(c) Multirate Sampling

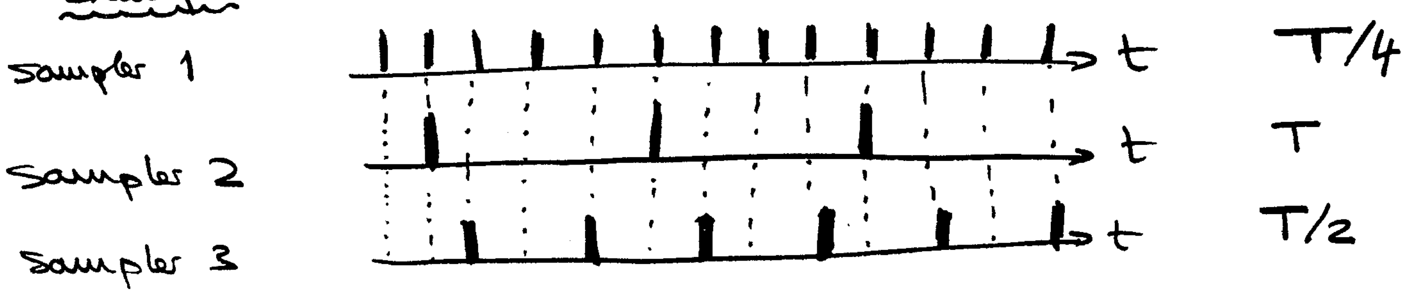
Multirate Sampling is important for many reasons:

- individual subsystems have a different bandwidth
- multilayered control systems require simple / fast algorithms for the innermost loops, and complicated / slow algorithms for the outermost loops.

The book (Kuo: p. 139 - 151) gives several techniques related to z-Transform. These are closely related to the topics mentioned in the last section.

Assume: Samplings occur synchronously. Otherwise, the analysis becomes a total mess. Thus:

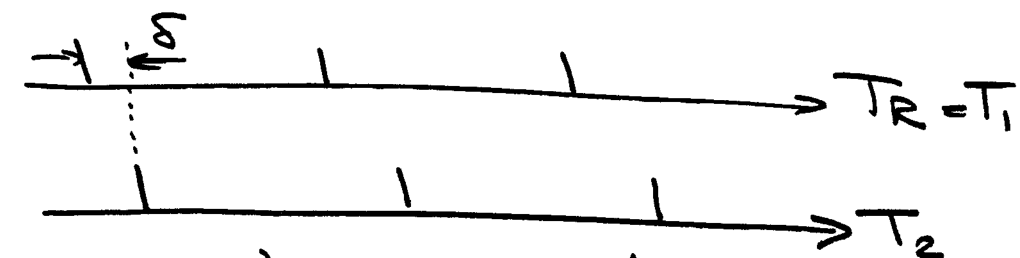
Example:



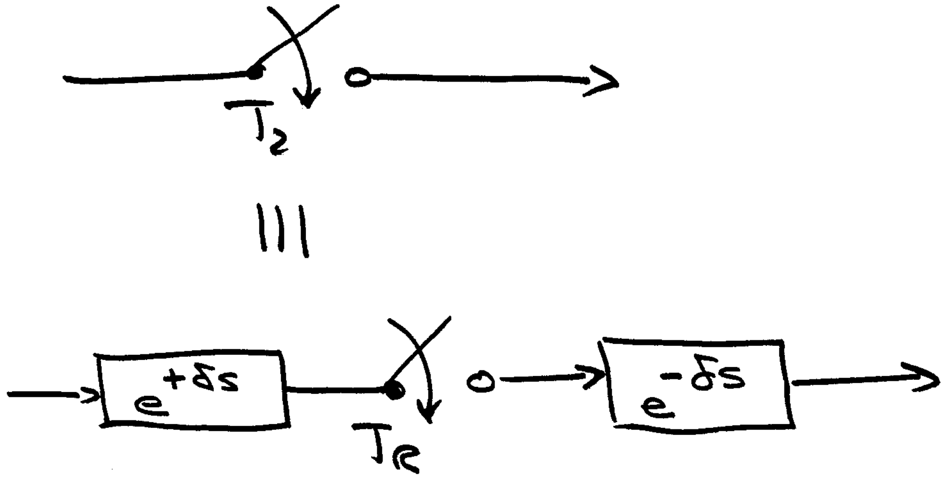
- The usual way to handle such a situation is to try to reduce a multirate sampling system to a single rate sampling system. For that purpose, we need a <u>reference sampler</u>.

- One common way would be to use the <u>slowest sampler</u> as the reference sampler. We shall call this T_R.

<u>Example</u>:

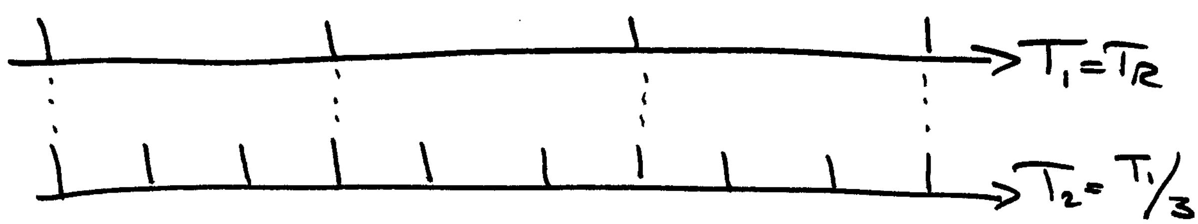


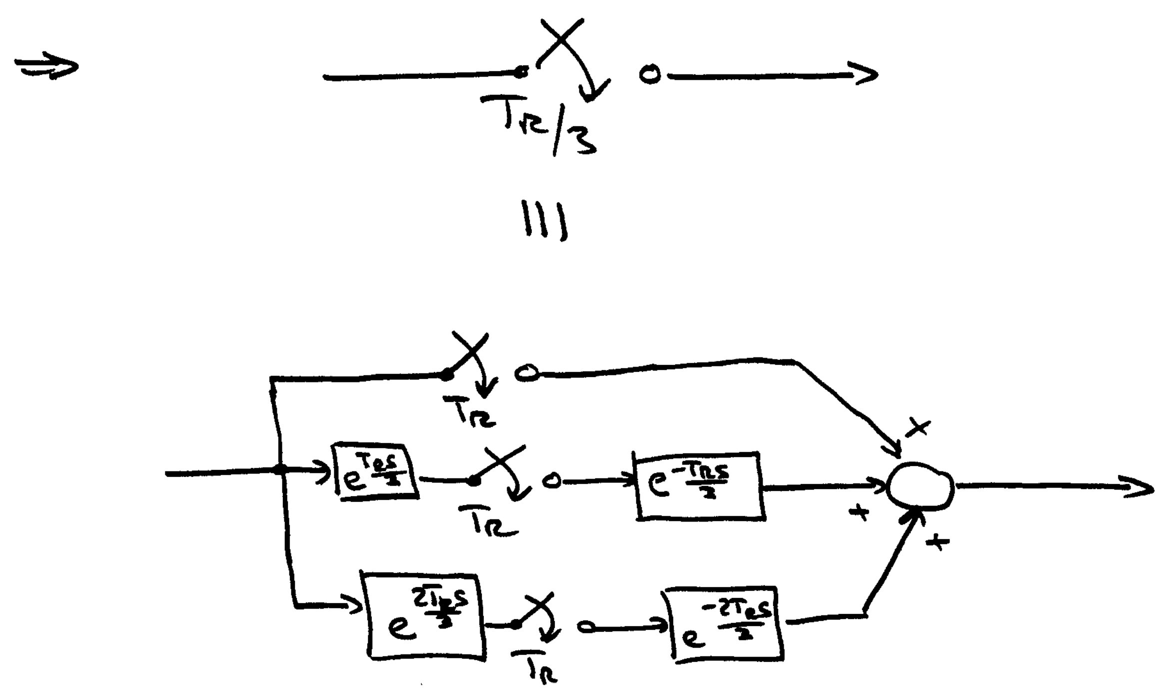
One sampler is by δ delayed as compared to the reference sampler.



We can now "add" these delays to the transfer functions to the left and right, and have effectively reduced the system to one that has the sampler synchronously with the reference sampler.

Example:

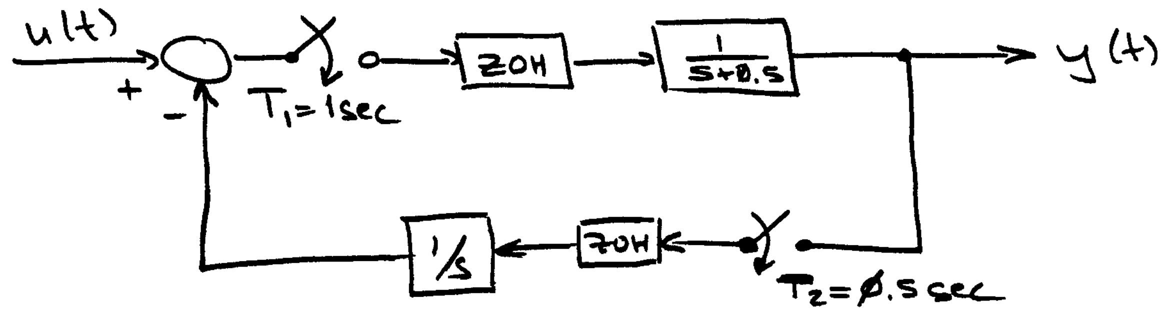




Again, at the price of a more complicated system, we were able to reduce the system to a singlerate sampling system.

- This technique relates closely to the modified z-Transform. We can read out the modified transfer functions from the table in Appendix A (Kuo).

Example:



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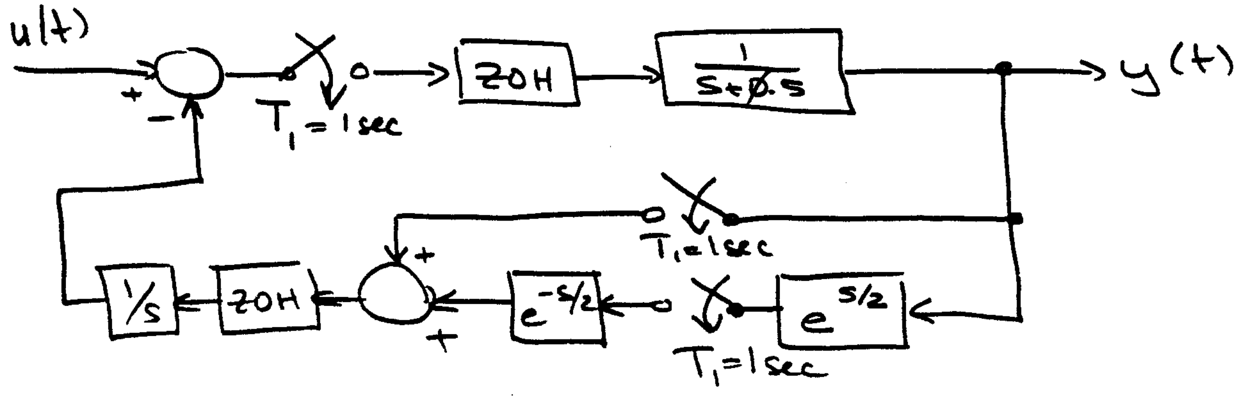
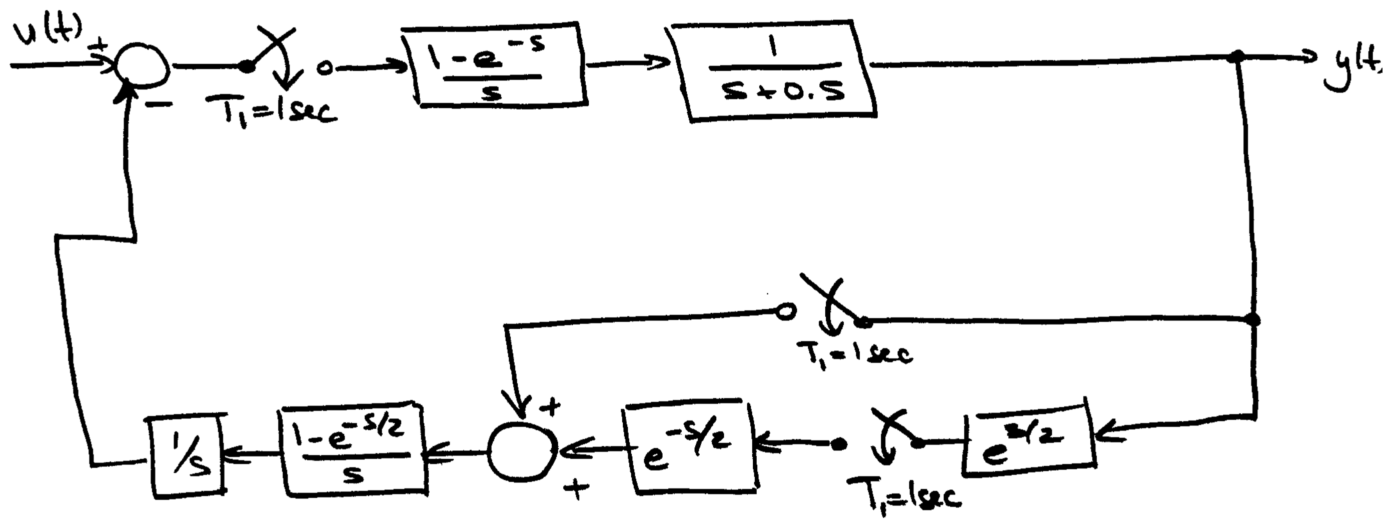


Now, we have again a singlerate system. However, watch that the ZOH is different in the two cases!



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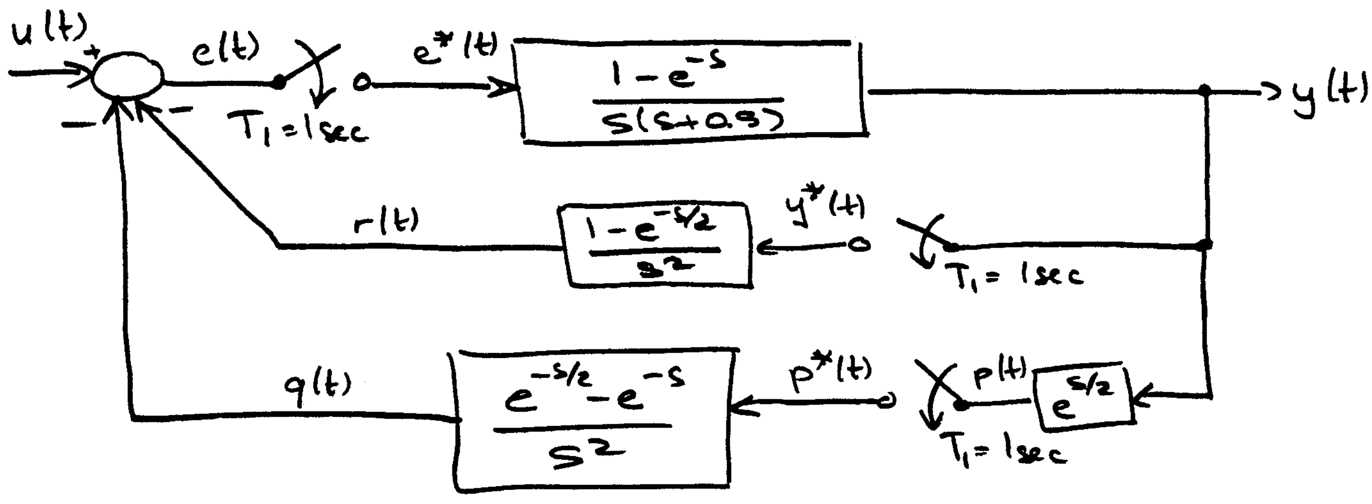


$$E(s) = U(s) - R(s) - Q(s)$$

$$= U(s) - \frac{1-e^{-s/2}}{s^2} Y^*(s) - \frac{e^{-s/2} - e^{-s}}{s^2} P^*(s)$$

$$\Rightarrow E^*(s) = U^*(s) - \left(\frac{1-e^{-s/2}}{s^2}\right)^* Y^*(s) - \left(\frac{e^{-s/2} - e^{-s}}{s^2}\right)^* P^*(s)$$

$$P(s) = e^{s/2} \cdot \frac{1-e^{-s}}{s(s+0.5)} \cdot E^*(s)$$

$$\Rightarrow P^*(s) = \left(\frac{e^{s/2} - e^{-s/2}}{s(s+0.5)}\right)^* \cdot E^*(s)$$

$$Y(s) = \frac{1-e^{-s}}{s(s+0.5)} \cdot E^*(s) \quad \Rightarrow Y^*(s) = \left(\frac{1-e^{-s}}{s(s+0.5)}\right)^* \cdot E^*(s)$$

Now, we can map these expressions into the z - domain by applying the modified z - Transform:

$$\frac{1-e^{-s/2}}{s^2} = \frac{1}{s^2} - \frac{1}{s^2} e^{-s/2}$$

$$\left[\frac{T_1 z}{(z-1)^2}\right]\Bigg|_{T_1=1} \qquad \left[\frac{mT_1}{z-1} + \frac{T_1}{(z-1)^2}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}}$$

$$\Rightarrow \frac{1-e^{-s/2}}{s^2} \xrightarrow{\mathcal{Z}} \frac{z}{(z-1)^2} - \frac{1/2}{z-1} - \frac{1}{(z-1)^2} = \frac{0.5z - 0.5}{(z-1)^2}$$

$$= 0.5 \cdot \frac{(z-1)}{(z-1)^2} = \underline{\underline{\frac{0.5}{z-1}}}$$

$$\frac{e^{-s/2} - e^{-s}}{s^2} = \frac{1}{s^2} e^{-s/2} - e^{-s}\left(\frac{1}{s^2}\right)$$

$$\xrightarrow{\mathcal{Z}} \left[\frac{mT_1}{z-1} + \frac{T_1}{(z-1)^2}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}} - z^{-1}\cdot\left[\frac{T_1 z}{(z-1)^2}\right]\Bigg|_{T_1=1}$$

$$= \frac{1/2}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^2} = \underline{\underline{\frac{0.5}{z-1}}}$$

$$\frac{e^{s/2} - e^{-s/2}}{s(s+0.5)} = \left[\frac{2}{s} - \frac{2}{s+0.5}\right]\cdot\left[e^{s/2} - e^{-s/2}\right]$$

$$= e^{s}\cdot\left(\frac{2}{s}\cdot e^{-s/2}\right) - \frac{2}{s}\cdot e^{-s/2} - e^{s}\cdot\left(\frac{2}{s+0.5}\cdot e^{-s/2}\right)$$

$$+ \frac{2}{s+0.5}\cdot e^{-s/2}$$

$$\xrightarrow{\mathcal{Z}} z\cdot\left[\frac{2}{z-1}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}} - \left[\frac{2}{z-1}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}} - z\left[\frac{2e^{-0.5mT_1}}{z-e^{-0.5T_1}}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}}$$

$$+ \left[\frac{2e^{-0.5mT_1}}{z-e^{-0.5T_1}}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}} = (z-1)\cdot\left[\frac{2}{z-1} - \frac{2e^{-0.5mT_1}}{z-e^{-0.5T_1}}\right]\Bigg|_{\substack{T_1=1 \\ m=1/2}}$$

$$= (z-1)\cdot\left[\frac{2}{z-1} - \frac{2e^{-0.25}}{z-e^{-0.5}}\right]$$

$$= \underline{\underline{\frac{2z\left(1-e^{-0.25}\right) + \left(2e^{-0.25} - 2e^{-0.5}\right)}{z-e^{-0.5}}}}$$

$$\frac{1-e^{-s}}{s(s+0.5)} = \left[\frac{2}{s} - \frac{2}{(s+0.5)}\right]\left(1-e^{-s}\right)$$

$$\xrightarrow{\mathcal{Z}} \left(1-z^{-1}\right)\cdot\left[\frac{2z}{z-1} - \frac{2z}{z-e^{-0.5T_1}}\right]\Bigg|_{T_1=1}$$

$$= \frac{z-1}{z}\cdot 2\cdot\frac{z}{z-1}\cdot\frac{1-e^{-0.5}}{z-e^{-0.5}} = \frac{2-2e^{-0.5}}{z-e^{-0.5}}$$

$$\left|\begin{aligned} &\mathcal{E}(z) = \mathcal{U}(z) - \frac{0.5}{z-1}\cdot\mathcal{Y}(z) - \frac{0.5}{z-1}\cdot\mathcal{P}(z) \\ &\mathcal{P}(z) = \frac{\left(2z+2e^{-0.25}\right)\cdot\left(1-e^{-0.25}\right)}{z-e^{-0.5}}\cdot\mathcal{E}(z) \\ &\mathcal{Y}(z) = \frac{2\left(1-e^{-0.5}\right)}{z-e^{-0.5}}\cdot\mathcal{E}(z) \end{aligned}\right|$$

$$\Rightarrow \mathcal{E}(z) = \mathcal{U}(z) - \frac{\left(1-e^{-0.5}\right)}{(z-1)\left(z-e^{-0.5}\right)}\mathcal{E}(z) - \frac{\left(z+e^{-0.25}\right)\left(1-e^{-0.25}\right)}{(z-1)\left(z-e^{-0.5}\right)}\mathcal{E}($$

$$\Rightarrow \left[1 + \frac{1-e^{-0.5} + z\left(1-e^{-0.25}\right) + e^{-0.25} - e^{-0.5}}{(z-1)\left(z-e^{-0.5}\right)}\right]\mathcal{E}(z) = \mathcal{U}(z)$$

$$\Rightarrow \left[\frac{z^2 \cancel{-z} - ze^{-0.5} + e^{-0.5} \cancel{+z} - ze^{-0.25} + 1 - \cancel{2}e^{-0.5} + e^{-0.25}}{(z-1)\left(z-e^{-0.5}\right)}\right]\mathcal{E}(z) = \mathcal{U}(z)$$

$$\Rightarrow \mathcal{E}(z) = \frac{(z-1)(z-e^{-0.5})}{z^2 - z(e^{-0.5}+e^{-0.25}) + (1-e^{-0.5}+e^{-0.25})} \cdot U(z)$$

$$\Rightarrow Y(z) = \frac{2(z-1)(1-e^{-0.5})}{z^2 - z(e^{-0.5}+e^{-0.25}) + (1-e^{-0.5}+e^{-0.25})} \cdot U(z)$$

$$G(z) = \frac{0.7869\,(z-1)}{z^2 - 1.3853\,z + 1.1723}$$

This is always possible, but it gets soon rather messy. A better way is to do the analysis again in the state-space.

For any system, we can write:

$$\underline{x}(t) = e^{A(t-t_0)}\underline{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

Assume: $t_0 = kT_1$; $t = kT_1 + \Delta T$; $\underline{u}$ = constant $= \underline{u}(kT_1)$

$$\Rightarrow \underline{x}(kT_1+\Delta T) = e^{A\Delta T}\cdot \underline{x}(kT_1) + \int_{kT_1}^{kT_1+\Delta T} e^{A(kT_1+\Delta T-\tau)} B\,u(kT_1)\,d\tau$$

Subst: $\sigma = \tau - kT_1 \iff \tau = \sigma + kT_1$

$d\tau = d\sigma$

τ	σ
kT_1	0
$kT_1+\Delta T$	ΔT

$$\Rightarrow \underline{x}(kT_1+\Delta T) = e^{A\Delta T} \cdot \underline{x}(kT_1) + \int_0^{\Delta T} e^{A(\Delta T-\sigma)} B \, d\sigma \cdot \underline{u}(kT_1)$$

We abbreviate:

$$\Phi(\Delta T) = e^{A\Delta T}$$

$$\vartheta(\Delta T) = \int_0^{\Delta T} e^{A(\Delta T-\sigma)} B \, d\sigma$$

$$\Rightarrow \underline{x}(kT_1+\Delta T) = \Phi(\Delta T)\cdot \underline{x}(kT_1) + \vartheta(\Delta T)\cdot \underline{u}(kT_1)$$

$\Phi(t)$ :: State transition matrix

<u>In particular</u>:

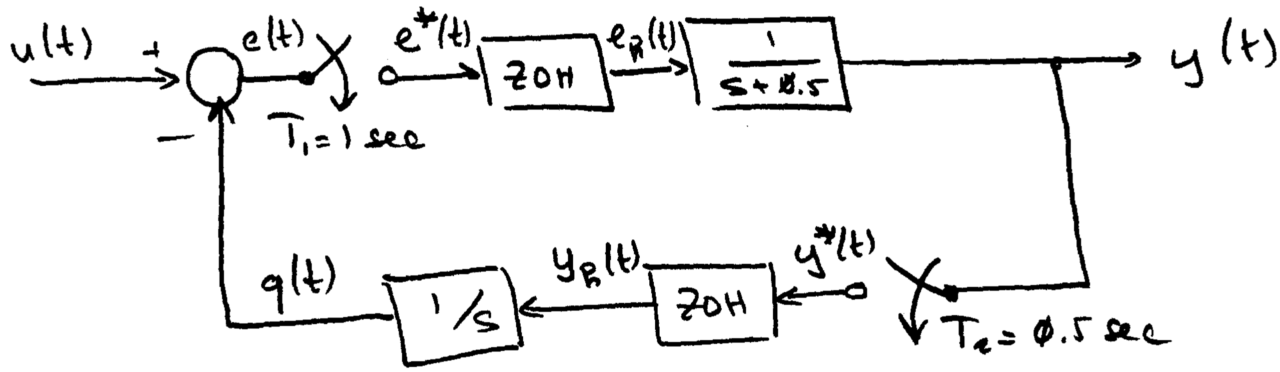
$\Delta T \equiv T \Rightarrow$

$$\underline{x}(kT_1+T) = \underline{x}((k+1)T) = \Phi(T)\cdot \underline{x}(kT_1) + \vartheta(T)\cdot \underline{u}(kT_1)$$
$$= F \cdot \underline{x}(kT_1) + G\, \underline{u}(kT_1)$$

$$\Rightarrow \boxed{\begin{array}{l} F \equiv \Phi(T) \\ G \equiv \vartheta(T) \end{array}}$$

However, for $\Delta T < T$, we can use this formulae to evaluate the output between sampling points. We can also use this formulae for multirate systems.

Example:



$$\underline{u} = \begin{bmatrix} u \\ e_R \\ y_R \end{bmatrix} \quad ; \quad \underline{y} = \begin{bmatrix} y \\ e \\ y \end{bmatrix}$$

$$\left| \begin{aligned} \dot{x}_1 &= -0.5 x_1 + u_2 \\ \dot{x}_2 &= u_3 \\ y_1 &= x_1 \\ y_2 &= u_1 - x_2 \\ y_3 &= x_1 \end{aligned} \right|$$

$(x_1 = y)$

$(x_2 = q)$

$$\Rightarrow \left| \begin{array}{l} \underline{\dot{x}} = \begin{bmatrix} -0.5 & 0 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{u} \\ \underline{y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{u} \end{array} \right|$$

$$\Rightarrow \Phi(t) = e^{At} = \mathcal{L}^{-1}\left\{ (sI-A)^{-1} \right\}$$

$$(sI-A) = \begin{bmatrix} (s+0.5) & 0 \\ 0 & s \end{bmatrix} \Rightarrow (sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+0.5} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \Phi(t) = \begin{bmatrix} e^{-0.5t} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vartheta(t) = \int_0^t \Phi(t-\tau)\, B \, d\tau$$

$$= \int_0^t \begin{bmatrix} e^{-0.5(t-\tau)} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} 0 & e^{-0.5(t-\tau)} & 0 \\ 0 & 0 & 1 \end{bmatrix} d\tau = \left. \begin{bmatrix} 0 & 2e^{-0.5(t-\tau)} & 0 \\ 0 & 0 & \tau \end{bmatrix} \right|$$

$$= \begin{bmatrix} 0 & (2-2e^{-0.5t}) & 0 \\ 0 & 0 & t \end{bmatrix} = \vartheta(t)$$

We now go to the next sampling point which is at $(k+0.5)T$:

$$\underline{x}(k+0.5) = \Phi(0.5) \cdot \underline{x}(k) + \vartheta(0.5) \cdot \underline{u}(k)$$

$$\Rightarrow \underline{x}(k+0.5) = \begin{bmatrix} e^{-0.25} & 0 \\ 0 & 1 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 & (2-2e^{-0.25}) & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \underline{u}(k)$$

$$\underline{y}(k) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{u}(k)$$

$$\Rightarrow \underline{y}_2(k) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1(k) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}_2(k)$$

$$\underline{u}_2(k) \equiv \underline{y}_2(k)$$

$$\Rightarrow \underline{x}(k+0.5) = \begin{bmatrix} e^{-0.25} & 0 \\ 0 & 1 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_1(k)$$

$$+ \begin{bmatrix} (2-2e^{-0.25}) & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{x}(k)$$

$$+ \begin{bmatrix} (2-2e^{-0.25}) & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1(k)$$

$$\Rightarrow \underline{x}(k+0.5) = \begin{bmatrix} e^{-0.25} & (2e^{-0.25}-2) \\ 0.5 & 1 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} (2-2e^{-0.25}) \\ 0 \end{bmatrix} u_1(k)$$

Now, we go to the next sampling point:

$$\underline{x}(k+1) = \Phi(0.5)\,\underline{x}(k+0.5) + \Theta(0.5) \cdot \underline{u}(k+0.5)$$

$$\Rightarrow \underline{x}(k+1) = \begin{bmatrix} e^{-0.25} & 0 \\ 0 & 1 \end{bmatrix} \underline{x}(k+0.5) + \begin{bmatrix} 0 & (2-2e^{-0.25}) & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \underline{u}(k+0.5)$$

$$\underline{y}(k+0.5) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{x}(k+0.5) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{u}(k+0.5)$$

We need only the y_3:

$$\Rightarrow \quad y_3(k+0.5) = [1 \quad 0]\,\underline{x}(k+0.5) + [0 \quad 0 \quad 0]\,\underline{u}(k+0.5)$$

$$\Rightarrow \quad u_3(k+0.5) \equiv y_3(k+0.5) = [1 \quad 0]\,\underline{x}(k+0.5)$$

The other inputs have maintained their previous values.

$$u_2(k+0.5) = u_2(k) = [0 \quad -1]\,\underline{x}(k) + [1]\,u_1(k)$$

$$\Rightarrow \underline{x}(k+1) = \begin{bmatrix} e^{-0.25} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-0.25} & (2e^{-0.25}-2) \\ 0.5 & 1 \end{bmatrix} \underline{x}(k)$$

$$+ \begin{bmatrix} e^{-0.25} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (2-2e^{-0.25}) \\ 0 \end{bmatrix} u_1(k)$$

$$+ \begin{bmatrix} (2-2e^{-0.25}) \\ 0 \end{bmatrix} \cdot [0 \quad -1] \cdot \underline{x}(k) + \begin{bmatrix} (2-2e^{-0.25}) \\ 0 \end{bmatrix} \cdot [1] \cdot u_1(k)$$

$$+ \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \cdot [1 \quad 0] \cdot \begin{bmatrix} e^{-0.25} & (2e^{-0.25}-2) \\ 0.5 & 1 \end{bmatrix} \cdot \underline{x}(k)$$

$$+ \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \cdot [1 \quad 0] \cdot \begin{bmatrix} (2-2e^{-0.25}) \\ 0 \end{bmatrix} \cdot u_1(k)$$

$$\underline{x}(k+1) = \begin{bmatrix} e^{-0.5} & (2e^{-0.5} - 2e^{-0.25} - 2 + 2e^{-0.25}) \\ (0.5 + 0.5e^{-0.25}) & (1 + e^{-0.25} - 1) \end{bmatrix} \underline{x}(k)$$

$$+ \begin{bmatrix} (2e^{-0.25} - 2e^{-0.5} + 2 - 2e^{-0.25}) \\ (1 - e^{-0.25}) \end{bmatrix} u_1(k)$$

$$y_1(k) = [1 \quad 0] \underline{x}(k)$$

This can easily be brought into e.g. controller - canonical form, and from there, we can determine the z - Transfer function.

$$\underline{x}(k+1) = \begin{bmatrix} e^{-0.5} & 2(e^{-0.5} - 1) \\ 0.5(1 + e^{-0.25}) & e^{-0.25} \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 2(1 - e^{-0.5}) \\ (1 - e^{-0.25}) \end{bmatrix} u_1$$

$$y(k) = [1 \quad 0] \underline{x}(k)$$

$$\Rightarrow \underline{x}(k+1) = \begin{bmatrix} 0.6065 & -0.7869 \\ 0.8894 & 0.7788 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0.7869 \\ 0.2212 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(k)$$

$$\Rightarrow G(z) = \frac{0.7869(z-1)}{z^2 - 1.3853z + 1.1723}$$