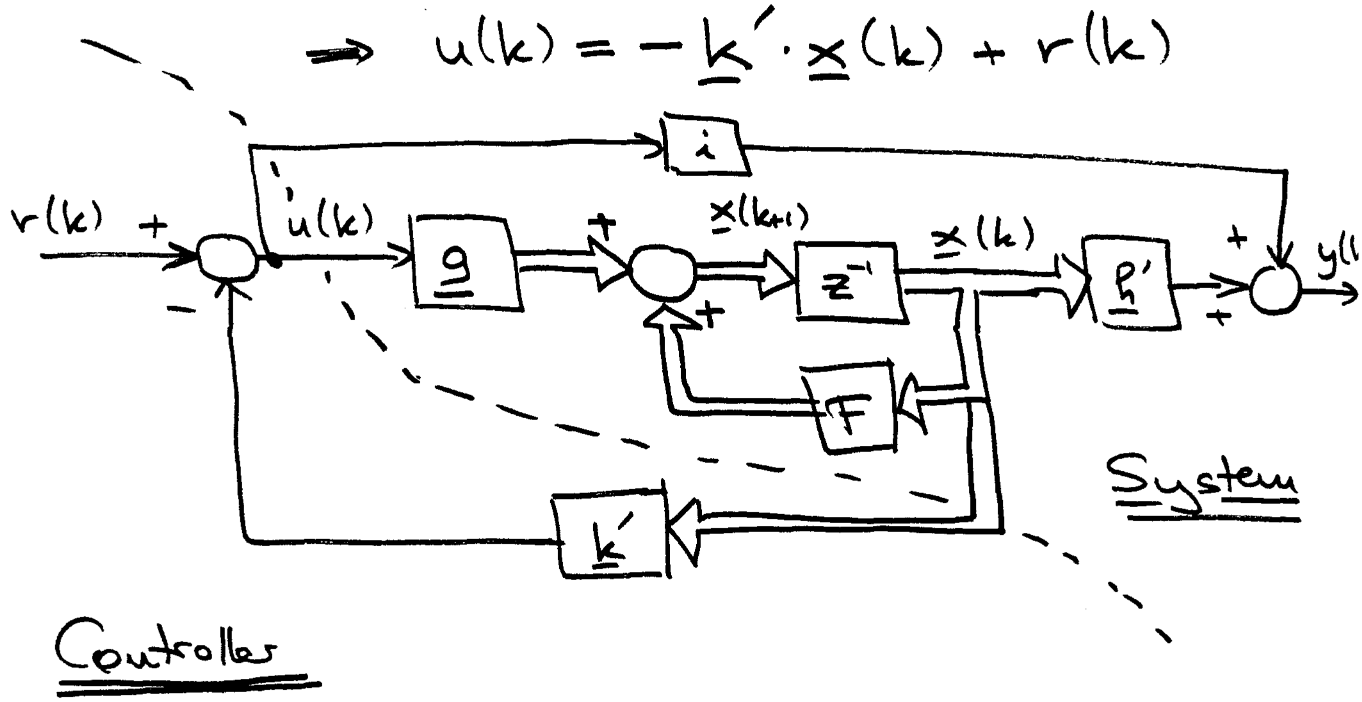
State Feedback:

Idea: As in the case of continuous systems, we can feed back a linear combination of all state variables to form the input:

$$\left| \begin{aligned} \underline{x}(k+1) &= F \cdot \underline{x}(k) + \underline{g} \cdot u(k) \\ y(k) &= \underline{h}' \underline{x}(k) + i \cdot u(k) \end{aligned} \right|$$

$$\Rightarrow \quad u(k) = -\underline{k}' \cdot \underline{x}(k) + r(k)$$



Lemma: If the plant is completely controllable, $\Longrightarrow$ the poles of the closed-loop system can be selected freely.

Proof (By construction of an algorithm)

- If the system is controllable, $\Longrightarrow$ we can bring it to controller-canonical form.

$$\hat{F} = \begin{bmatrix} \emptyset & 1 & & \emptyset \\ & & \ddots & \\ \emptyset & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \qquad \hat{\underline{g}} = \begin{bmatrix} \emptyset \\ \vdots \\ \emptyset \\ 1 \end{bmatrix} \qquad \hat{\underline{k}} = \begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \\ \vdots \\ \hat{k}_n \end{bmatrix}$$

unknown

$$\underline{x}(k+1) = \hat{F}\cdot\underline{x}(k) + \hat{\underline{g}}\cdot\left[-\hat{\underline{k}}'\underline{x}(k) + r(k)\right]$$

$$\Rightarrow \underline{x}(k+1) = \underbrace{\left(\hat{F} - \hat{\underline{g}}\hat{\underline{k}}'\right)}_{\hat{F}_{CL}}\underline{x}(k) + \hat{\underline{g}}\cdot r(k)$$

$$\Rightarrow \hat{F}_{CL} = \hat{F} - \underline{\hat{g}} \cdot \underline{\hat{k}}'$$

$$\underline{\hat{g}} \cdot \underline{\hat{k}}' = \begin{bmatrix} & \emptyset & \\ \hat{k}_1 & \hat{k}_2 & \cdots & \hat{k}_n \end{bmatrix}$$

$$\Rightarrow \hat{F}_{CL} = \begin{bmatrix} & 1 & & & \emptyset \\ & & 1 & & \\ & \emptyset & & \ddots & 1 \\ -(a_0 + \hat{k}_1) & -(a_1 + \hat{k}_2) & \cdots & & -(a_{n-1} + \hat{k}_n) \end{bmatrix}$$

$\Rightarrow$ again in controller-canonical form

Example:

$$G(z) = \frac{z^2}{z^3 - 6z^2 + 11z - 6}$$

$$Q(z) = z^3 - 6z^2 + 11z - 6 = (z-1)(z-2)(z-3)$$

$\Longrightarrow$ instable

We want the closed-loop poles at:

$$\left| \begin{array}{l} \lambda_1 = 1 \\ \lambda_{2,3} = 0.5 \pm 0.5j \end{array} \right|$$

$$\Rightarrow Q_{CL}(z) = (z-1)(z-0.5+0.5j)(z-0.5-0.5j)$$

$$= z^3 - 2z^2 + 1.5z - 0.5$$

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

$$F_{CL} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & -1.5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0+\hat{k}_1) & -(a_1+\hat{k}_2) & -(a_2+\hat{k}_3) \end{bmatrix}$$

$$\Rightarrow a_0 = -6 \quad ; \quad a_0 + \hat{k}_1 = -0.5$$

$$\Rightarrow \underline{\underline{\hat{k}_1}} = -0.5 + 6 = \underline{\underline{5.5}}$$

$$a_1 = 11 \quad ; \quad a_1 + \hat{k}_2 = 1.5$$

$$\Rightarrow \underline{\underline{\hat{k}_2}} = \underline{\underline{-9.5}}$$

$$a_2 = -6 \quad ; \quad a_2 + \hat{k}_3 = -2$$

$$\Rightarrow \underline{\underline{\hat{k}_3}} = \underline{\underline{4}}$$

$$\Rightarrow \underline{\underline{\hat{k}' = [5.5 \quad -9.5 \quad 4]}}$$

will place the poles where I want them to be.

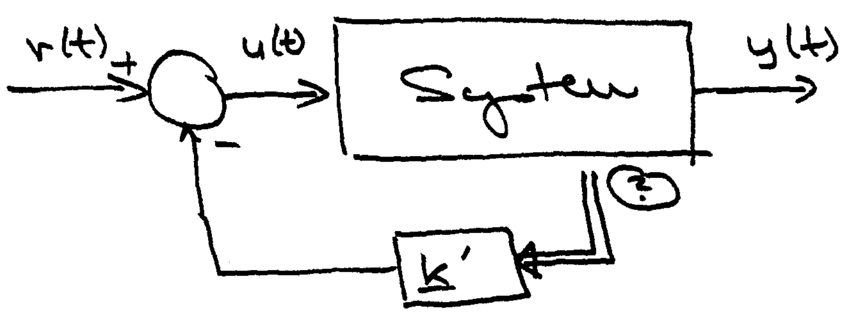
⟹ The controllable poles of a system can be placed freely by state feedback. This is called the <u>pole placement</u> or <u>pole assignment</u> problem.

⟹ The uncontrollable poles cannot be affected at all by state feedback.

⟹ Also the zeroes remain the same. if no direct coupling.

Design of Observers.

a) Problem: We have seen that and how we can design a state feedback to place controllable poles wherever we want them to be. Unfortunately in a real system, what we can measure are the outputs and not the states.



⟹ We need output feedback.

$$\left|\begin{array}{l} \underline{x}(k+1) = F\,\underline{x}(k) + G\,\underline{u}(k) \\ \underline{y}(k) = H\,\underline{x}(k) + I\,\underline{u}(k) \end{array}\right|$$

If H is square $(p=n)$ and non-singular:

$$\Longrightarrow \quad \underline{\hat{x}}(k) = H^{-1}\left(\underline{y}(k) - I\,\underline{u}(k)\right)$$

will be a perfect observer.

However, this case is rare. E.g. our SISO-system:

$$\left|\begin{array}{l} \underline{x}(k+1) = F\,\underline{x}(k) + \underline{g}\,u(k) \\ y(k) = \underline{h}'\,\underline{x}(k) + i\,u(k) \end{array}\right|$$

will not satisfy the condition $p=n$ except for first-order systems.

$\Longrightarrow$ We need something better.

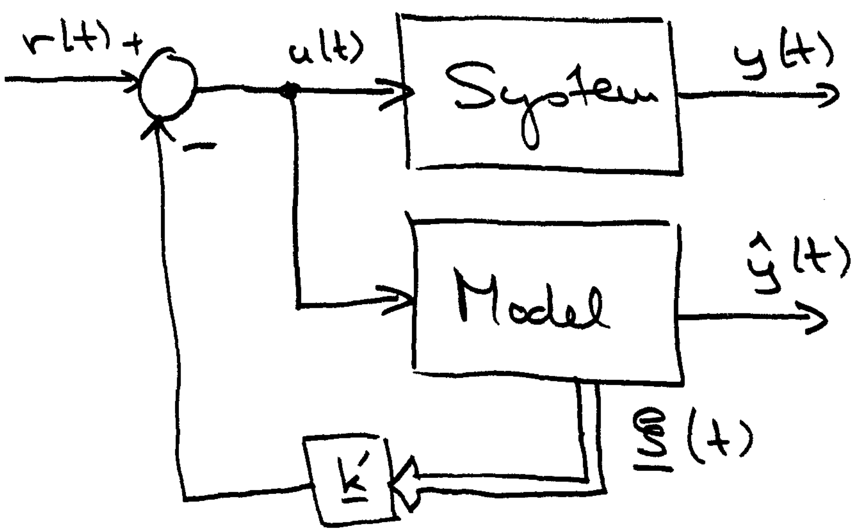
Idea: Create a model of the system:

$$\begin{aligned} \underline{\xi}(k+1) &= F \cdot \underline{\xi}(k) + \underline{g} \cdot u(k) \\ \hat{y}(k) &= \underline{h}' \cdot \underline{\xi}(k) + i \cdot u(k) \end{aligned}$$

If the system matrices and the initial conditions are the same

$$\Longrightarrow \quad \underline{\xi}(k) \equiv \underline{x}(k)$$

is an "observer" for $\underline{x}(k)$.



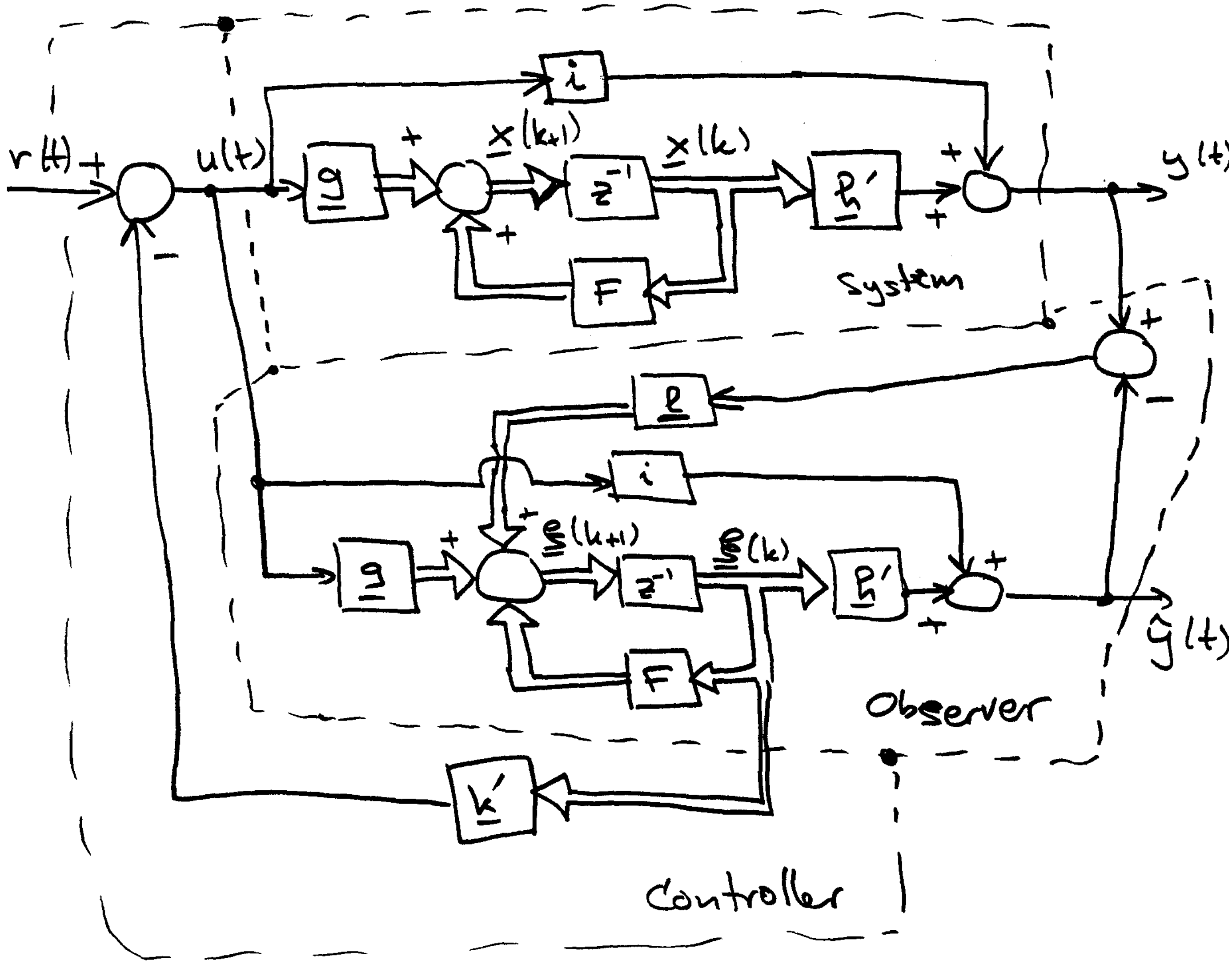
Of course, the smallest disturbance will make $\underline{\xi}(k)$ drift away from $\underline{x}(k)$.

$\Rightarrow$ We need to ensure that:

$$\underline{e}(k) = \underline{x}(k) - \underline{\xi}(k) \longrightarrow \emptyset$$

as time goes to infinity.

Idea: Create a second "control"-loop to make $\underline{\xi}(k)$ track $\underline{x}(k)$ even if the initial conditions are different.



$$\left| \begin{array}{l} \underline{x}(k+1) = F \cdot \underline{x}(k) + \underline{g} \cdot u(k) \\ y(k) = \underline{h}' \cdot \underline{x}(k) + i \cdot u(k) \end{array} \right| \quad \text{System}$$

$$\left| \begin{array}{l} \underline{\xi}(k+1) = F \cdot \underline{\xi}(k) + \underline{g}\, u(k) + \underline{\ell}\left(y(k) - \hat{y}(k)\right) \\ \hat{y}(k) = \underline{h}' \cdot \underline{\xi}(k) + i \cdot u(k) \end{array} \right|$$

Observer

$$\left| u(k) = r(k) - \underline{k}' \cdot \underline{\xi}(k) \right| \quad \text{Controller}$$

Proof that $\underline{e}(k) \longrightarrow \phi$:

$$\underline{e}(k+1) = \underline{x}(k+1) - \underline{\xi}(k+1) = F \cdot \underline{x}(k) \cancel{+ \underline{g}\, u(k)}$$
$$- F \underline{\xi}(k) \cancel{- \underline{g}\, u(k)} - \underline{\ell}\,(y(k) - \hat{y}(k))$$

$$\Rightarrow \underline{e}(k+1) = F(\underline{x}(k) - \underline{\xi}(k)) - \underline{\ell}\,(\underline{h}'\underline{x}(k) \cancel{+ i\, u(k)}$$
$$- \underline{h}'\underline{\xi}(k) \cancel{- i\, u(k)})$$

$$= F\underline{e}(k) - \underline{\ell}\,\underline{h}'(\underline{x}(k) - \underline{\xi}(k))$$

$$\Rightarrow \boxed{\underline{e}(k+1) = (F - \underline{\ell}\,\underline{h}')\,\underline{e}(k)}$$

$\lim_{k\to\infty} \underline{e}(k) \longrightarrow \phi$ <u>iff</u> $(F - \underline{\ell}\,\underline{h}')$ is stable.

Lemma: The poles of $(F - \underline{\ell}\,\underline{h}')$ can be chosen freely as long as the system is completely observable.

Proof (by construction)

- Bring the system into observer-canonical form.

$$\hat{F} = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & \emptyset & -a_1 \\ & 1 & \ddots & & \vdots \\ & \emptyset & \ddots & 0 & \\ & & & 1 & -a_{n-1} \end{bmatrix} \qquad \hat{\underline{g}} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \qquad \hat{\underline{\ell}} = \begin{bmatrix} \hat{\ell}_1 \\ \hat{\ell}_2 \\ \vdots \\ \hat{\ell}_n \end{bmatrix}$$

$$\hat{\underline{h}}' = [\,\emptyset \; \cdots \; \emptyset \;\; 1\,]$$

$$\Rightarrow \hat{\underline{\ell}}\,\hat{\underline{h}}' = \begin{bmatrix} & \hat{\ell}_1 \\ \emptyset & \hat{\ell}_2 \\ & \vdots \\ & \hat{\ell}_n \end{bmatrix}$$

$$\Rightarrow (\hat{F} - \hat{\underline{\ell}}\,\hat{\underline{h}}') = \begin{bmatrix} 0 & & & & -(a_0+\hat{\ell}_1) \\ 1 & 0 & & \emptyset & -(a_1+\hat{\ell}_2) \\ & 1 & \ddots & & \vdots \\ & \emptyset & \ddots & 0 & \\ & & & 1 & -(a_{n-1}+\hat{\ell}_n) \end{bmatrix}$$

$\Rightarrow$ yields one equation for each of the $\hat{l}_i$

q.e.d.

This looks very similar to the controller design. Indeed, if the "observer"-design is done on the dual system:

$$\left|\begin{array}{l} F \to F' \\ \underline{g} \to \underline{h}' \\ \underline{h} \to \underline{g}' \\ i \to i \end{array}\right|$$

$$\Rightarrow \quad (F' - \underline{l}\cdot\underline{h}') \equiv (\underbrace{F - \underline{g}\,\underline{l}'}_{\text{pole placement problem.}})'$$

$\Rightarrow$ The observer-problem and the pole-placement-problem are

Lemma: The observer modes are all uncontrollable.

Proof:

System: $\left| \begin{array}{l} \underline{x}(k+1) = F\,\underline{x}(k) + \underline{g}\,u(k) \\ y(k) = \underline{h}'\,\underline{x}(k) + i\,u(k) \end{array} \right|$

Observer: $\left| \begin{array}{l} \underline{\xi}(k+1) = F\,\underline{\xi}(k) + \underline{g}\,u(k) + \underline{\ell}\big(y(k) - \hat{y}(k)\big) \\ \hat{y}(k) = \underline{h}'\,\underline{\xi}(k) + i\,u(k) \end{array} \right|$

Controller: $\left| \; u(k) = r(k) - \underline{k}' \cdot \underline{\xi}(k) \; \right|$

Let us choose the following state vector of our overall system:

$$\underline{\varphi}(k) = \begin{bmatrix} \underline{x}(k) \\ \underline{e}(k) \end{bmatrix}$$

where: $\underline{e}(k) = \underline{x}(k) - \underline{\xi}(k)$

We know already that:

$$\underline{e}(k+1) = (F - \underline{\ell}\,\underline{h}') \cdot \underline{e}(k)$$

$$\underline{x}(k+1) = F \cdot \underline{x}(k) + \underline{g}\left(r(k) - \underline{k}'\, \underline{\xi}(k)\right)$$

$$\underline{\xi}(k) = \underline{x}(k) - \underline{e}(k)$$

$$\Rightarrow \underline{x}(k+1) = F \cdot \underline{x}(k) + \underline{g} \cdot r(k) - \underline{g}\,\underline{k}'\,\underline{x}(k) + \underline{g}\,\underline{k}'\,\underline{e}(k)$$

$$\Rightarrow \underline{x}(k+1) = (F - \underline{g} \cdot \underline{k}')\,\underline{x}(k) + \underline{g}\,\underline{k}'\,\underline{e}(k) + \underline{g} \cdot r(k)$$

$$\Rightarrow \left|\begin{array}{l} \underline{\zeta}(k+1) = \left[\begin{array}{c|c} (F - \underline{g}\,\underline{k}') & \underline{g}\,\underline{k}' \\ \hline \phi & (F - \underline{\ell}\,\underline{h}') \end{array}\right] \underline{\zeta}(k) + \left[\begin{array}{c} \underline{g} \\ \hline \phi \end{array}\right] \cdot r(k) \\ y(k) = \left[\begin{array}{c|c} (\underline{h}' - i\,\underline{k}') & i\,\underline{k}' \end{array}\right] \underline{\zeta}(k) + \left[\, i \,\right] \cdot r(k) \end{array}\right|$$

$\Rightarrow$ From the structure, it becomes obvious that:

(1) the F_{CL}-matrix is block-triangular

$$\Rightarrow \operatorname{eig}(F_{CL}) = \Big\{ \underbrace{\operatorname{eig}(F - \underline{g}\,\underline{k}')}_{\text{controller modes}} ; \underbrace{\operatorname{eig}(F - \underline{\ell}\,\underline{h}')}_{\text{observer modes}} \Big\}$$

(2) the observer-modes are uncontrollable.

<u>Lemma</u>: The particular set of state variables chosen e.g. for the control (feedback) is immaterial.

<u>Proof</u> (trivial): As the observer is only fed by inputs and outputs, the actual states do not appear explicitely. Thus, it is perfectly okay to write:

System:
$$\left| \begin{array}{l} \underline{x}(k+1) = F \cdot \underline{x}(k) + \underline{g} \cdot u(k) \\ y(k) = \underline{h}' \cdot \underline{x}(k) + i \cdot u(k) \end{array} \right|$$

Observer:
$$\left| \begin{array}{l} \underline{\xi}(k+1) = \hat{F} \cdot \underline{\xi}(k) + \underline{\hat{g}} \cdot u(k) + \underline{\hat{\ell}}(y - \hat{y}) \\ \hat{y}(k) = \underline{\hat{h}}' \cdot \underline{\xi}(k) + i \cdot u(k) \end{array} \right|$$

Controller:
$$\left| \; u(k) = r(k) - \underline{\hat{k}}' \cdot \underline{\xi}(k) \; \right|$$

<u>where</u>:
$$\left| \begin{array}{l} \hat{F} = T \cdot F \cdot T^{-1} \\ \underline{\hat{g}} = T \cdot \underline{g} \\ \underline{\hat{h}}' = \underline{h}' \cdot T^{-1} \end{array} \right|$$

that is: we can <u>choose</u> a base representation, and then transform everything into this one representation.

<u>E.g.</u> We might choose the controller-canonical form as a base representation

$\Longrightarrow$ The $\underline{k}'$-vector found earlier will be <u>correct</u> as it assumes the $\underline{S}(k)$ in this form.

$\Longrightarrow$ The $\hat{\underline{\ell}}$-vector, however, will be <u>incorrect</u>, as it assumes the $\hat{\underline{S}}(k)$ in observer-canonical form.

$\Longrightarrow$ We need to transform it back to controller-canonical form.

Let us assume, we were in the "beginning" in controller-canonical form ($\underline{x}$ are state-variables in controller-canonical form)

$$\Rightarrow \quad \underline{z} = T \cdot \underline{x}$$

is the transformation that gets us into observer-canonical form.

After this transformation, we can build an observer for $\underline{z}$:

$$\underline{\zeta}(k+1) = \hat{F} \cdot \underline{\zeta}(k) + \underline{\hat{g}}\, u(k) + \underline{\hat{\ell}}\,(y(k) - \hat{y}(k))$$

Obviously, if $\underline{\zeta}$ observes $\underline{z}$

$$\Rightarrow \quad \underline{\xi} = T^{-1} \cdot \underline{\zeta} \quad \text{observes } \underline{x}$$

$$\Rightarrow \quad \underline{\zeta}(k) = T \cdot \underline{\xi}(k)$$

$$\Rightarrow T\,\underline{\xi}(k+1) = \hat{F}\cdot T\,\underline{\xi}(k) + \underline{\hat{g}}u(k) + \underline{\hat{\ell}}\,(y(k)-\hat{y}(k))$$

$$\Rightarrow \underline{\xi}(k+1) = \underbrace{T^{-1}\cdot\hat{F}\cdot T}_{F}\cdot\underline{\xi}(k) + \underbrace{T^{-1}\underline{\hat{g}}}_{\underline{g}}\,u(k) + T^{-1}\hat{\ell}\,(y(k)-\hat{y}(k))$$

is an observer for $\underline{x}(k)$

$$\Rightarrow \underline{\xi}(k+1) = F\cdot\underline{\xi}(k) + \underline{g}u(k) + \underline{\ell}\,(y(k)-\hat{y}(k))$$

$$\Rightarrow \boxed{\underline{\ell} = T^{-1}\underline{\hat{\ell}}}$$

is the back-transformation into the original (controller-canonical) coordinates.