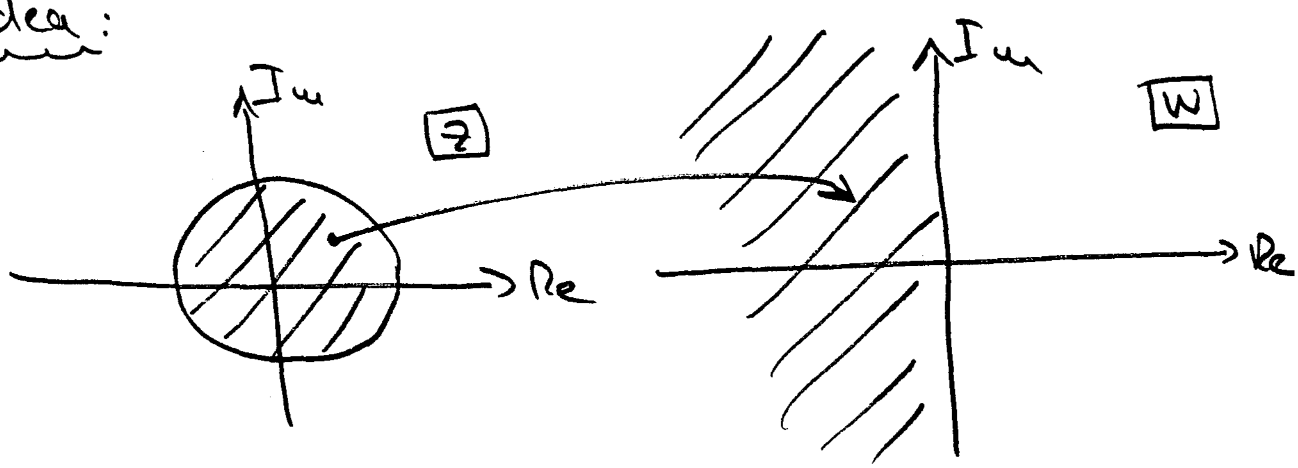
Stability Tests:

Unfortunately, we cannot directly apply the Routh criterion as we have to test on something else than LHP.

A) Indirect Method.

Idea:



We find a transformation that maps the unity circle back onto the LHP while maintaining the algebraic structure of rational functions.

A particular transformation that will accomplish this would be:

$$z = \frac{1+w}{1-w}$$

Proof:

Unity circle: $|z| = 1$ where $w = a + jb$

$$\Rightarrow |z| = 1 = \left| \frac{1+a+jb}{1-a-jb} \right| = \frac{|1+a+jb|}{|1-a-jb|}$$

$$= \frac{\sqrt{(1+a)^2 + b^2}}{\sqrt{(1-a)^2 + b^2}} = 1$$

$$\Rightarrow \sqrt{(1+a)^2 + b^2} = \sqrt{(1-a)^2 + b^2}$$

$$\Rightarrow (1+a)^2 + b^2 = (1-a)^2 + b^2$$

$$\Rightarrow (1+a)^2 = (1-a)^2$$

$$\Rightarrow a = \emptyset$$

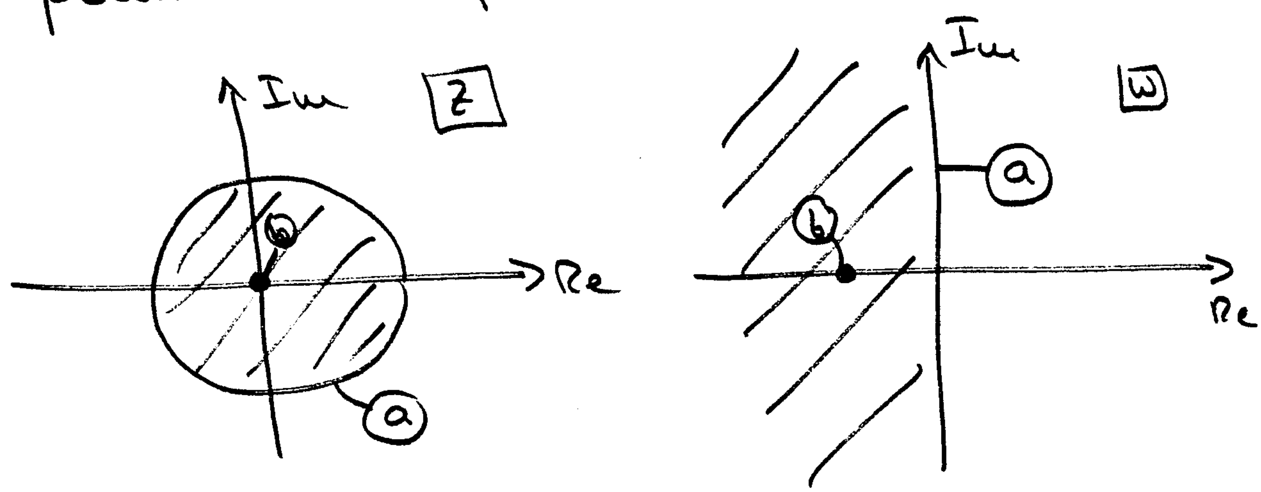
- The unity circle of $\boxed{z}$ ~~is mapped~~ is mapped to the imaginary axis of $\boxed{w}$.

Let us look at $w = -1$

$$\Rightarrow \quad z = \frac{1-1}{1+1} = \frac{\phi}{2} = \phi$$

$\Rightarrow$ The origin of z maps onto the point -1 of w.

$\Longrightarrow$



In the new variable $\boxed{w}$, we can thus apply easily the Routh criterion.

Example:

$$G(z) = \frac{z(z-1)}{(z-0.5)(z-2)}$$

$$= \frac{z^2 - z}{z^2 - 2.5z + 1}$$

(We know, this system is instable)

$$z^2 = \frac{(1+w)^2}{(1-w)^2} = \frac{w^2 + 2w + 1}{(1-w)^2}$$

$$\Rightarrow G(w) = \frac{\dfrac{w^2+2w+1}{(1-w)^2} - \dfrac{(1+w)(1-w)}{(1-w)^2}}{\dfrac{w^2+2w+1}{(1-w)^2} - 2.5\,\dfrac{(1+w)(1-w)}{(1-w)^2} + \dfrac{(1-w)^2}{(1-w)^2}}$$

$$= \frac{w^2 + 2w + 1 - w^2 + 1}{w^2 + 2w + 1 + 2.5w^2 - 2.5 + w^2 - 2w + 1}$$

$$\Rightarrow G(w) = \frac{2w + 2}{4.5w^2 - 0.5} = \frac{\frac{4}{9}w + \frac{4}{9}}{w^2 - \frac{1}{9}}$$

$$\Rightarrow Q(w) = w^2 - 1/9$$

$$\begin{array}{c|cc} w^2 & 1 & -1/9 \\ w^1 & 0 & \\ w^0 & & \end{array}$$

$$\Rightarrow w^2 - 1/9 = 0 \Rightarrow \underline{\underline{w_{1,2} = \pm 1/3}}$$

$\Rightarrow$ There is one pole in RHP

$\Rightarrow$ system is instable ✓.

Example: $G(z) = \dfrac{z(z-2)}{(z-0.5)^2}$

$$= \frac{z^2-2z}{z^2-z+0.25}$$ (we know, this system is stable.)

$$\Rightarrow G(w) = \frac{\dfrac{w^2+2w+1}{(1-w)^2} - 2\,\dfrac{1-w^2}{(1-w)^2}}{\dfrac{w^2+2w+1}{(1-w)^2} - \dfrac{1-w^2}{(1-w)^2} + 0.25\,\dfrac{1-2w+w^2}{(1-w)^2}}$$

$$= \frac{w^2+2w+1-2+2w^2}{w^2+2w+1-1+w^2+0.25-0.5w+0.25w^2}$$

$$= \frac{3w^2+2w-1}{2.25w^2+1.5w+0.25} = \frac{\frac{4}{3}w^2+\frac{8}{9}w-\frac{4}{9}}{w^2+\frac{2}{3}w+\frac{1}{9}}$$

$\Rightarrow Q(w) = w^2+\frac{2}{3}w+\frac{1}{9}$

w^2	1	1/9
w^1	2/3	
w^0	1/9	

no sign change

Disadvantage of the indirect method:

Although the overall mapping is correct, the $\boxed{w}$-plane is **not** the same as the $\boxed{s}$-plane.

E.g. maps $z = \emptyset \longrightarrow s = -\infty$
$\longrightarrow w = -1$

$\Rightarrow$ The pole location in the $\boxed{w}$-plane are **not** the same as in the $\boxed{s}$-plane.

$\Rightarrow$ We **cannot** use this technique to guarantee a minimum stability reserve (σ) by replacing $w \longrightarrow (w - \sigma)$ in $Q(w)$.

B) Necessary Conditions for Stability:

$$Q(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_\emptyset$$

(1) $Q(1) = 1 + \sum_{i=\emptyset}^{n-1} a_i \in (\emptyset, 2^n)$

(2) $(-1)^n \cdot Q(-1) = 1 + (-1)^n \cdot \sum_{i=\emptyset}^{n-1} (-1)^i a_i \in (\emptyset, 2^n)$

(3) $|a_\emptyset| < 1$

(4) $|a_i| < \binom{n}{i}$

For polynomials with real coefficients, the last condition can be put even more stringently:

$$m < a_i < M$$

where:

		$a_\emptyset$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
n=2	M	1	2						
	m	−1	−2						
n=3	M	1	3	3					
	m	−1	−1	−3					
n=4	M	1	4	6	4				
	m	−1	−4	−2	−4				
n=5	M	1	5	1∅	1∅	5			
	m	−1	−3	−1∅	−2	−5			
n=6	M	1	6	15	2∅	15	6		
	m	−1	−6	−5	−2∅	−3	−6		
n=7	M	1	7	21	35	35	21	7	
	m	−1	−5	−21	−5	−35	−3	−7	
n=8	M	1	8	28	56	7∅	56	28	8
	m	−1	−8	−14	−56	−1∅	−56	−4	−8

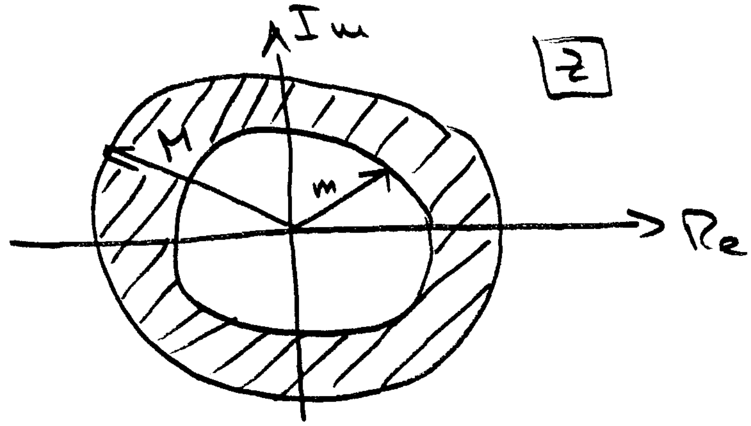
- If a single of these conditions is not satisfied $\Longrightarrow$ the system is certainly instable.
- Always check these conditions first.

C) Sufficient Conditions for Stability:

(1) $$\sum_{i=\emptyset}^{n-1} |a_i| < 1$$

(2) If : $a_i > \emptyset$; $\forall i \in [\emptyset, n-1]$

$\Longrightarrow$ all zeros$\{Q(z)\} = \{z_i\}$ are located in a ring around the origin:



$$m \leq |z_i| \leq M$$

where: m is the smallest and M is the largest of the values:

$$a_{n-1}, \frac{a_{n-2}}{a_{n-1}}, \frac{a_{n-3}}{a_{n-2}}, \dots, \frac{a_1}{a_2}, \frac{a_\emptyset}{a_1}$$

For stability: $\underline{\underline{M<1}}$

$\Rightarrow$ simplified condition:

$$\emptyset < a_0 < a_1 < a_2 < \dots < a_{n-1} < 1$$

- If any of these conditions is satisfied, the system is certainly stable.
- Check these conditions next.

D) Direct Method:

- Several people (Schur, Cohn, Jury, Raible) have designed schemes similar to the Routh criterion that allow to check stability of discrete systems directly.
- I shall introduce <u>one</u> such scheme by E. Jury:

Given: $$\mathcal{G}(z) = \frac{\mathcal{P}(z)}{\mathcal{Q}(z)}$$

where: $$\mathcal{Q}(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_\emptyset$$

We write the following scheme:

$$\begin{array}{c|cccccc}
z^n & a_\emptyset & a_1 & a_2 & \cdots & a_{n-1} & 1 \\
z^n & 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_\emptyset \\
z^{n-1} & b_\emptyset & b_1 & b_2 & \cdots & b_{n-1} & \\
z^{n-1} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_\emptyset & \\
z^{n-2} & c_\emptyset & c_1 & c_2 & \cdots & c_{n-2} & \\
z^{n-2} & c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_\emptyset & \\
 & \vdots & & & \iddots & & \\
z^3 & p_\emptyset & p_1 & p_2 & p_3 & & \\
z^3 & p_3 & p_2 & p_1 & p_\emptyset & & \\
z^2 & q_\emptyset & q_1 & q_2 & & &
\end{array}$$

$\longleftarrow$ stop here.

where: $$b_{n-1} = \begin{vmatrix} a_\emptyset & a_1 \\ 1 & a_{n-1} \end{vmatrix}; \quad b_{n-2} = \begin{vmatrix} a_\emptyset & a_2 \\ 1 & a_{n-2} \end{vmatrix}; \cdots$$

$$c_{n-2} = \begin{vmatrix} b_\emptyset & b_1 \\ b_{n-1} & b_{n-2} \end{vmatrix}; \quad c_{n-3} = \begin{vmatrix} b_\emptyset & b_2 \\ b_{n-1} & b_{n-3} \end{vmatrix}; \cdots$$

etc.

Now, we can derive conditions for stability that are both necessary <u>and</u> sufficient :

(1) $Q(1) = 1 + \sum_{i=\emptyset}^{n-1} a_i \in (\emptyset, 2^n)$

(2) $(-1)^n \cdot Q(-1) = 1 + (-1)^n \sum_{i=\emptyset}^{n-1} (-1)^i a_i \in (\emptyset, 2^n)$

(3) $|a_\emptyset| < 1$

(4) $|b_\emptyset| > |b_{n-1}|$

$|c_\emptyset| > |c_{n-2}|$

$\vdots$

$|p_\emptyset| > |p_3|$

$|q_\emptyset| > |q_2|$

- 2^{nd} order systems have only one row in their Jury-scheme :

$$Q(z) = z^2 + a_1 z + a_0$$

z^2	$a_\emptyset$	a_1	1

$\Rightarrow$ The fourth condition does not exist.

Example: $Q(z) = z^3 - 1.8z^2 + 1.05z - 0.2$

(1) $Q(1) = 1 - 1.8 + 1.05 - 0.2 = 0.05 > 0$ ✓

(2) $Q(-1) = -1 - 1.8 - 1.05 - 0.2 = -4.05$

$\Rightarrow -Q(-1) = 4.05 \in (0, 8)$ ✓

(3) $|a_0| = 0.2 < 1$ ✓

(4)

z^3	-0.2	$+1.05$	-1.8	$+1$
z^3	$+1$	-1.8	$+1.05$	-0.2
z^2	-0.96	$+1.59$	-0.69	
z^2	-0.69	$+1.59$	-0.96	

$\Rightarrow |b_0| = 0.96 > |b_2| = 0.69$ ✓

$\Rightarrow$ System is stable

Example: $Q(z) = (z-0.5)^2 \cdot (z+0.1)^2$

$$= z^4 - 0.8z^3 + 0.06z^2 + 0.04z + 0.0025$$

(1) $Q(1) = 0.3025 \in (0, 16)$ ✓

(2) $Q(-1) = 1.8225 \in (0, 16)$ ✓

(3) $|a_0| = 0.0025 < 1$ ✓

(4)

z^4	0.0025	0.04	0.06	−0.8	1
z^4	1	−0.8	0.06	0.04	0.0025
z^3	−1	0.8001	−0.0598	−0.042	
z^3	−0.042	−0.0598	0.8001	−1	
z^2	0.9982	−0.8026	0.0935		
z^2	0.0935	−0.8026	0.9982		

$|b_0| = 1 > |b_3| = 0.042$ ✓

$|c_0| = 0.9982 > |c_2| = 0.0935$ ✓

$\Rightarrow$ System is stable

- If there are any poles placed exactly on the unity circle, some of the inequalities will become equalities.

Example: $Q(z) = (z-0.5)^2(z+j)(z-j)$

$$= z^4 - z^3 + 1.25z^2 - z + 0.25$$

(1) $Q(1) = 0.5 \in (0, 16)$ ✓

(2) $Q(-1) = 4.5 \in (0, 16)$ ✓

(3) $|a_0| = 0.25 < 1$ ✓

(4)

z^4	0.25	−1	1.25	−1	1
z^4	1	−1	1.25	−1	0.25
z^3	−0.9375	0.75	−0.9375	0.75	
z^3	0.75	−0.9375	0.75	−0.9375	
z^2	0.3164	0	0.3164		
z^2	0.3164	0	0.3164		

$\Rightarrow |b_0| = 0.9375 > |b_3| = 0.75$ ✓

$|c_0| = 0.3164 = |c_2| = 0.3164$

Borderline case.

Recipe:

Replace $z \longrightarrow (1 \pm \varepsilon) z$

This is easy as:

$$z^n \longrightarrow (1 \pm \epsilon)^n \cdot z^n \approx (1 \pm n\varepsilon) \cdot z^n$$

This moves all poles a little out or in. $\Rightarrow$ They will no longer be on the unity circle, and the analysis can continue.

Exponential stability:

Problem: We want to guarantee a certain maximum settling time T_s.

$$\Rightarrow \quad \sigma \approx \frac{4}{T_s}$$

$$\Rightarrow \quad |z| \leq e^{T\sigma} = e^{-4T/T_s} = r$$

$\Rightarrow$ All poles should be inside the circle with $r = e^{-4T/T_s}$.

Recipe:

Replace $z \longrightarrow r * z$

- Use the Jury-Test on the modified $Q^*(z)$ to see whether all poles are inside the unity circle. If so, the poles of the original problem are inside a circle of radius r around the origin.

Example: $Q(z) = (z+0.3)^2 \cdot (z-0.3)^2$

$$= z^4 - 0.18 z^2 + 0.0081$$

$$T = 1 \text{ sec}$$

Let: $T_s = 4$ sec $\Rightarrow r = e^{-4T/T_s} = e^{-1}$

$$= 0.3679$$

$$\Rightarrow Q^*(z) = r^4 z^4 - 0.18 r^2 z^2 + 0.0081$$

Normalize a_n again to 1

$$\Rightarrow Q^*(z) = z^4 - \frac{0.18}{r^2} z^2 + \frac{0.0081}{r^4}$$

$$= z^4 - 1.33 z^2 + 0.4422$$

$$Q^*(z) = (z + 0.8155)^2 (z - 0.8155)^2$$

$\Rightarrow$ stable

$\Rightarrow$ exponential damping (stability reserve) is guaranteed

(1) $Q^*(1) = 0.1122 \in [0, 16]$ ✓

(2) $Q^*(-1) = 0.1122 \in (0, 16)$ ✓

(3) $|a_0| = 0.4422 < 1$ ✓

(4)

z^4	0.4422	0	−1.33	0	1
z^4	1	0	−1.33	0	0.4422
z^3	−0.8044	0	0.7418	0	
z^3	0	0.7418	0	−0.8044	
z^2	0.6471	0	−0.5967		
z^2	−0.5967	0	0.6471		

$|b_0| = 0.8044 > |b_3| = 0$ ✓

$|c_0| = 0.6471 > |c_2| = 0.5967$ ✓

$\Rightarrow$ stable

$\Rightarrow$ stability reserve is guaranteed.