# Advice Complexity of Adaptive Priority Algorithms: Part 2 – Lower Bounds

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# Overview

1. Old lower bound for priority algorithms without advice
2. Gadget pairs
3. Lower bound for optimality — Vertex Cover
4. Lower bounds for approximation
    1. Template for proving hardness results
    2. Example results

Section 1

# Old Lower Bound Result for Priority Algorithms without Advice

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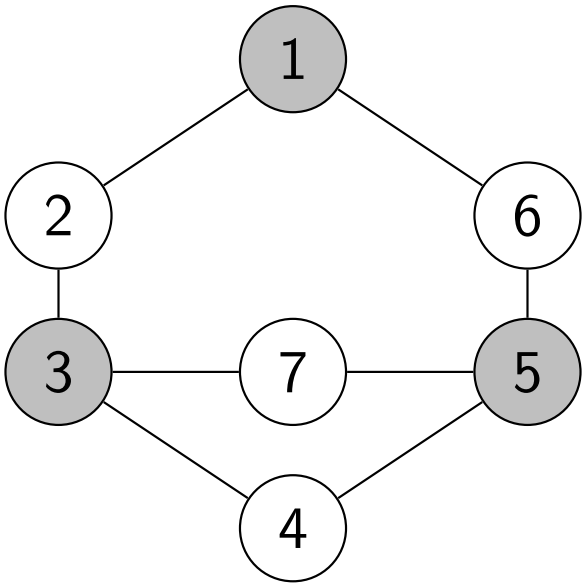


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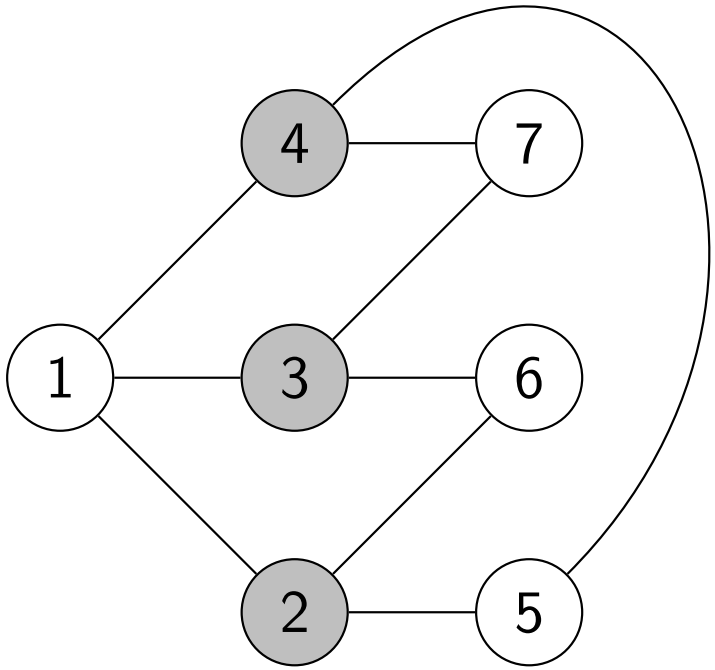


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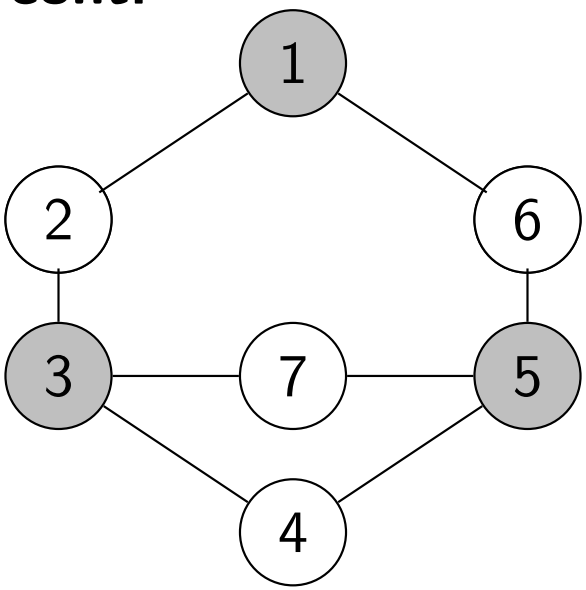


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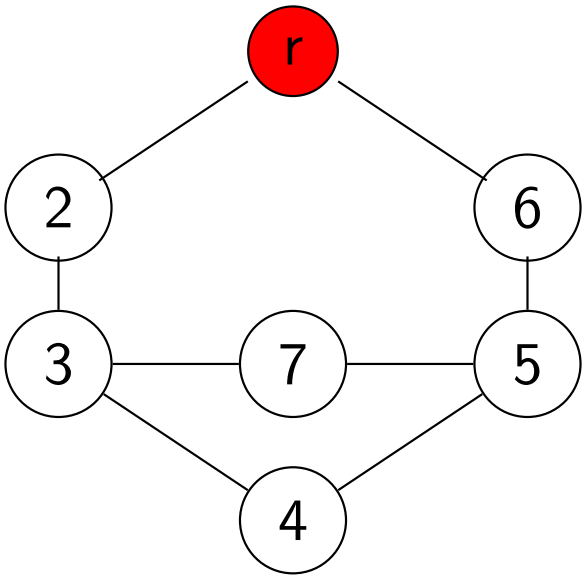


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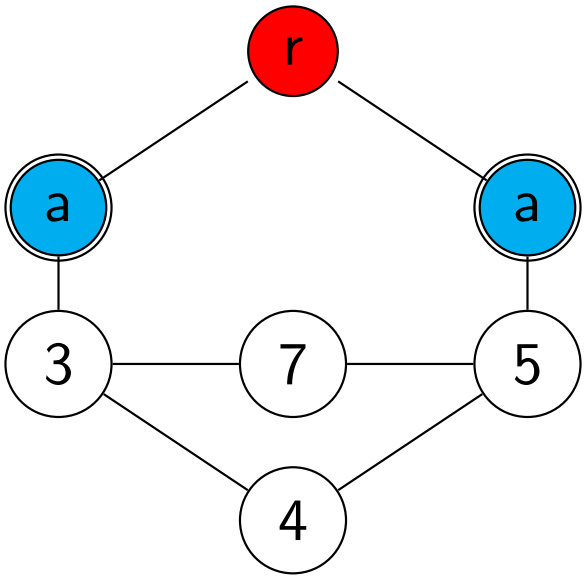


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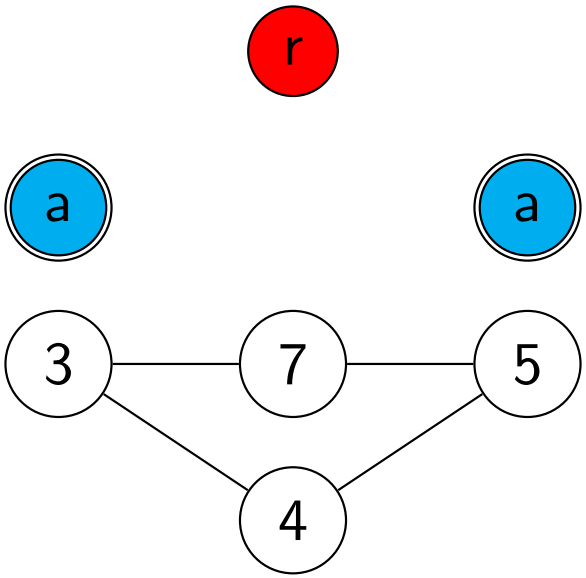


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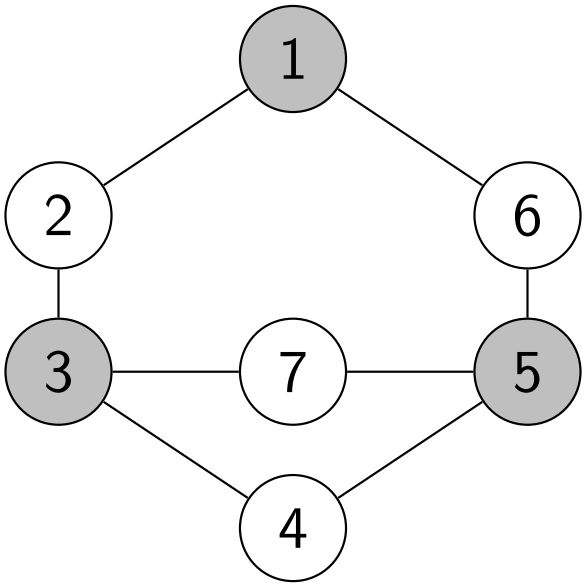
- If $v$ has degree 2 and ALG rejects, ADV chooses vertex 1 in $G_1$.
- If $v$ has degree 3, and ALG accepts, ADV chooses vertex 1 in $G_2$.
- If $v$ has degree 3 and ALG rejects, ADV chooses vertex 3 in $G_1$.

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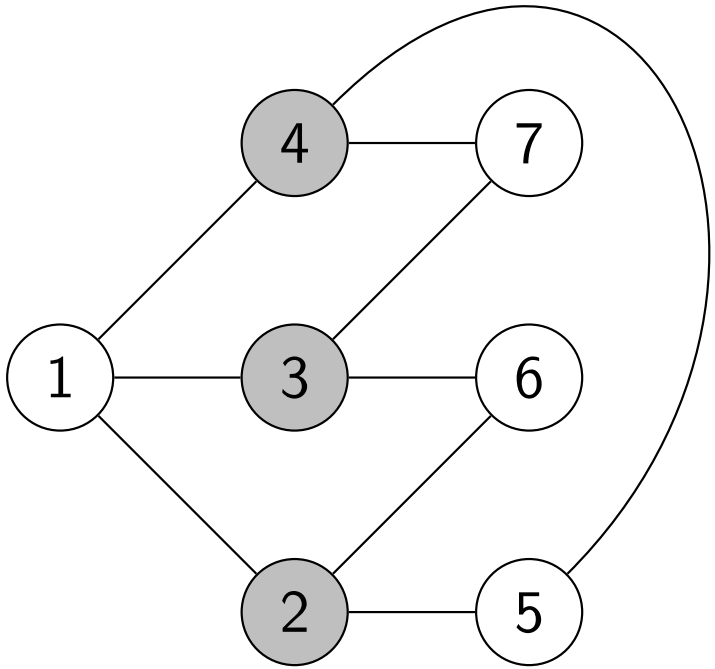


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In all cases, ALG accepts $\geq 4$ vertices, but 3 is optimal. $\square$

Section 2

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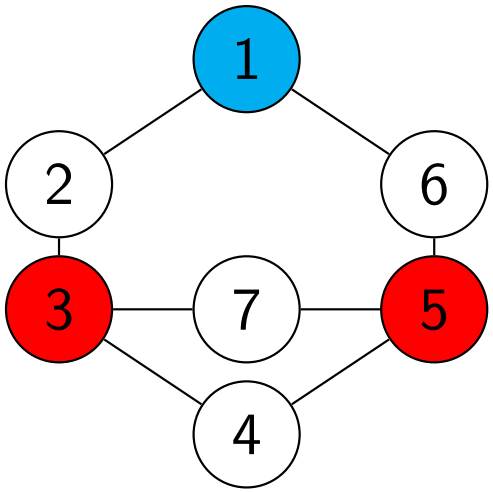


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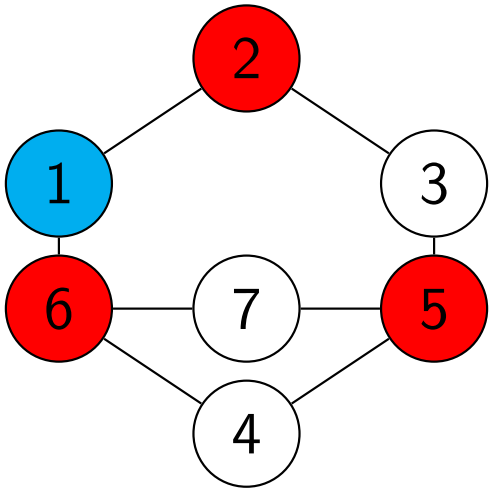


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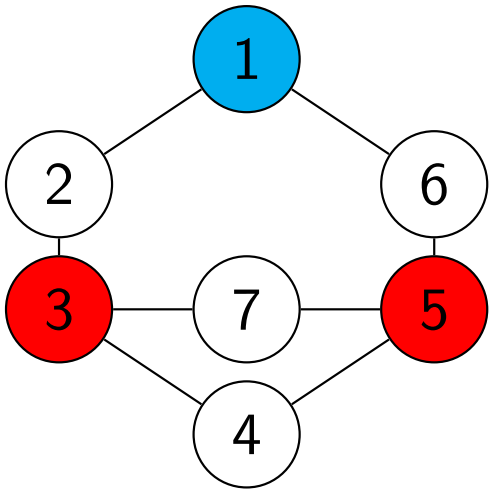


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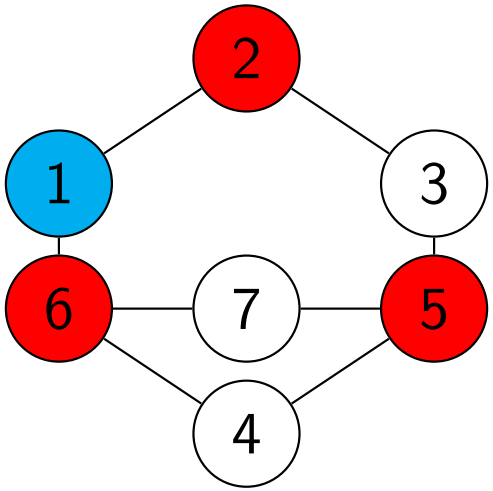


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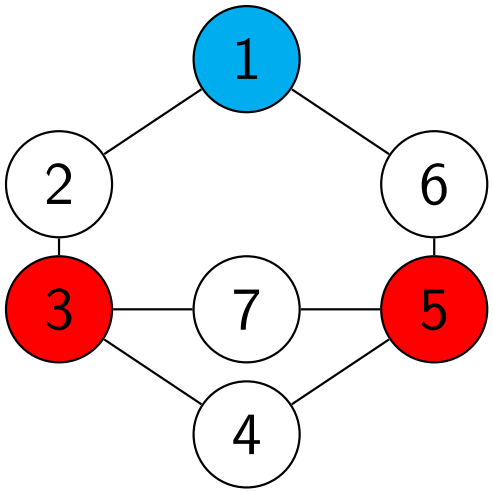


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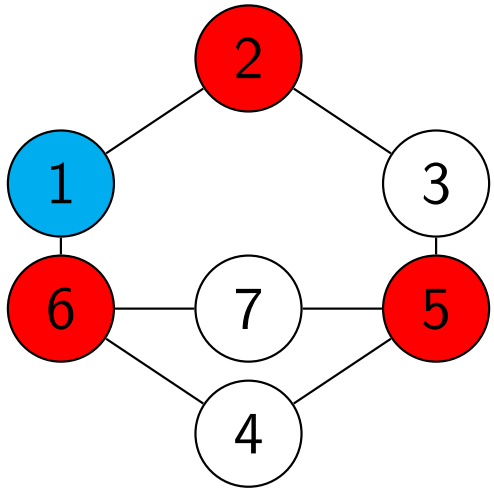


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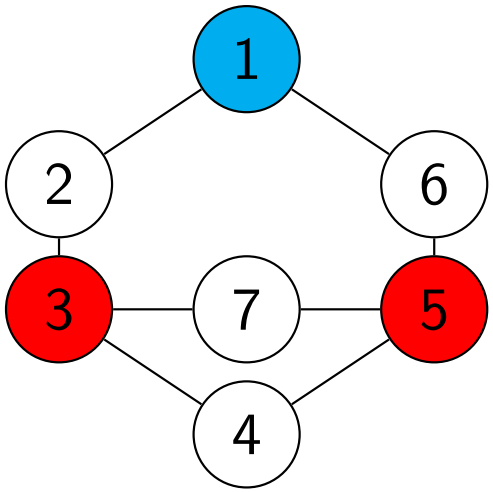


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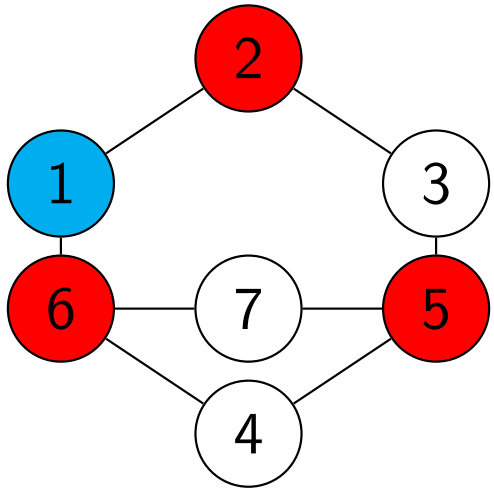


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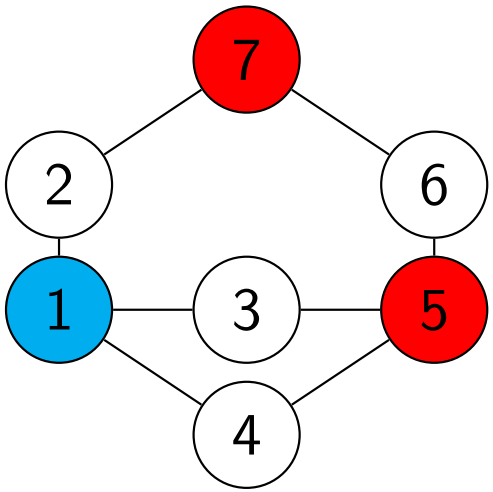


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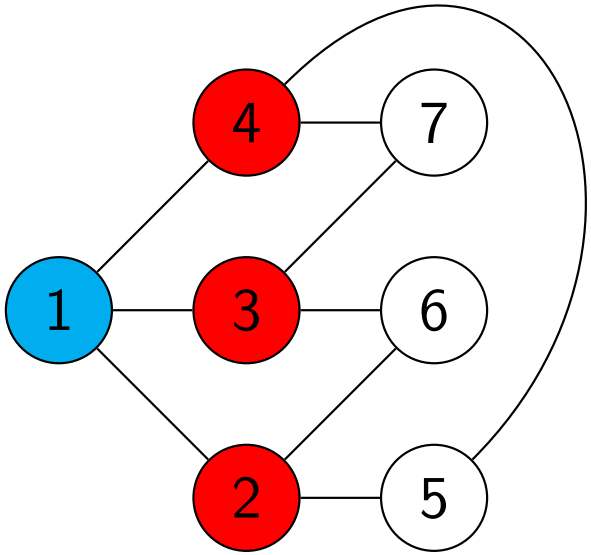


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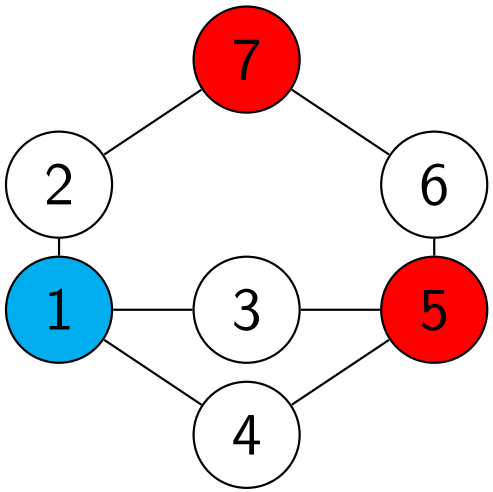


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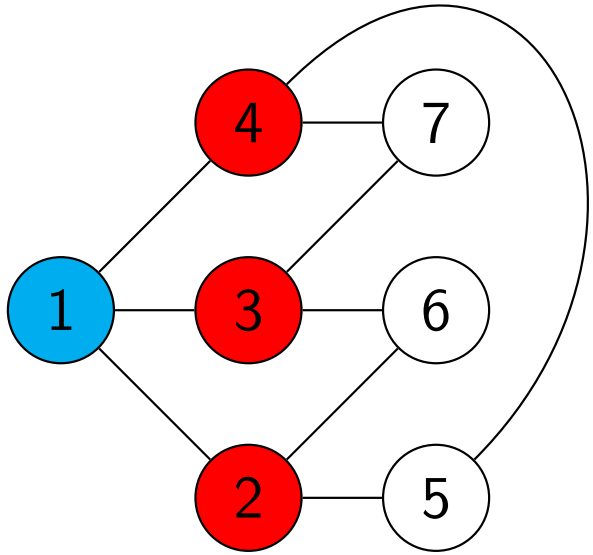


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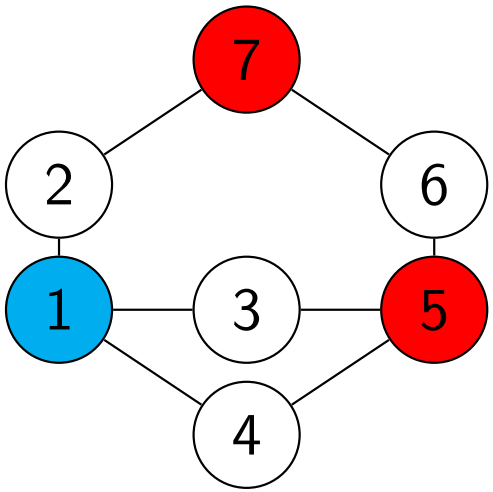


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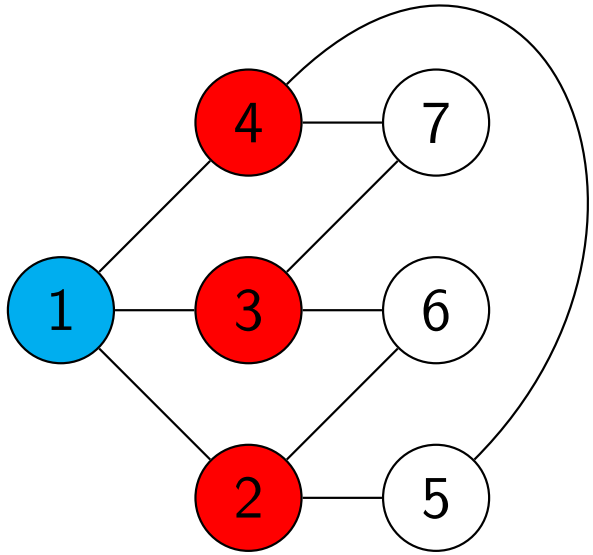


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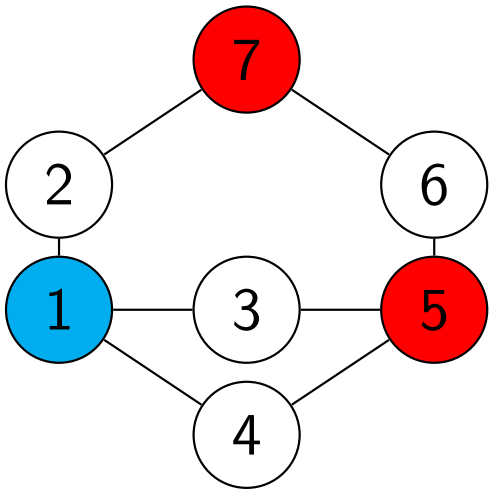


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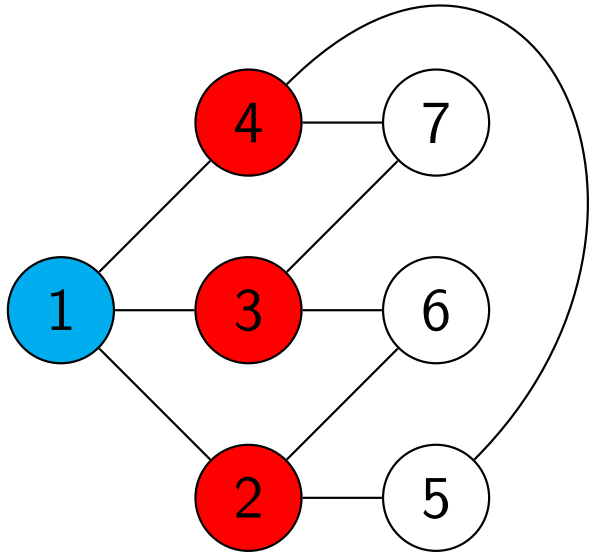


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# Lower Bound for Optimality – Vertex Cover

# Vertex Cover – lower bound for optimality

**Theorem**

For Vertex Cover, in order to achieve optimality, an adaptive priority algorithm with advice (Model 2) requires at least $\lfloor |V|/7 \rfloor$ bits of advice.

**Pf.** Create $m$ disjoint universes for $m$ gadgets, so the resulting input can be $H_1, H_2, \ldots, H_m$, where $H_i$ is an isomorphic copy of either $G_1$ or $G_2$.

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Each $H_i$ has 7 vertices, so $m = |V|/7$. $\square$

# General theorem – lower bound for optimality

**Theorem**

Let $B$ be a problem with some reasonable properties and gadget pairs. Let $s = \max_j(|G_j^a|, |G_j^r|)$, where the cardinality of a gadget is the number of input items it consists of.

Any optimal adaptive priority algorithm with advice (Model 2) for $B$ must use at least $\lfloor n/s \rfloor$ advice bits on worst case instances with $n$ input items.

Section 4

# Lower Bounds for Approximation

# Approximation – Binary String Guessing Problem

Binary string guessing problem (with known history): 2-SGKH
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**Theorem (Böckenhauer,Hromkovič,Komm,Krug,Smula,Sprock, 2014)**

On inputs of length $n$, any deterministic algorithm for 2-SGKH that is guaranteed to guess more than $\epsilon n$ bits correctly, for $0 < \epsilon \leq 1/2$, needs at least $(1 + (1-\epsilon)\log(1-\epsilon) + \epsilon \log(\epsilon))n = (1 - H(\epsilon))n$ bits of advice.

Note: Often used for lower bounds for online algorithms with advice.

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The lower bound results for optimality also apply to fixed priority and oblivious priority algorithms.

# Another gadget pair – Independent Set example

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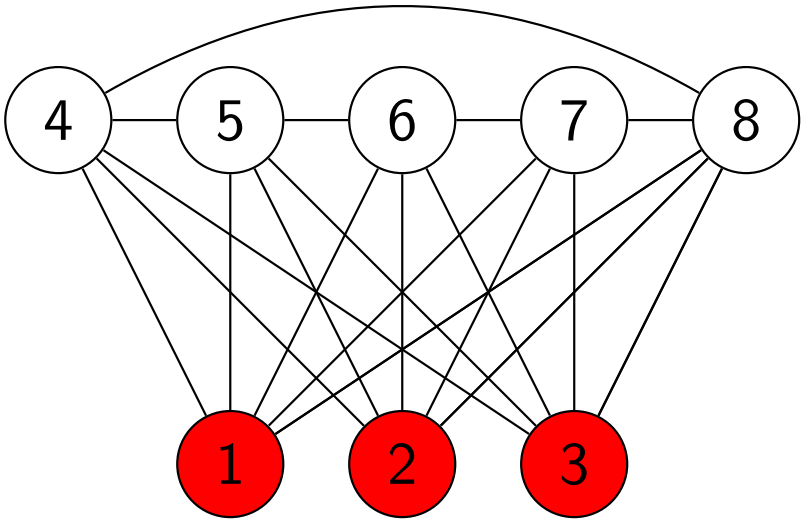


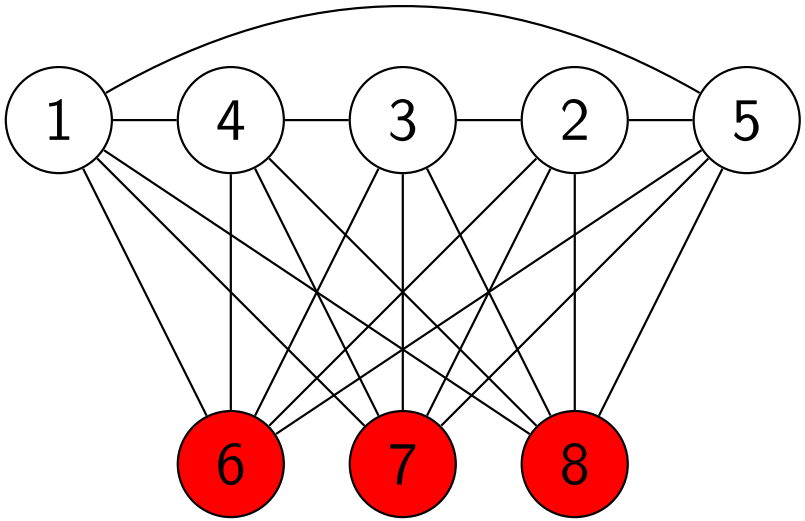

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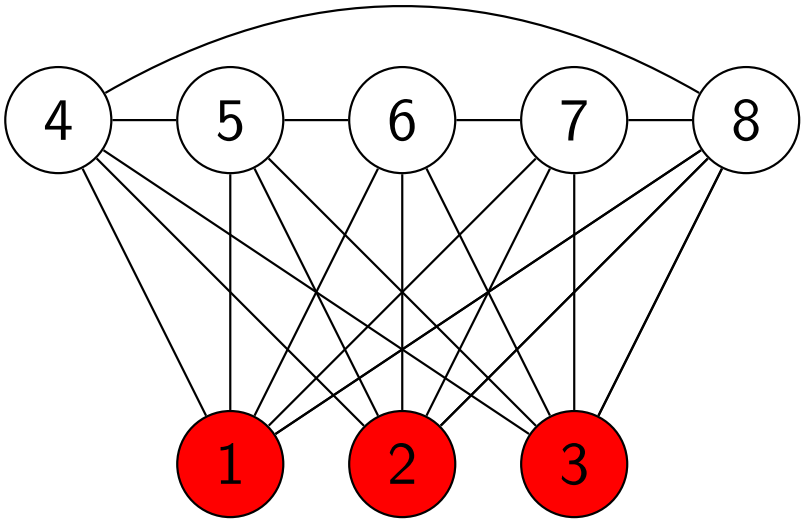


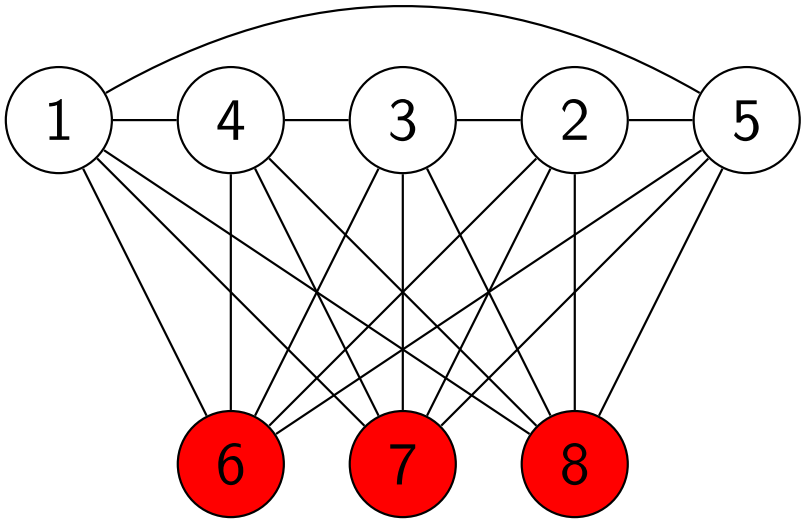


Adversary will make the first vertex selected be 1.

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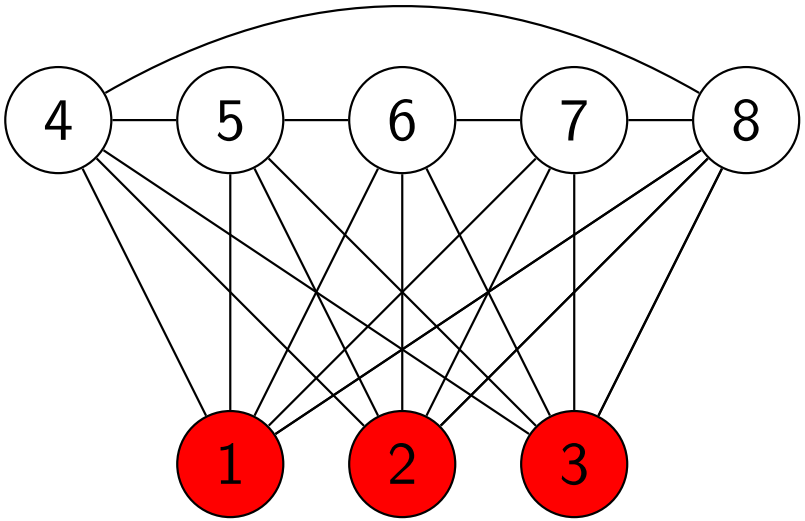


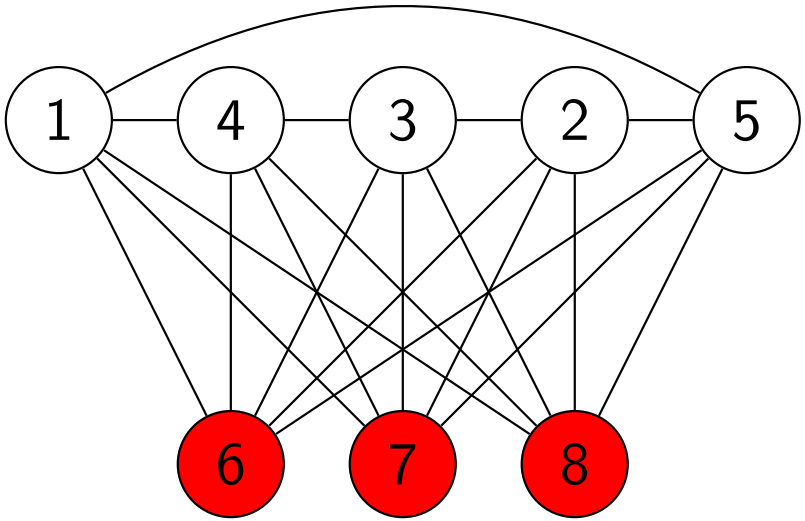


Adversary will make the first vertex selected be 1.
To get an independent set of size 3, need to accept in one gadget and reject in the other.
Advantage over Vertex Cover gadget pair: Optimal solution is smaller.

# Maximization problems – lower bound

The maximization part of our general corollary:
(Let $s$ be the number of input items per gadget.)

## Theorem

For a maximization problem, if $\mathrm{OPT}(G_1) = \mathrm{OPT}(G_2) = \mathrm{BAD}(G_1) + 1 = \mathrm{BAD}(G_2) + 1$, then for any $\epsilon \in (0, 1/2]$, no oblivious priority algorithm reading fewer than $(1 - H(\epsilon))n/s$ advice bits can achieve an approximation ratio smaller than $1 + \frac{\epsilon}{OPT(G_1) - \epsilon}$.

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For Independent Set:

**Theorem**

For Maximum Independent Set and any $\epsilon \in (0, \frac{1}{2}]$, no oblivious priority algorithm reading fewer than $(1 - H(\epsilon))n/8$ advice bits can achieve an approximation ratio smaller than $1 + \frac{\epsilon}{3-\epsilon}$.

# Minimization problems – lower bound

The minimization part of our general corollary:
(Let $s$ be the number of input items per gadget.)

**Theorem**

For a minimization problem, if $\mathrm{OPT}(G_1) = \mathrm{OPT}(G_2) = \mathrm{BAD}(G_1)$ - 1 $= \mathrm{BAD}(G_2)$ - 1, then for any $\epsilon \in (0, 1/2]$, no oblivious priority algorithm reading fewer than $(1 - H(\epsilon))n/s$ advice bits can achieve an approximation ratio smaller than $1 + \frac{\epsilon}{OPT(G_1)}$.

# Other results

For PROBLEM and any $\epsilon \in (0, \frac{1}{2}]$, no priority algorithm reading fewer than BITS advice bits can achieve an approximation ratio smaller than RATIO.

| PROBLEM | BITS | RATIO |
|---|---|---|
| Maximum Independent Set* | $(1 - H(\epsilon))n/8$ | $1 + \frac{\epsilon}{3-\epsilon}$ |
| Maximum Independent Set** | $(1 - H(\epsilon))n/7$ | $1 + \frac{\epsilon}{4-\epsilon}$ |
| Maximum Bipartite Matching | $(1 - H(\epsilon))n/3$ | $1 + \frac{\epsilon}{3-\epsilon}$ |
| Maximum Cut | $(1 - H(\epsilon))n/8$ | $1 + \frac{\epsilon}{15-\epsilon}$ |
| Minimum Vertex Cover | $(1 - H(\epsilon))n/7$ | $1 + \frac{\epsilon}{3}$ |
| Maximum 3-Satisfiability | $(1 - H(\epsilon))n/3$ | $1 + \frac{\epsilon}{8-\epsilon}$ |
| Unit Job Scheduling with PC | $(1 - H(\epsilon))n/9$ | $1 + \frac{\epsilon}{6-\epsilon}$ |

# Comments

These results are the same as those obtained previously for fixed priority algorithms with advice, but a factor 2 better for number of bits.

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The amount of advice required for optimality is the same as the advice required for approximation, without the $(1 - H(\epsilon))$ term.

| PROBLEM | Approximation | Optimality |
|---|---|---|
| Maximum Independent Set** | $(1 - H(\epsilon))n/7$ | $n/7$ |
| Maximum Bipartite Matching | $(1 - H(\epsilon))n/3$ | $n/3$ |
| Maximum Cut | $(1 - H(\epsilon))n/8$ | $n/8$ |
| Maximum 3-Satisfiability | $(1 - H(\epsilon))n/3$ | $n/3$ |
| Unit Job Scheduling with PC | $(1 - H(\epsilon))n/9$ | $n/9$ |

Thank you for your attention.