we determine the next rectangle r_{i+1} and H_{i+1}. Our goal is to find a *augmenting* pair $([abcd], ef)$ where $[abcd] \in \overline{R_i}$ and $ef \in E(H_i)$ such that $afeb$ is a simple quadrilateral which does not intersect the interior of any rectangle of R nor the interior of

H_i. If such a pair $([abcd], ef)$ exists, then we let $r_{i+1} = [abcd]$. Replacing the edge ef of H_i by the polygonal chain (e, a, d, c, b, f), we obtain a circuit H_{i+1} satisfying properties (1)–(4) stated above.

For finding an augmenting pair, we consider only the polygon circumscribed by H_i and the rectangles in $\overline{R_i}$. We maintain two geometric properties of H_i. They clearly hold for H_1:

(5) For any side v_1v_2 of H_i, the axis-parallel rectangle spanned by v_1v_2 is disjoint from every rectangle in $\overline{R_i}$. (See Figure 4.)

(6) For every rectangle $r \in \overline{R_i}$, there is a side ab of r and a side ef of H_i such that the interior of the quadrilateral $afeb$ is disjoint from H_i.

In order to maintain Property (5), we require a bit more from an augmenting pair.

Definition 5 *A pair $([abcd], ef)$, where $[abcd] \in \overline{R_i}$ and ef is a side of H_i is an* augmenting *pair if the segments af and af are disjoint, and the quadrilateral $afeb$ and the axis-parallel rectangles spanned by af and eb are disjoint from the interior of every rectangle in $\overline{R_i}$ and from the interior of H_i.*

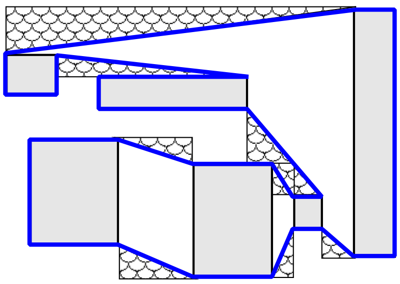


Figure 4: Rectangles spanned by sides of H_i are disjoint from the rectangles of $\overline{R_i}$.

If Property (6) is not maintained, then possibly there is no augmenting pair, as indicated in Figure 5. Similarly, if the first rectangle in our algorithm is $[abcd]$ in Figure 5 then there is, again, no augmenting pair. That is, the choice of r_1 is used by our algorithm.

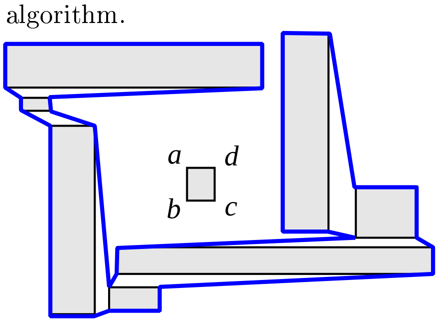


Figure 5: There is no augmenting pair if Property (6) is violated.

Let $[a_1b_1c_1d_1]$ be the rectangle in $\overline{R_i}$ whose top side, a_1b_1, has highest y-coordinate. Ideally, we would like to find an augmenting pair with $[abcd]_1$. This is possible if the top side of $[a_1b_1c_1d_1]$ is below H_i. Denoting by e_1f_1 the lowest side of $[a_1b_1c_1d_1]$, necessarily $[abcd]_1, e_1f_1)$ is an augmenting pair (see Figure 6).

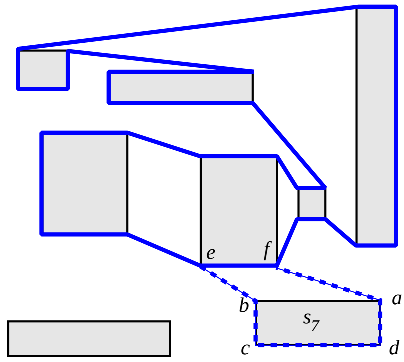


Figure 6: $[a_1b_1c_1d_1]$ forms an augmenting pair with the lowest side of H_i.

If the top side of $[a_1b_1c_1d_1]$ is above the lowest side of H_i, then the horizontal slab of $[a_1b_1c_1d_1]$ intersects H_i. Let e_1f_1 be the side of H_i closest to $[a_1b_1c_1d_1]$ within its horizontal slab. If $([a_1b_1c_1d_1], e_1f_1)$ is an augmenting pair, the recursion step is complete (see Figure 7).

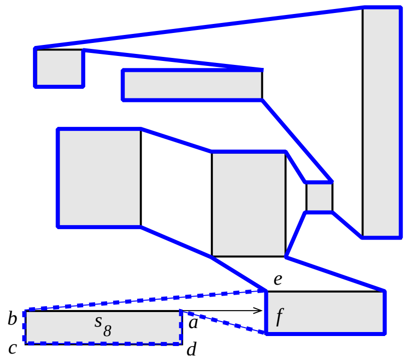


Figure 7: $[a_1b_1c_1d_1]$ forms an augmenting pair with the closest side of H_i in its horizontal slab.

Otherwise we consider recursively a series of pairs $([a_jb_jc_jd_j], e_jf_j)$, $j = 1, 2, \ldots$ until we find eventually an augmenting pair. There are two possible reasons why a pair $([a_jb_jc_jd_j], e_jf_j)$ is not augmenting: (A) the quadrilateral $a_jf_je_jb_j$ or the axis-parallel rectangles spanned by a_je_j or b_jf_j intersects a rectangle in $R_i \setminus \{[a_jb_jc_jd_j]\}$; (B) the one of the segments a_je_j and b_jf_j crosses H_i. We consider the two cases separately.

In case (A), we preserve $e_{j+1}f_{j+1} = e_jf_j$, but we choose a new rectangle. Let $[a_{j+1}b_{j+1}c_{j+1}d_{j+1}] \in$

$\overline{R_i}$ be the rectangle closest to e_jf_j among all rectangles of $\overline{R_i}$ intersected by the quadrilateral $a_jf_je_jb_j$ or the axis-parallel rectangles spanned by a_je_j or b_jf_j such that the side $a_{j+1}b_{j+1}$ faces to e_jf_j (see an example in Figure 8).

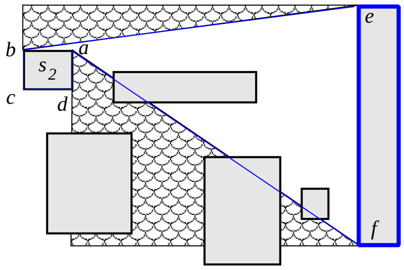


Figure 8: $a_1b_1f_1e_1$ intersects a rectangle from $\overline{R_1}$.

In case (B), we choose another side $e_{j+1}f_{j+1}$ but keep the rectangle $[a_{j+1}b_{j+1}c_{j+1}d_{j+1}] = [a_jb_jc_jd_j]$. Let $e_{j+1}f_{j+1}$ be the side of H_i closest to $[a_jb_jc_jd_j]$ among all sides of H_i intersected by the quadrilateral $a_jf_je_jb_j$ or the axis-parallel rectangles spanned by a_je_j or b_jf_j (see an example in Figure 9).

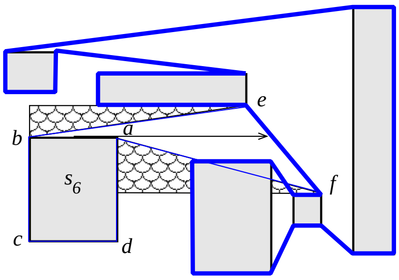


Figure 9: $a_1b_1f_1e_1$ intersects the interior of H_5.

After a finite number steps, we can find an augmenting pair and all six invariants are maintained for the augmented circuit H_{i+1}.

5 Open questions

If S is a set of disjoint segments and P is a set of points: Is there always a spanning tree T_P such that $\text{cr}(T_P, s) \leq 3$ for every $s \in S$? Is there always a spanning tree T_P such that $\text{cr}(T_P, S) < 2|S|$?

If T is a set of disjoint triangles: Is there always a Hamiltonian circuit $H_{V(T)}$ with $\text{cr}(H_{V(T)}, T) = 0$? If Q is a set of disjoint quadrilaterals: Is there always a Hamiltonian circuit $H_{V(Q)}$ with $\text{cr}(H_{V(Q)}, q) \leq 1$ for every $q \in Q$? Given a set B of disjoint polygons, is there a polynomial algorithm to decide if there exists a Hamiltonian circuit $H_{V(B)}$ with $\text{cr}(H_{V(B)}, B) = 0$?

References

[1] T. Asano, M. de Berg, O. Cheong, L. J. Guibas, J. Snoeyink, and H. Tamaki, Spanning trees crossing few barriers, in: *Proc. 15th ACM Symp. Comput. Geom.*, 1999, ACM Press, pp. 41–48.

[2] M. Hoffmann and Cs. D. Tóth, Segment endpoint visibility graphs are Hamiltonian, in: *Proc. 13th Canad. Conf. Comput. Geom.* 2001, pp. 109–112.

[3] J. Matoušek, Spanning trees with low crossing number, *Inform. Théor. Appl.* **25** (1991), 103–123.

[4] A. Mirzaian, Hamiltonian triangulations and circumscribing polygons of disjoint line segments, *Comput. Geom. Theory Appl.* **2** (1) (1992), 15–30.

[5] D. Rappaport, Computing simple circuits from a set of line segments is NP-complete, *SIAM J. Comput.* **18** (6) (1989), 1128–1139.

[6] M. Urabe and M. Watanabe, On a counterexample to a conjecture of Mirzaian, *Comput. Geom. Theory Appl.* **2** (1) (1992), 51–53.