Frequency - Domain Analysis

In ECE 441, we have learned how to use Bode- and Nyquist plots to analyze / design continuous control systems. It was found that:

(1) Frequency - Domain techniques are often more appropriate than Root-Locus techniques to assess and influence the behavior of higher-order systems

(2) Bode - diagrams are easy to sketch (by straight-line approximation). They work well for Lead/Lag - compensato. design, but not so well for stability assessment except for minimum-phase systems.

(3) Nyquist - diagrams work well for global stability assessment.

(4) Bode-diagrams are better than Nyquist for <u>quantitative</u> analysis / design, that is: determination of parameter values.

(5) Nyquist-diagrams work better than Bode-diagrams for <u>qualitative</u> analysis/design, that is: for taking structural decisions.

(6) Bode-diagrams are better for looking at a limited frequency range.

(7) Nyquist-diagrams are better for the total picture over all frequencies.

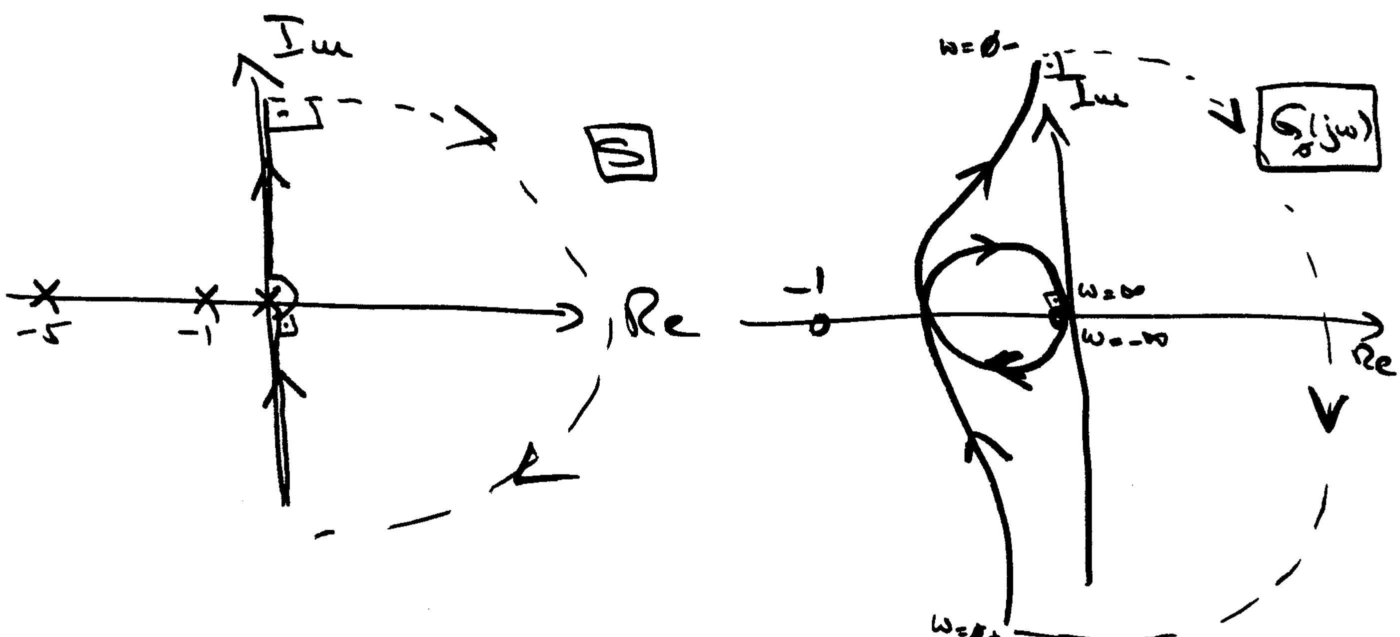
$\Rightarrow$ We need both representations, although they contain the same information.

Nyquist-Diagram: (Stability)

Lemma: A system is stable iff the number of counterclockwise encirclements of the point (-1) by the complete Nyquist diagram of $G_o(s)$ equals the number of instable poles of $G_o(s)$

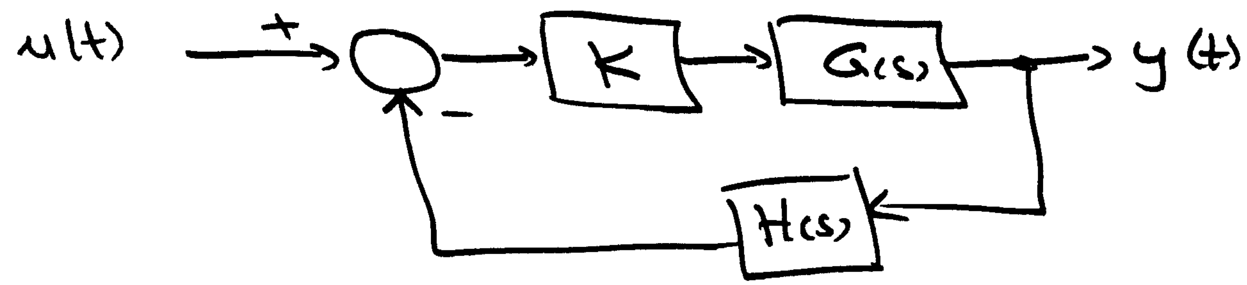
Example:

$$G_\emptyset(s) = \frac{K}{s(s+1)(s+5)}$$



$\#\text{poles}_{>\emptyset}\{G_\emptyset(s)\} = \emptyset$ $\qquad \# \overset{\curvearrowleft}{E}_{-1} = \emptyset \implies$ stable

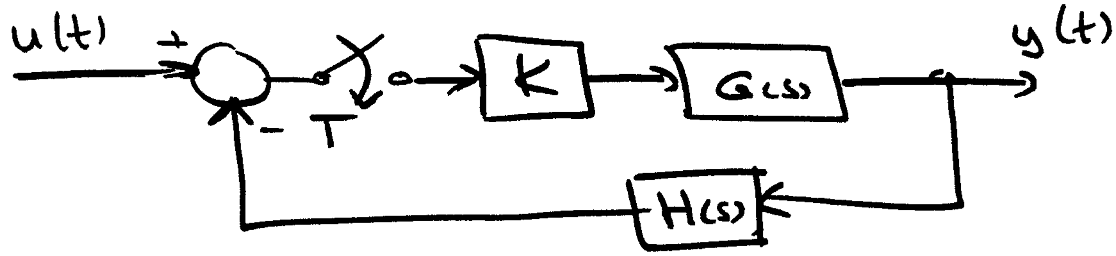
⇒ We look at the behavior of the open-loop system, and from there judge the stability of the closed-loop system.



$$G_{tot}(s) = \frac{KG(s)}{1+KG(s)H(s)} = \frac{KG(s)}{1+G_o(s)}$$

$$\left[\ G_o(s) = KG(s)H(s)\ \right]$$

Question: Can this be extended to the case of the sampled-data system?



$$\Rightarrow G^*_{tot}(s) = \frac{KG^*(s)}{1 + K\left[G(s)H(s)\right]^*} = \frac{KG^*(s)}{1 + G_o^*(s)}$$

$$\left[\; G_o^*(s) = K\left[G(s)H(s)\right]^* \;\right]$$

This is basically still the same kind of system. $\Longrightarrow$ All the derivatives taken for the previous case, still hold here as well.

$\Rightarrow$ We draw the Nyquist-diagram by letting $s = j\omega$ in $G_o^*(s)$.

However, it may be simpler to use the z-domain.

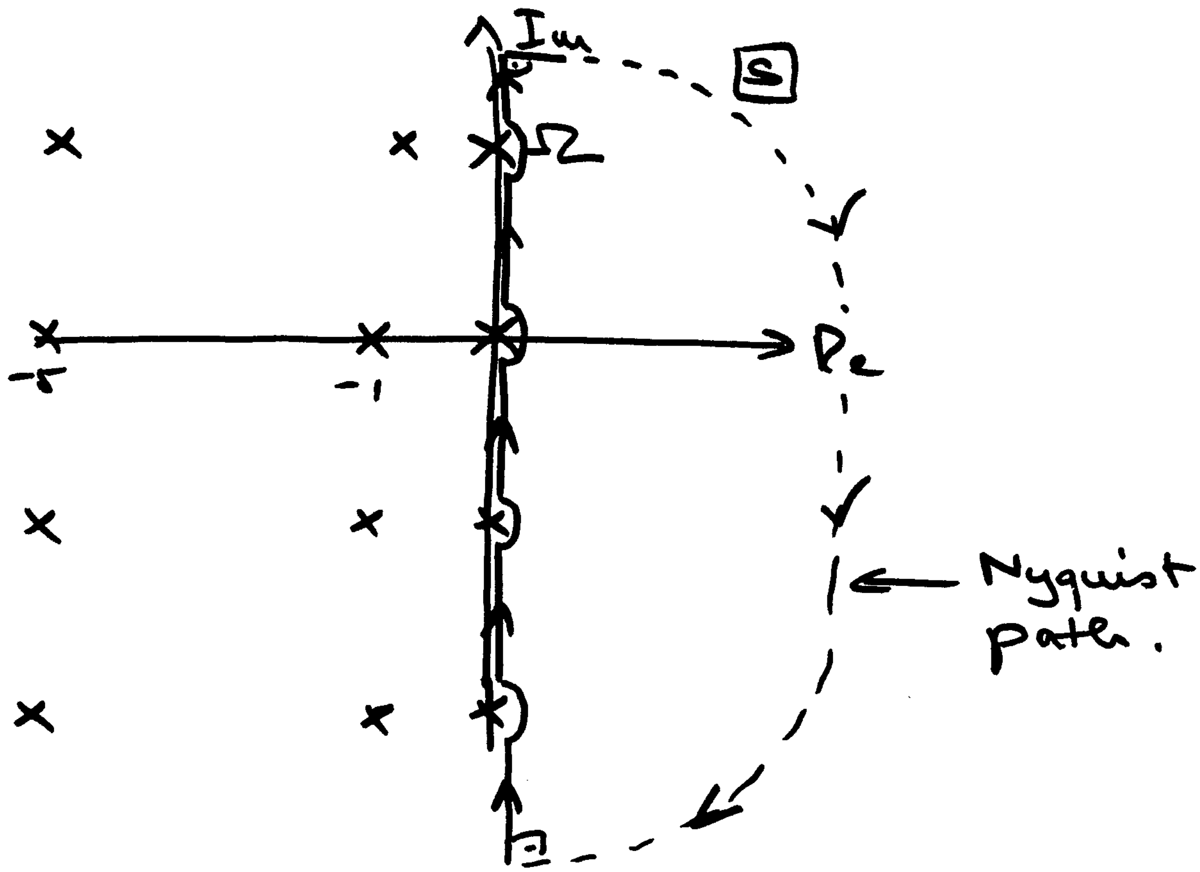
$$\underset{\text{in } G_o(s)}{s \rightarrow j\omega} \implies \underset{\text{in } \mathcal{G}_o(z)}{z \rightarrow e^{j\omega T}}$$

Example: $G_o(s) = \dfrac{K}{s(s+1)(s+5)}$

$$\Rightarrow \Lambda = \begin{bmatrix} -5 & \phi & \phi \\ \phi & -1 & \phi \\ \phi & \phi & \phi \end{bmatrix} \Rightarrow e^{\Lambda T} = \begin{bmatrix} e^{-5T} & \phi & \phi \\ \phi & e^{-T} & \phi \\ \phi & \phi & 1 \end{bmatrix}$$

$$\Rightarrow \mathcal{G}_o(z) = \frac{\mathcal{P}(z)}{(z-1)(z-e^{-T})(z-e^{-5T})}$$

$\Rightarrow$ $G_o^*(s)$ has the same poles as $G_o(s)$, but duplicated along the imaginary axis with a distance of $\Omega = 2\pi/T$.



Let us discuss, how this path looks like in the z-plane:

$$s = \phi \longrightarrow z = 1$$

$$s = j\Omega \longrightarrow z = e^{j\Omega T} = e^{j\frac{2\pi}{T}\cdot T} = e^{j2\pi} = 1$$

etc.

⇒ The imaginary axis maps into the unity circle infinitely many times.

$s = re^{j\varphi}$ with $r \to \infty$:

$$z = e^{sT} = e^{rTe^{j\varphi}} = e^{rT(\cos\varphi + j\sin\varphi)}$$

$$= e^{rT\cos\varphi} \cdot e^{jrT\sin\varphi}$$

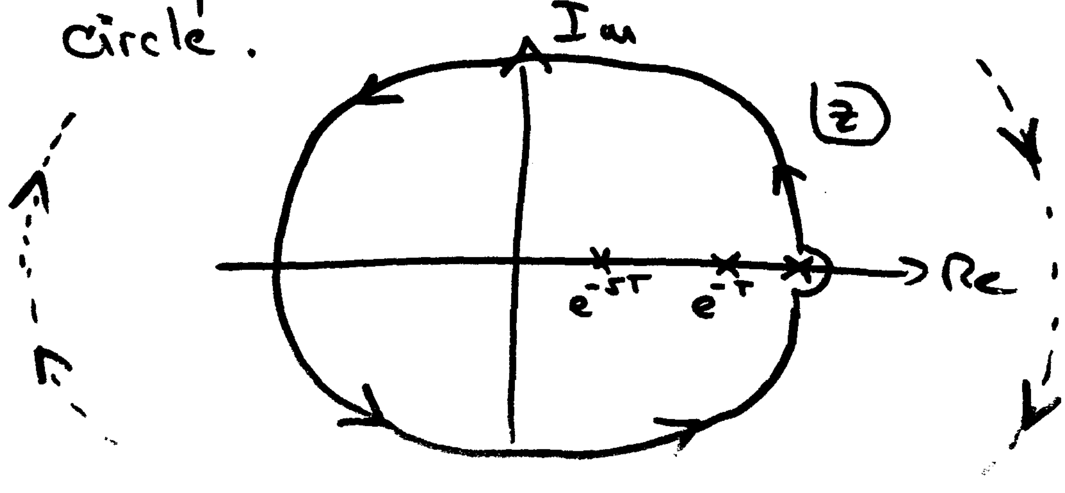
$\varphi = 0$, $r \to \infty$: $z = \infty \angle ?$

$\varphi = \frac{\pi}{4}$; $r \to \infty$: $z = \infty \angle ?$

$\varphi = \frac{\pi}{2}$; $r \to \infty$; $z = \, ? \angle ?$

$\Rightarrow$ Looks somewhat odd !!!
However, for all angles $\neq \pm \pi/4$:
$|z| \to \infty$; $\angle z$ turns infinitely fast !!!

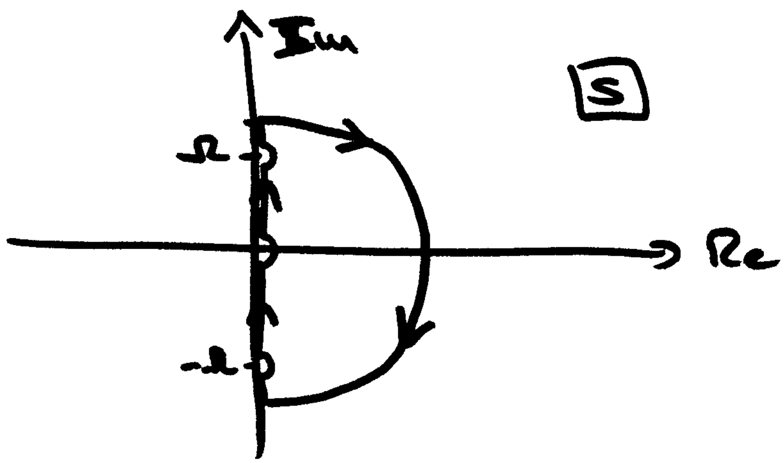
$\Rightarrow$ The positive infinity circle maps into a total infinity circle.



To envisage a little better what happens: let r remain finite,

e.g. $r = 1.35\,\Omega$

at the upper edge: $s = j \cdot 1.35\,\Omega$



$$\Rightarrow \quad z = e^{sT} = e^{j\,1.25\,\Omega T} = e^{j\,1.35 \cdot \frac{2\pi}{T} \cdot T}$$

$$\Rightarrow \quad \angle z = 1.35 \cdot 360^\circ = 486^\circ \mathrel{\hat{=}} 126^\circ$$

$$|z| = 1$$

at $\varphi = 0$: $\quad s = 1.35\,\Omega \;\Rightarrow\; z = e^{1.35 \cdot 2\pi}$

$$\Rightarrow \angle z = 0 \quad ; \quad |z| = e^{2.7\pi} = 4828.5$$

For any angle φ:

$$s = 1.35\,\Omega\,(\cos\varphi + j\sin\varphi)$$

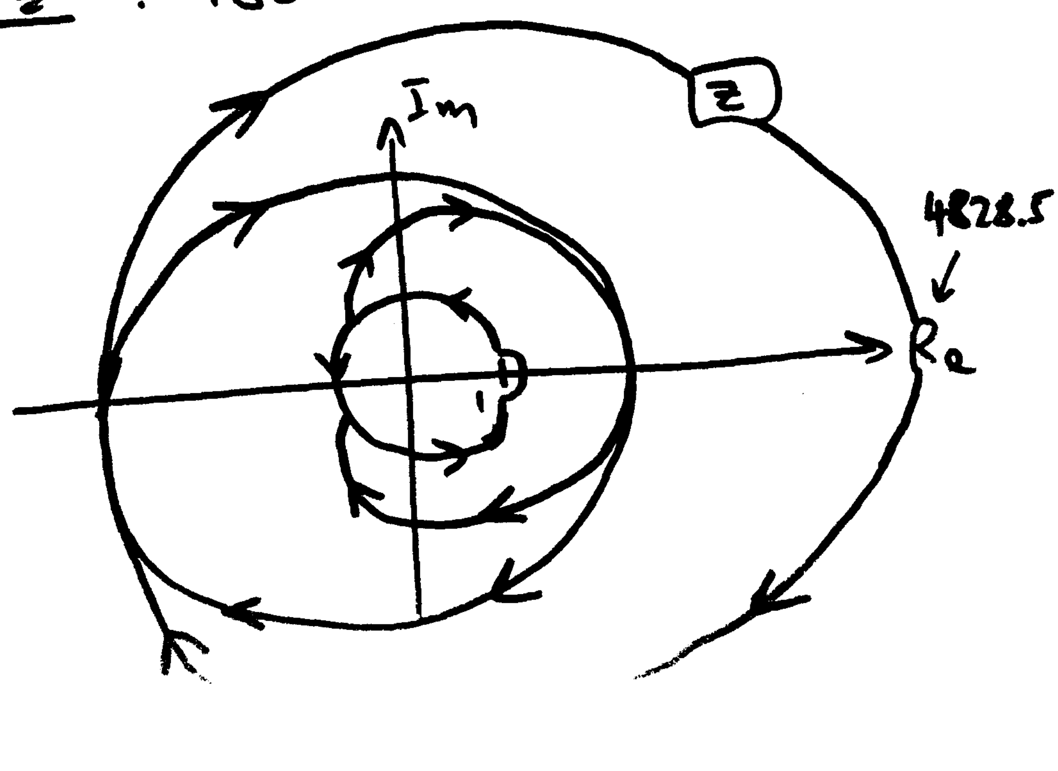
$$\Rightarrow z = e^{1.35 \cdot 2\pi \cdot \cos\varphi} \cdot e^{j1.35 \cdot 2\pi \cdot \sin\varphi}$$

$$= e^{2.7\pi \cdot \cos\varphi} \cdot e^{j\,2.7\pi \sin\varphi}$$

while $\varphi : \pi/2 \longrightarrow -\pi/2$:

$$\angle z : e^{j\,2.7\pi} \longrightarrow e^{-j\,2.7\pi}$$

$$\Rightarrow \angle z : 486° \longrightarrow -486°$$

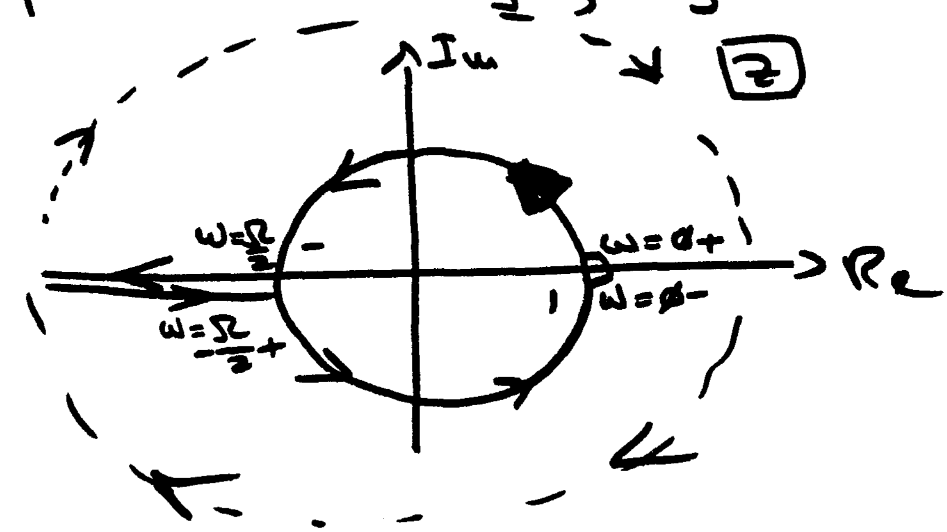


As $r \to \infty$, the locus in the z-plane turns faster and faster.

In the limit, the unity circle is passed infinitely often in the mathematically positive sense while $\omega: 0+ \to \infty$, then there occurs a sudden jump to the infinity circle which is passed twice as often in the mathematically negative sense while $\omega: \infty \to -\infty$, then another jump brings the locus back to the unity circle which is again passed infinitely often in the positive direction for $\omega: -\infty \to 0-$.

Of course, while the Nyquist path in the z-plane goes around infinitely often, also the Nyquist diagram in the G(z)-planed is passed infinitely often, which makes "counting" of the encirclements of (-1) difficult (!)

A more practical proposition seems to reestablish the stability criterion for the discrete path by designing another Nyquist path in the z-plane directly, e.g.



$G_o(z)$ has the same algebraic structure as $G_o(s)$ $\Rightarrow$ the same properties apply:

$$\Delta_{\Gamma_s} \angle G(z) = 2\pi (P - S)$$

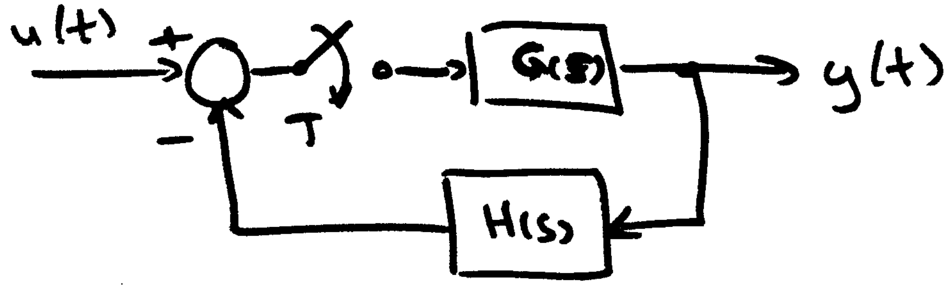
$P ::=$ # of zeroes inside Γ_s

$S ::=$ # of poles inside Γ_s

⇒ The # of mathematically positive circles around the origin of $\mathcal{G}(z)$ by Γ_g can be computed as:

$$\overset{\curvearrowleft}{N} = \frac{1}{2\pi} \Delta_{\Gamma_s} \angle \mathcal{G}(z) = P - S$$

Let us now look at the sampled-data control system:



$$\Rightarrow \quad \mathcal{G}_{tot}(z) = \frac{\mathcal{G}(z)}{\mathcal{F}(z)} \; ; \quad \mathcal{F}(z) = 1 + \mathcal{G}_0(z)$$

$$\mathcal{G}_0(z) = \mathcal{Z}\{G(s)H(s)\}$$

Let $\mathcal{G}(z) = \dfrac{N_G(z)}{D_G(z)}$; $\mathcal{G}_0(z) = \dfrac{N_{GH}(z)}{D_{GH}(z)}$

<u>However</u>: $\mathcal{D}_{GH}(z) = \mathcal{D}_G(z) \cdot \mathcal{D}_H(z)$

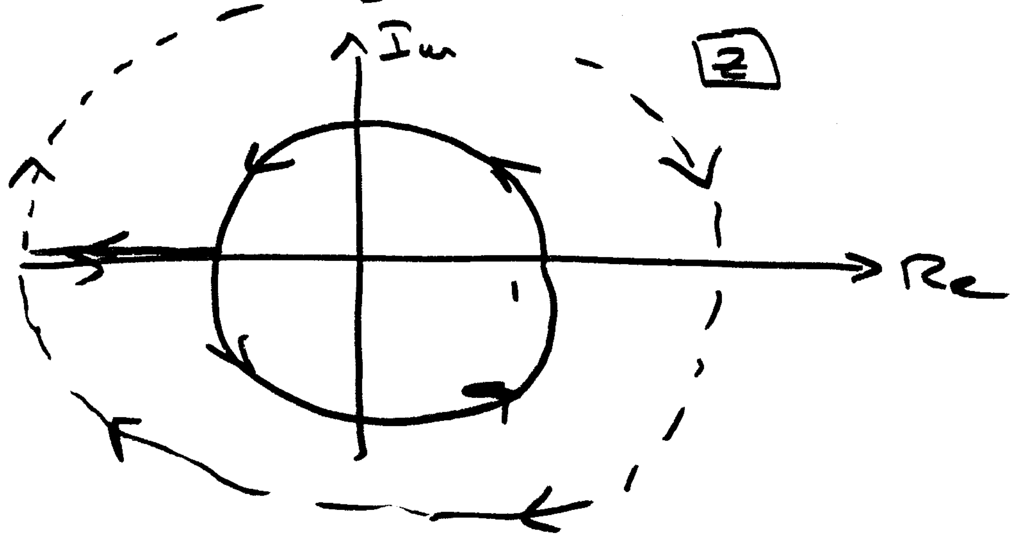
$$\left(\mathcal{N}_{GH}(z) \neq \mathcal{N}_G(z) \cdot \mathcal{N}_H(z) \right)$$

$$\Rightarrow \mathcal{F}(z) = 1 + \mathcal{G}_0(z) = 1 + \frac{\mathcal{N}_{GH}(z)}{\mathcal{D}_G(z) \cdot \mathcal{D}_H(z)}$$

$$= \frac{\mathcal{N}_{GH}(z) + \mathcal{D}_G(z)\,\mathcal{D}_H(z)}{\mathcal{D}_G(z) \cdot \mathcal{D}_H(z)}$$

$$\Rightarrow \mathcal{G}_{tot}(z) = \frac{\mathcal{G}(z)}{\mathcal{F}(z)} = \frac{\mathcal{N}_G(z) \cdot \mathcal{D}_H(z)}{\mathcal{N}_{GH}(z) + \mathcal{D}_G(z)\,\mathcal{D}_H(z)}$$

Let us now draw a pole/zero-diagram of $\mathcal{F}(z)$, and apply the discrete Nyquist path:



This is a closed loop clockwise:

$$\Rightarrow \overrightarrow{N}_0^{F} = \underset{>1}{\#\,\text{zeroes}}\left(F(z)\right) - \underset{>1}{\#\,\text{poles}}\left(F(z)\right)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\overrightarrow{N}_{-1}^{G_0} = \underset{>1}{\#\,\text{poles}}\left(G_{tot}(z)\right) - \underset{>1}{\#\,\text{poles}}\left(G_0(z)\right)$$

$$\Rightarrow \underset{>1}{\#\text{poles}}\left(G_{tot}(z)\right) = \overrightarrow{N}_{-1}^{G_0} + \underset{>1}{\#\,\text{poles}}\left(G_0(z)\right)$$

For stability: $\underset{>1}{\#\,\text{poles}}\left(G_{tot}(z)\right) \equiv \emptyset$

$$\Rightarrow \quad \boxed{\overset{\curvearrowleft}{N}{}^{G_0}_{-1} \equiv \#\ \text{poles}_{>1}\left(\mathcal{G}_0(z)\right)}$$

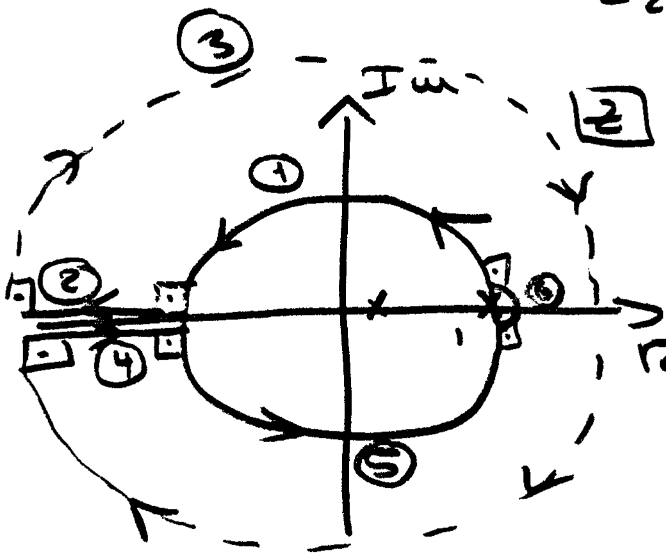
<u>Lemma</u>: A system is stable iff the # of counterclockwise encirclements of the point $(-1, 0)$ by the complete discrete Nyquist-diagram of $\mathcal{G}_0(z)$ equals the # of instable poles of $\mathcal{G}_0(z)$.

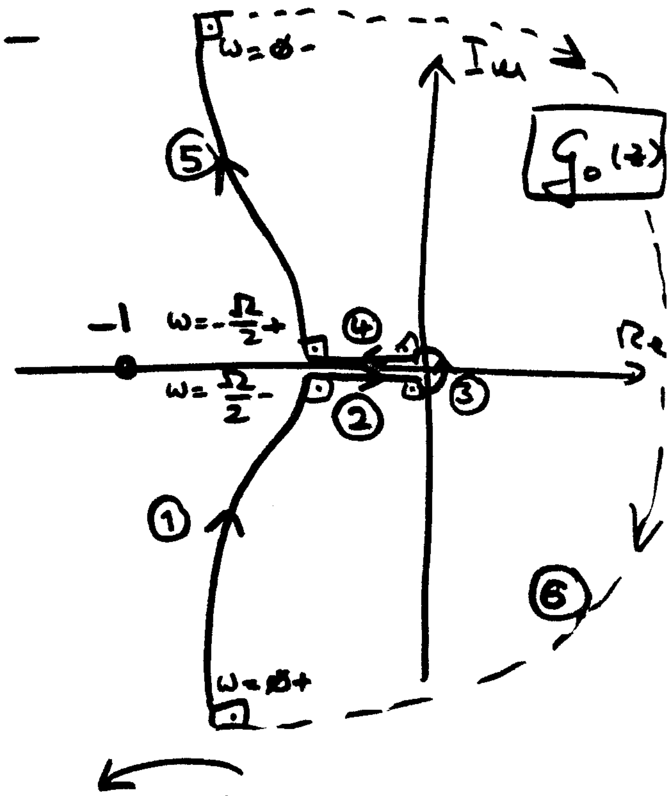
<u>Notice</u>: The discrete Nyquist-diagram uses a different Nyquist path from the continuous Nyquist diagram.

<u>Example</u>: $$G_0(s) = \frac{1.57}{s(s+1)}$$

$$\Rightarrow \mathcal{G}_0(z) = 1.57 \cdot \frac{0.792\,z}{(z-1)(z-0.208)}$$

(for a particular T)





$\#\text{poles}_{>1} \equiv 0$ $\qquad$ $N_{-1}^{G_0} \equiv 0$

$\Rightarrow$ stable.

If K grows, the Nyquist-diagram moves further to the left. At K = 3.05: The Nyquistdiagram goes exactly through the point (-1, 0).

Proof:
$$G_0(z) = K \cdot \frac{0.792\,z}{(z-1)(z-0.208)} = -1$$

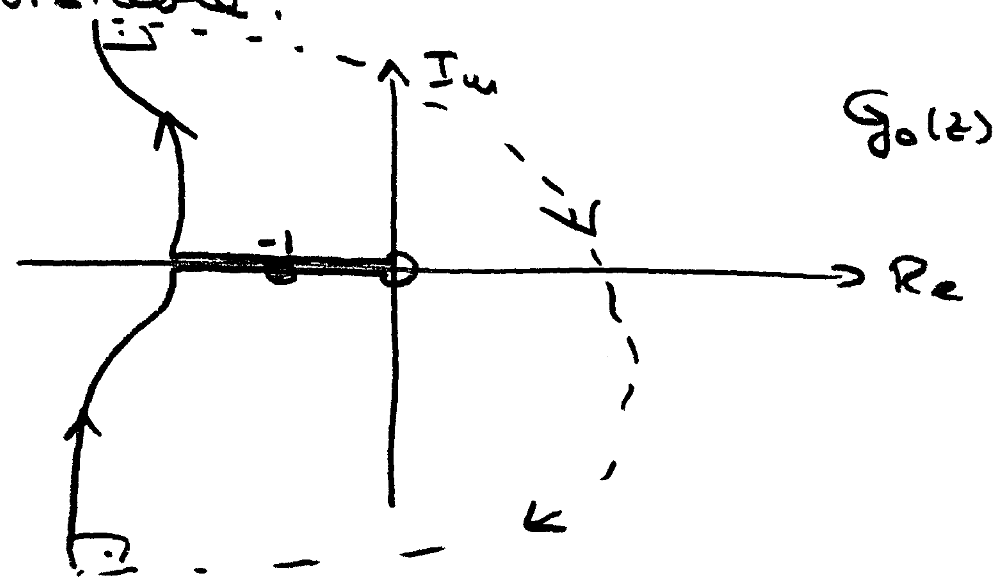
where: $z = -1$

$$\Rightarrow K \cdot 0.792(-1) = (-1)(-2)(-1.208)$$

$$\Rightarrow 0.792K = 2.416$$

$$\Rightarrow K = \frac{2.416}{0.792} = 3.05$$

For larger values of K: The discrete Nyquist diagram goes always exactly through $(-1, 0)$
$\Rightarrow$ instable.



$\Rightarrow$ instable