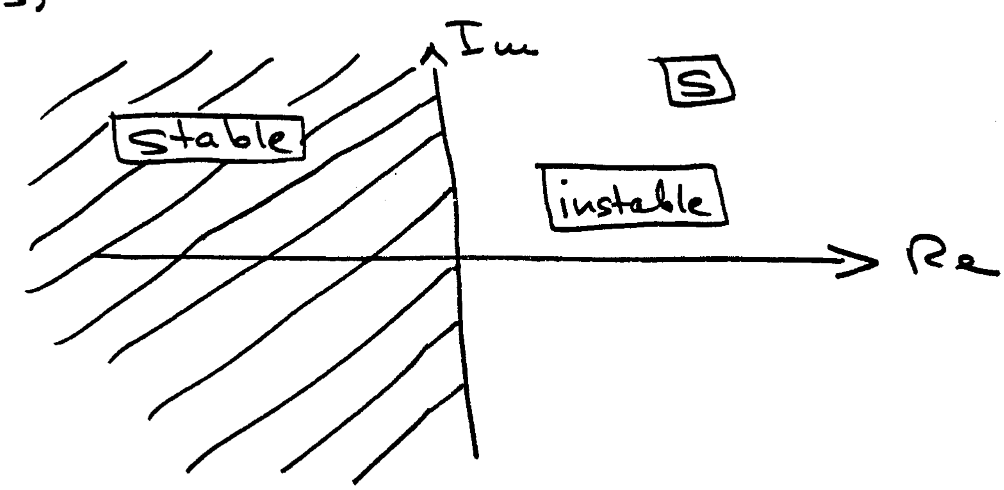
Stability of Discrete & Sampled Data Systems

When we sampled a continuous system:

$$G(s) \xrightarrow{\text{sampling}} G^*(s)$$

we still have a "continuous" system.
$\Longrightarrow$ The same properties hold as before:

$G^*(s)$ is stable iff all "poles" (in s) are in the left half plane (of s)



Now, we introduced the abbreviation:

$$z = e^{Ts}.$$

How does stability look in this new variable?

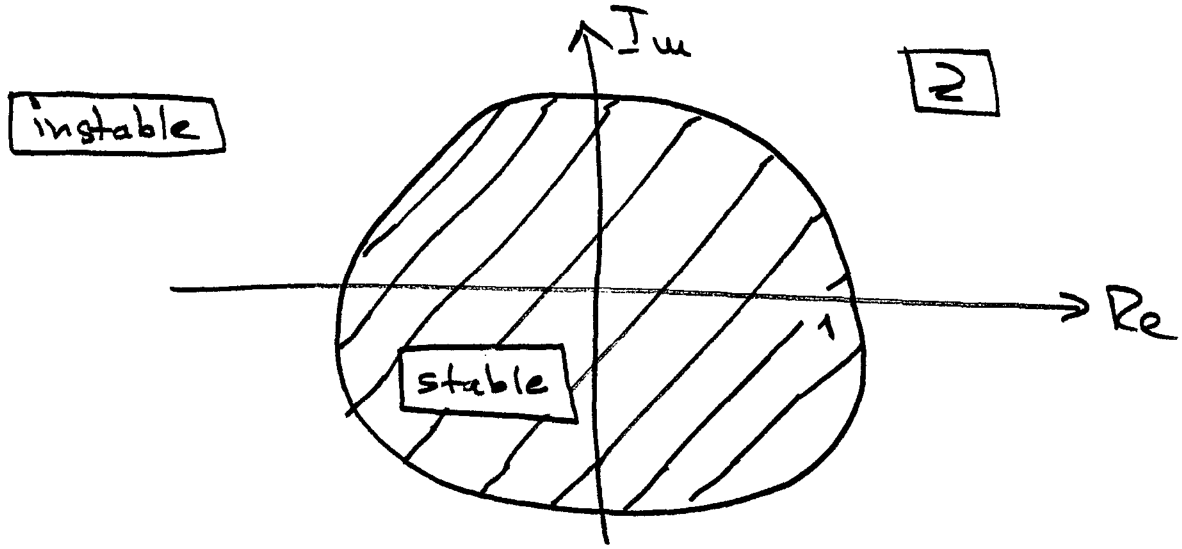
$$s = \sigma + j\omega$$

$$\Rightarrow \quad z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\sigma} \cdot e^{jT\omega}$$

$$= |z| \cdot \underline{\angle z}$$

$$|z| = e^{T\sigma} \quad ; \quad \underline{\angle z} = T\omega$$

Stability is granted if $\underline{\underline{\sigma < 0 \iff |z| < 1}}$



Marginal stability in $\boxed{s}$ is the imaginary axis:

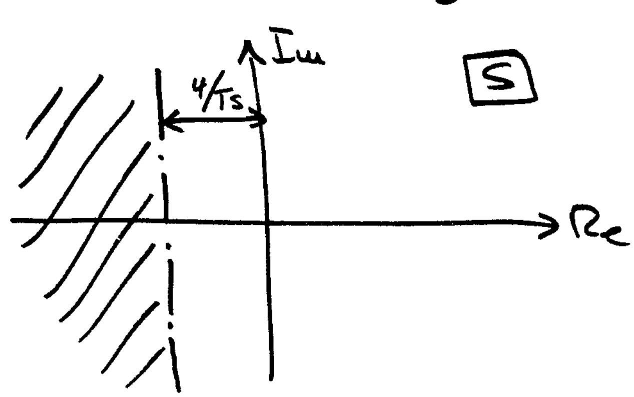
$$\sigma = 0 \longleftrightarrow |z| = 1$$

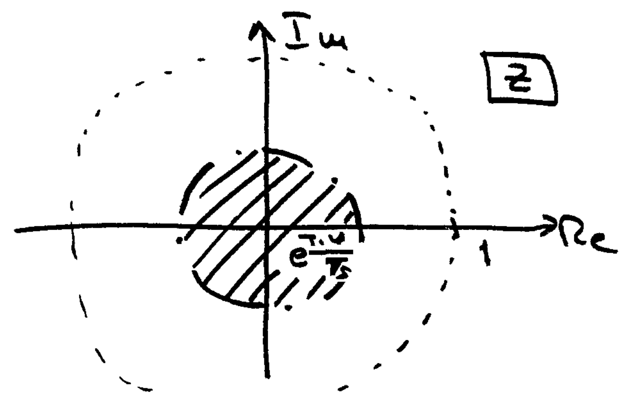
$\Rightarrow$ For Lyapunov-stability, we can tolerate single poles on the unity circle (in z), whereas multiple poles on the unity circle (in z) make the system instable.

Stability reserve:

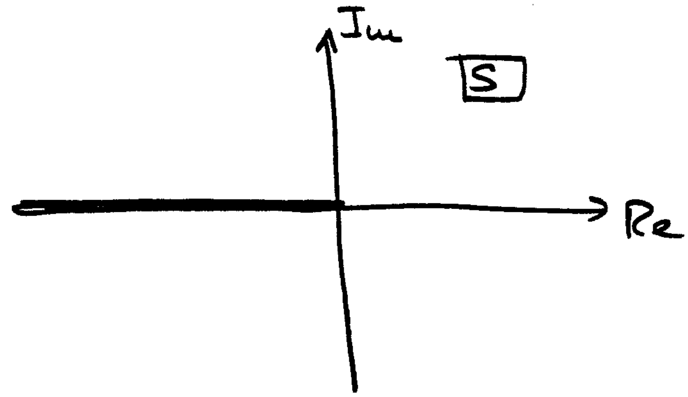
If we want to guarantee that a certain settling time is not surpassed:

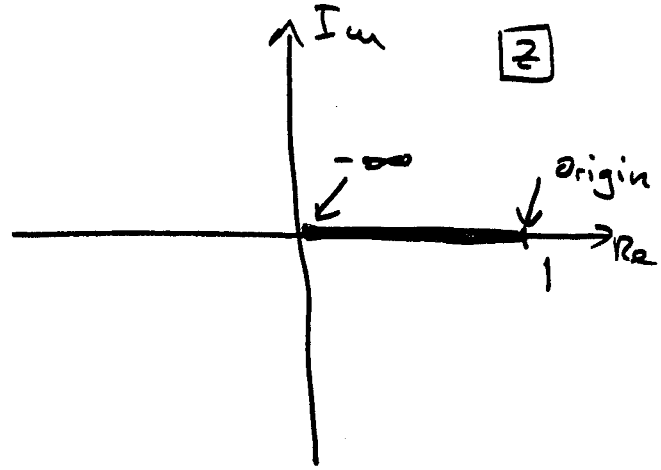
$$|\sigma_i| \geq \frac{4}{T_s} \longleftrightarrow |z_i| \leq e^{T \cdot \frac{4}{T_s}}$$





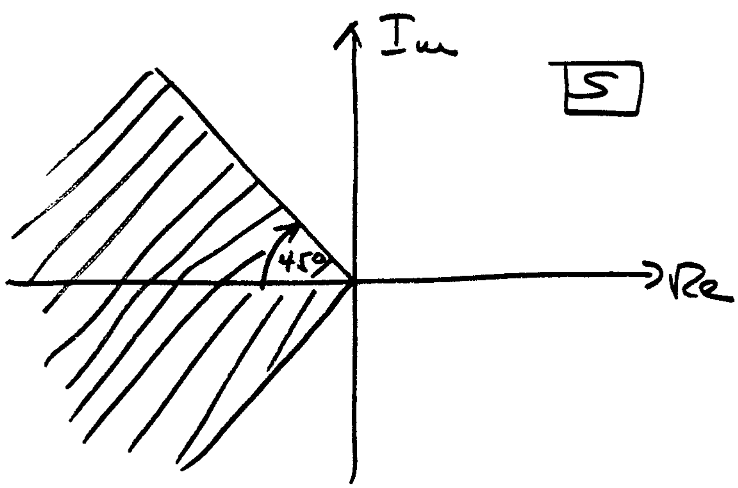
Real poles:

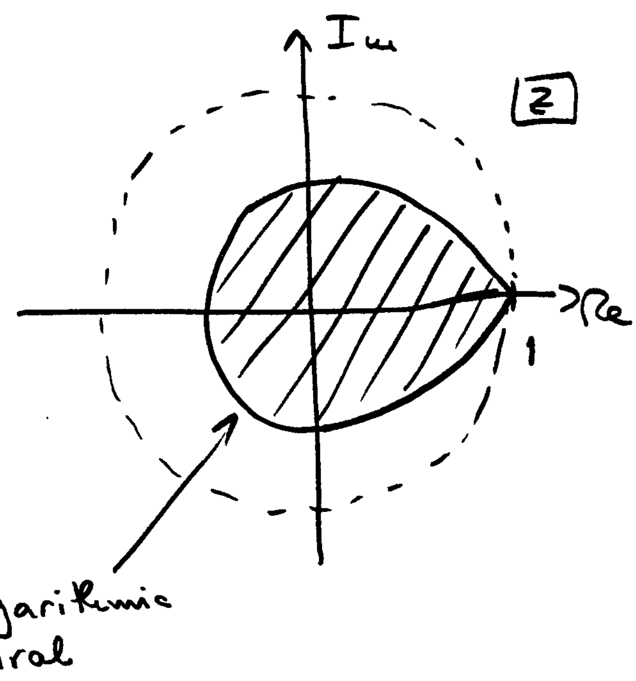




5% Overshooting:

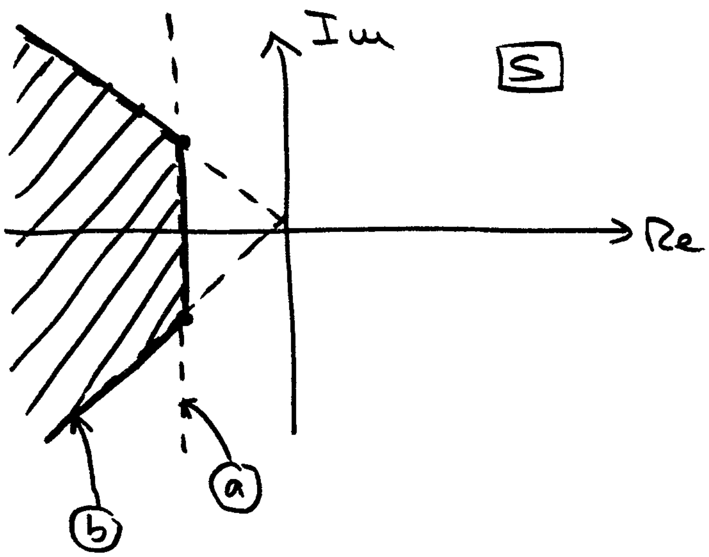
We usually asked not more than 5% overshooting ⟹

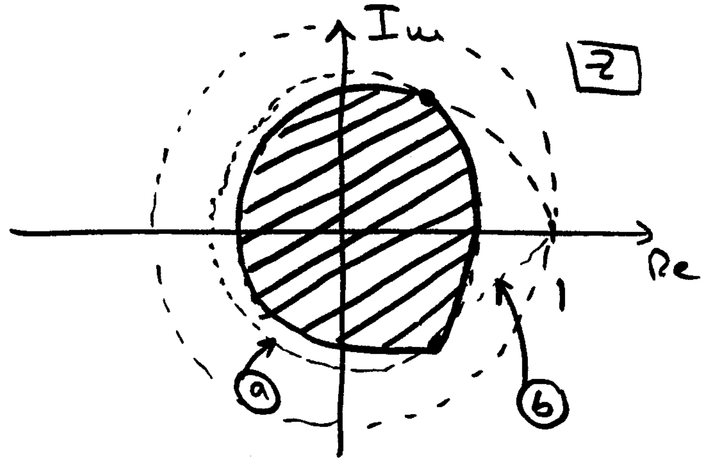




⟹ curves of constant damping are less convenient in the z-domain.

⇒ "Good" pole locations:





Construction of the logarithmic spiral:

Let $\Omega = \frac{2\pi}{T}$

$$\Rightarrow \angle z = T \cdot \omega = 2\pi \cdot \frac{\omega}{\Omega}$$

45° damping $\Rightarrow$ $|\sigma| \equiv |\omega|$

$$\angle z = 180° = \pi \Rightarrow \omega = \Omega/2 = |\sigma|$$

$$\Rightarrow |z| = e^{-T\sigma} = e^{-T \cdot \frac{\Omega}{2}} = e^{-\pi} = \underline{\underline{0.0432}}$$

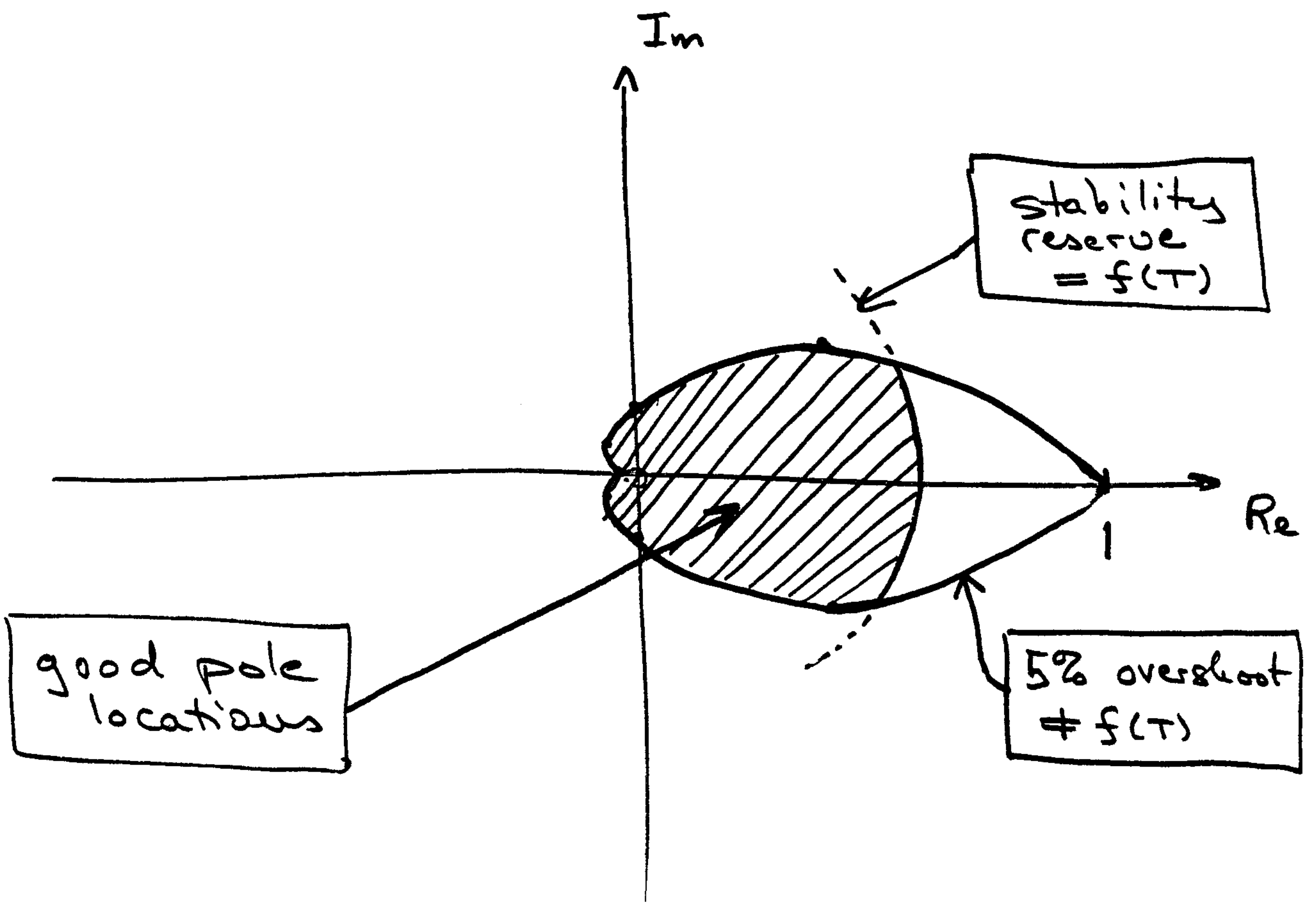
$$\angle z = 90° = \pi/2 \quad \Rightarrow \quad \omega = \frac{\Omega}{4} = |\sigma|$$

$$\Rightarrow |z| = e^{-T\sigma} = e^{-T\cdot\frac{\Omega}{4}} = e^{-\pi/2} = \underline{\underline{0.2079}}$$

$$\angle z = 45° = \pi/4 \quad \Rightarrow \quad |z| = e^{-\pi/4} = \underline{\underline{0.4559}}$$

$$\angle z = 135° = 3\pi/4 \quad \Rightarrow \quad |z| = e^{-3\pi/4} = \underline{\underline{0.0948}}$$

<u>etc.</u>



Assume: $|\sigma_i| \geq 4$ $\quad (\hat{=}\ T_s \leq 1\,\text{sec})$

$$\Rightarrow |z| \leq e^{-4T}$$; Assume $T = 0.1 \Rightarrow |z| \leq 0.6703$

<u>Problem</u>: In the past, we mostly looked at real poles.

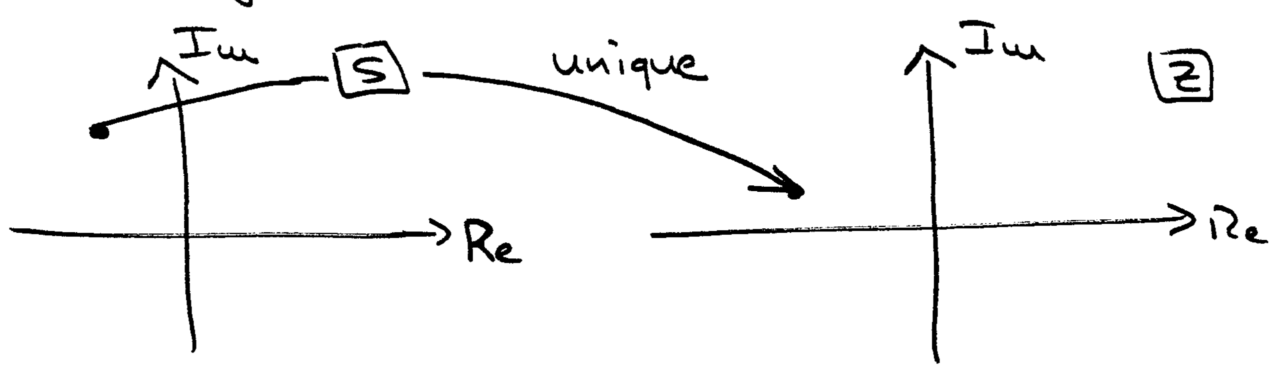
$$s=-a \longleftrightarrow z=e^{-aT}$$

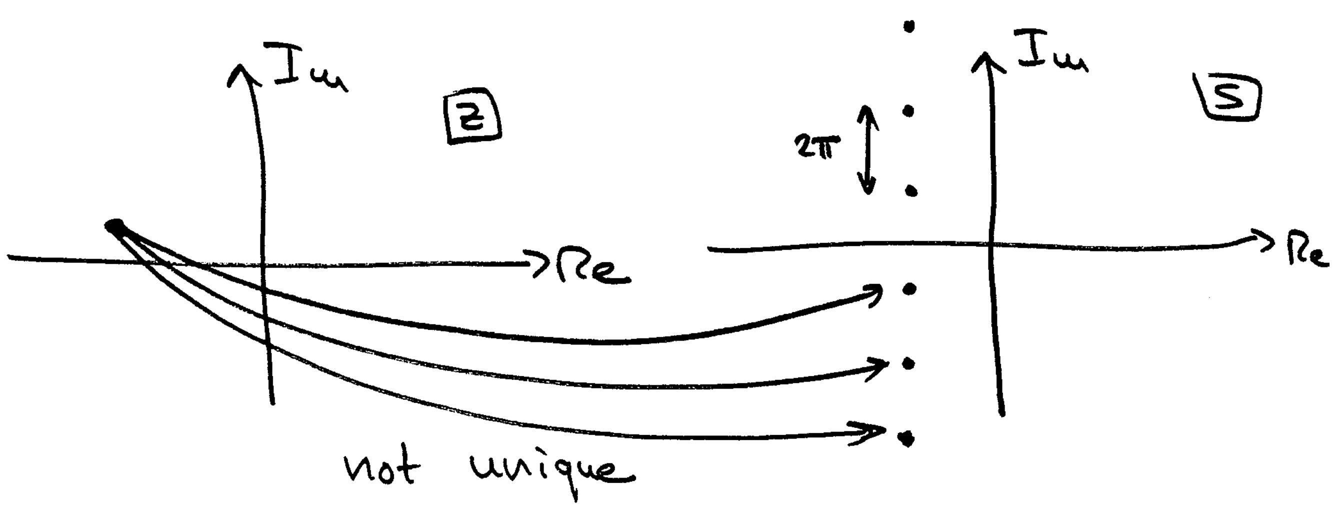
This mapping is unique both ways.

<u>However</u>:

$$s=-a\pm jb \longleftrightarrow z=e^{-aT}\angle \pm bT$$

The angle is 2π-periodic, thus:





$$\hat{A} = P \cdot A \cdot P^{-1} = \Lambda$$

Pole locations don't change (eigenvalues are invariant to similarity transformations)

$$\Rightarrow \quad F = e^{AT} \quad \Rightarrow \quad \hat{F} = e^{\Lambda T} = Q \cdot F \cdot Q^{-1}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \emptyset \\ & \lambda_2 & & \\ & & \ddots & \\ \emptyset & & & \lambda_n \end{bmatrix} \longrightarrow e^{\Lambda T} = \begin{bmatrix} e^{\lambda_1 T} & & & \emptyset \\ & e^{\lambda_2 T} & & \\ & & \ddots & \\ \emptyset & & & e^{\lambda_n T} \end{bmatrix}$$

If the continuous system has its poles at $\{\lambda_i\}$ $\Rightarrow$ the discretized system (with or without ZOH) has its poles at $\{e^{\lambda_i T}\}$

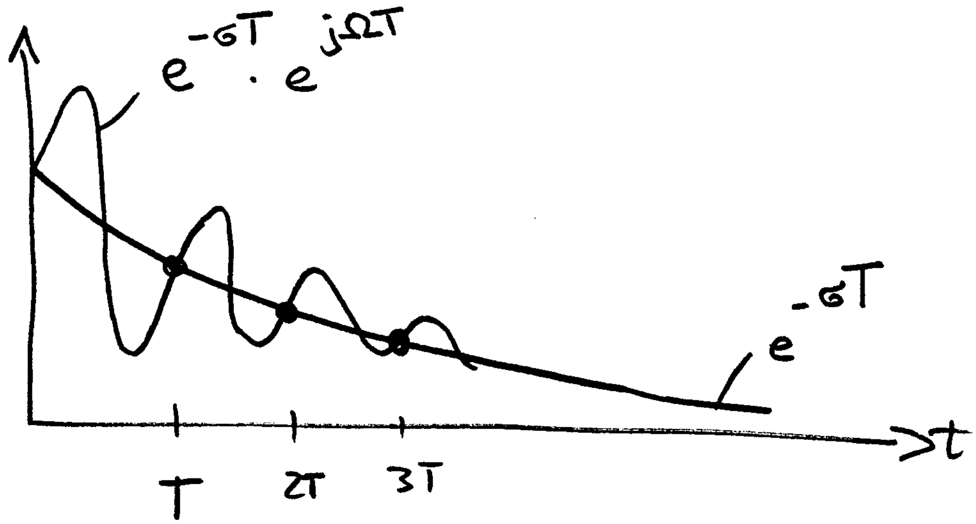
$\Rightarrow$ If $G(s)$ is stable $\Rightarrow$ $G(z)$ will be stable also (with or without ZOH).

<u>However</u>: Given A, we can find F in a unique manner. The opposite is unfortunately not true.

What does this mean in practice?

Assume: $\angle z = \omega T = 2\pi \frac{\omega}{\Omega} = \emptyset$

$\Rightarrow \omega = \{\emptyset, \Omega, 2\Omega, \dots\}$



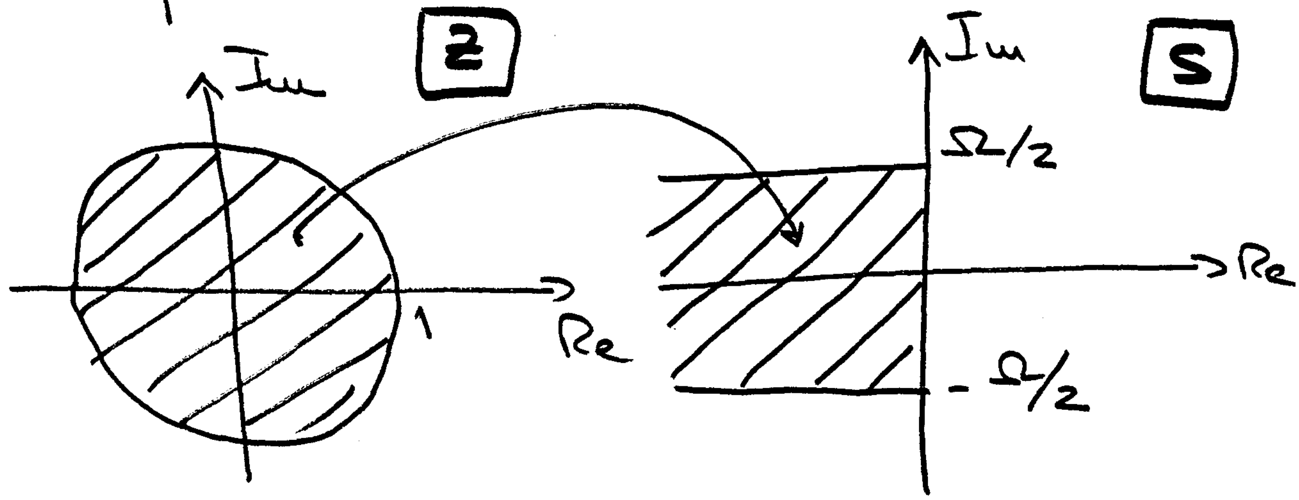
By sampling with a sampling period of T, the two signals cannot be distinguished at all.

Remember: Sampling Theorem: $\Omega \geq 2\omega_{max}$

This is not satisfied for $\omega = \Omega$

$\Longrightarrow$ We always assume that the engineer was reasonable enough to satisfy the sampling theorem.

In that case, the mapping from $z \longrightarrow s$ is also unique :



$$\left(\omega_{max} \leq \Omega/2\right)$$

Symmetry :

We remember that complex poles in the $\boxed{s}$- plane always appear as conjugate complex pairs $\Rightarrow$ the real axis is a symmetry axis.

$$s_1 = -a + jb \Longrightarrow \angle z_1 = Tb$$

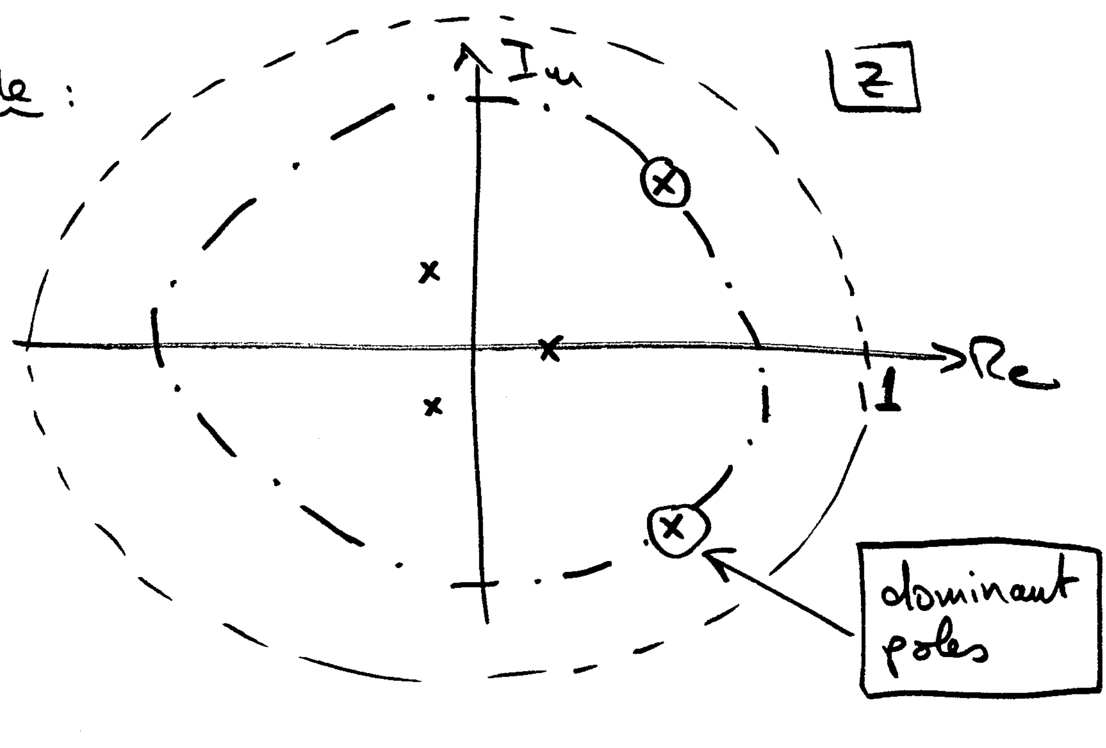
$$s_2 = -a - jb \Longrightarrow \angle z_2 = -Tb$$

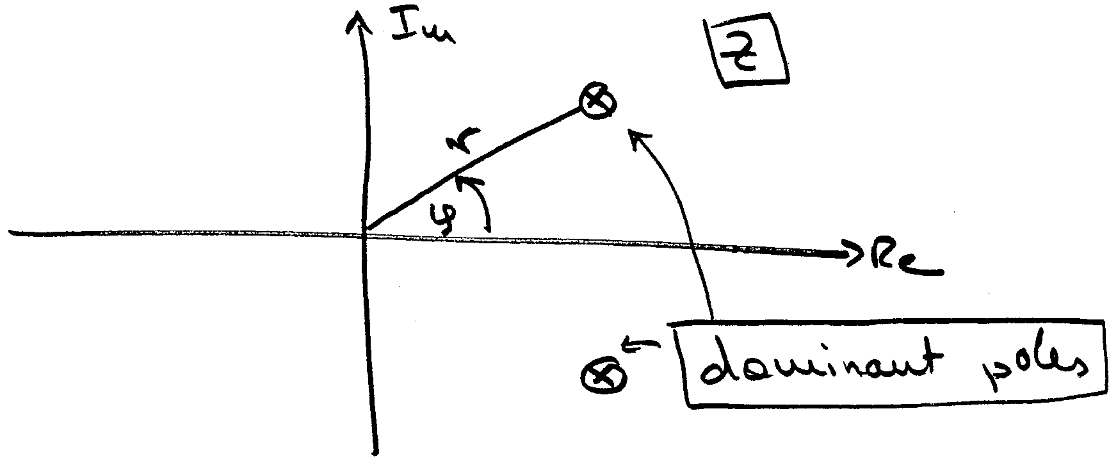
⇒ Complex poles appear also in the $\boxed{z}$-plane always as conjugate complex pairs ⇒ the real axis is a symmetry axis.

Approximation by dominant poles:

The poles with the largest distance from the origin (inside the unity circle) are the dominant poles of a stable system.

Example:

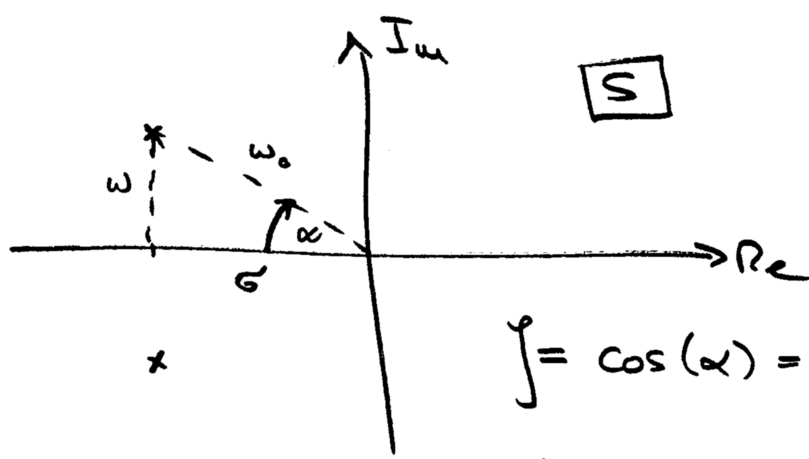




$$\Rightarrow \quad r = e^{\sigma T} \quad \Rightarrow \quad \boxed{\sigma = \frac{1}{T} \cdot \ln(r)}$$

$$\varphi = \omega T \quad \Rightarrow \quad \boxed{\omega = \varphi / T}$$

$$\Rightarrow \quad \omega_0 = \sqrt{\sigma^2 + \omega^2} \quad \Rightarrow \quad \zeta = \left| \frac{\sigma}{\omega_0} \right|$$



$$\zeta = \cos(\alpha) = \left| \frac{\sigma}{\omega_0} \right|$$

ζ : damping factor

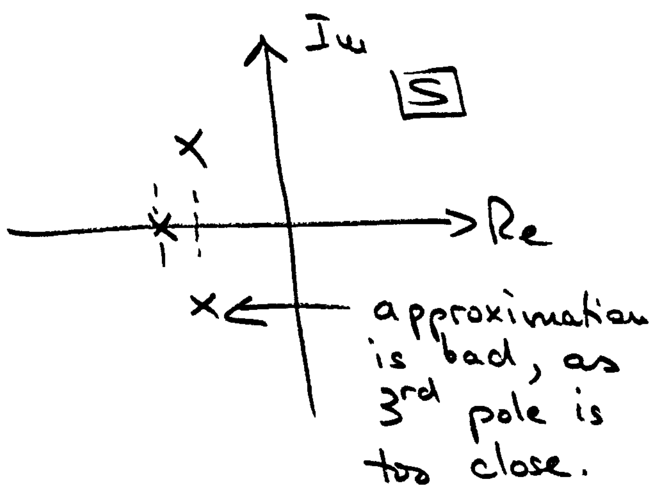
Remember: The approximation by dominant poles was justified iff

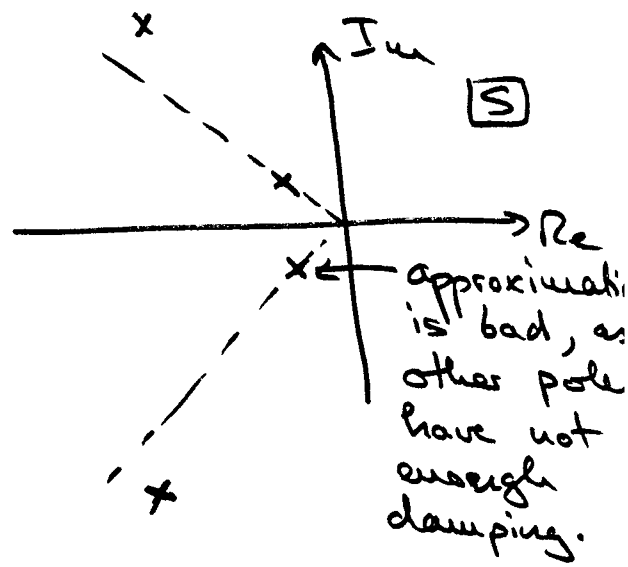
→ all other poles were "sufficiently" further to the left

→ no other poles had a smaller damping

→ **dominant poles have no zeros close by**

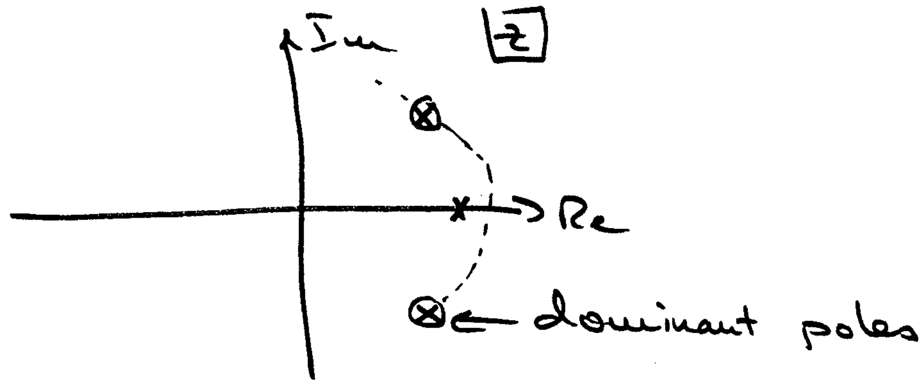
⇒ Otherwise, consider all poles and simulate.





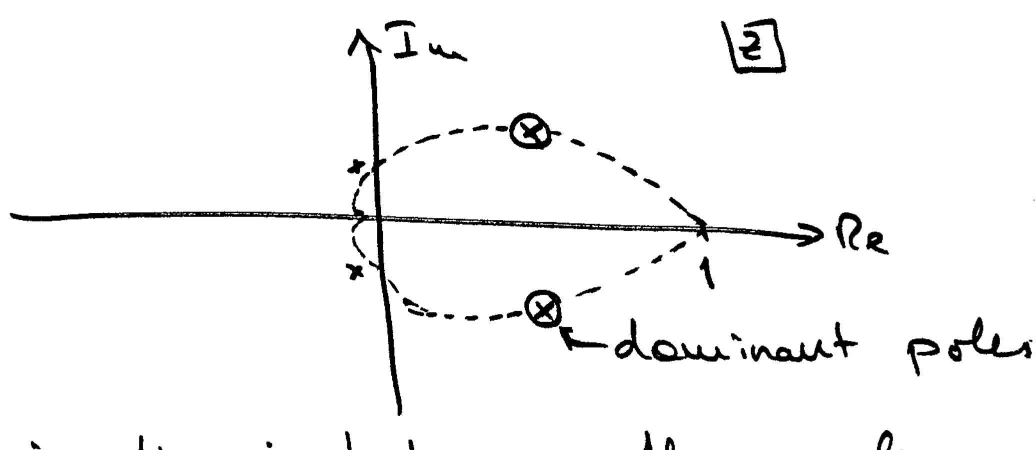
In the $\boxed{z}$-plane, the first condition is easy to check.

→ There may not exist other poles with similar distance to the origin.



⇒ approximation is bad
⇒ 3rd pole is too close.

The second condition requires to sketch the logarithmic spiral that leads through the dominant poles:



⇒ approximation is bad, as other poles are outside the logarithmic spiral.

As the algebraic structure of continuous & discrete systems is the same, the definitions for stability remain the same as well:

a) BIBO - stability: (stability with respect to the input):

$\Rightarrow$ A system is BIBO-stable iff all poles of $G(z)$ lie inside the unity circle:

$$|z_i| < 1$$

b) Lyapunov - stability: (stability with respect to the initial conditions):

$\Rightarrow$ A system is Lyapunov-stable iff all eigenvalues of F lie inside or on the unity circle:

$$|\lambda_i| \leq 1$$

and all eigenvalues on the unity circle are single :

$$|\lambda_i| = 1 \longrightarrow m_i = 1$$

Example :

$$\left| \begin{array}{l} x(k+1) = x(k) \\ y(k) = x(k) \end{array} \right| \quad ; \quad x(\emptyset) = 1$$

$\Rightarrow \quad F = [1] \quad \Rightarrow \quad \lambda_1 = 1$ on unity circle:

$\Rightarrow \quad x(i) = 1$ for all i
$\qquad y(i) = 1$ for all i

$\Rightarrow$ stable.

Example :

$$\left| \begin{array}{l} \underline{x}(k+1) = \begin{bmatrix} \emptyset & 1 \\ -1 & 2 \end{bmatrix} \underline{x}(k) \\ y(k) = \begin{bmatrix} 1 & \emptyset \end{bmatrix} \underline{x}(k) \end{array} \right| \quad ; \quad \underline{x}(\emptyset) = \begin{bmatrix} 1 \\ \emptyset \end{bmatrix}$$

$\Rightarrow$ System has double pole at $(+1)$

$$F = \begin{bmatrix} \emptyset & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 1 \; ; \; m_1 = 2$$

$$\underline{x}(\emptyset) = \begin{bmatrix} 1 \\ \emptyset \end{bmatrix} \Rightarrow \underline{x}(1) = \begin{bmatrix} \emptyset \\ -1 \end{bmatrix} \Rightarrow \underline{x}(2) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \Rightarrow \underline{x}(3) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \underline{x}(4) = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \quad \underline{\text{etc.}}$$

$y(\emptyset) = 1$; $y(1) = \emptyset$; $y(2) = -1$; $y(3) = -2$; $\underline{\text{etc.}}$

$\Rightarrow$ y goes to $-\infty$

$\Rightarrow$ $\left\{ \begin{array}{l} \text{system is indeed} \\ \text{instable.} \end{array} \right.$

[Warning: It is not sufficient to check **one** initial condition to ensure stability. E.g., the last example with $\underline{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does **not** go to infinity.]

- For a system to be <u>stable</u>, it must both be BIBO-stable <u>and</u> Lyapunov-stable.

Example:

$$\left| \begin{array}{l} x(k+1) = x(k) + u(k) \\ y(k) = x(k) + u(k) \end{array} \right| \qquad x(\emptyset) = x_\emptyset$$

<u>is</u> Lyapunov-stable (input $u(k) = \emptyset$)
but is <u>not</u> BIBO-stable ($x_\emptyset = \emptyset$)
which can be easily verified by computing the step response:

$$x(\emptyset) = \emptyset \Rightarrow x(1) = 1 \rightarrow x(2) = 2 \quad \text{etc.}$$
$$y(\emptyset) = 1 \Rightarrow y(1) = 2 \rightarrow y(2) = 3 \quad \text{etc.}$$

$$\underline{\underline{y \longrightarrow +\infty}}$$

Example:

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 2.5 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -1 & 0.5 \end{bmatrix} \underline{x}(k) + [1]\, u(k)$$

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 2.5 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_1 = 0.5 \\ \lambda_2 = 2 \end{array}$$

$\Rightarrow$ System is not Lyapunov-stable.

$$G(z) = 1 + \frac{0.5z - 1}{z^2 - 2.5z + 1}$$

$$= \frac{z^2 - 2.5z + 1 + 0.5z - 1}{z^2 - 2.5z + 1} = \frac{z^2 - 2z}{z^2 - 2.5z + 1}$$

$$\Rightarrow G(z) = \frac{z(z-2)}{(z-0.5)(z-2)} = \frac{z}{z-0.5}$$

$\rightarrow$ System is BIBO-stable.