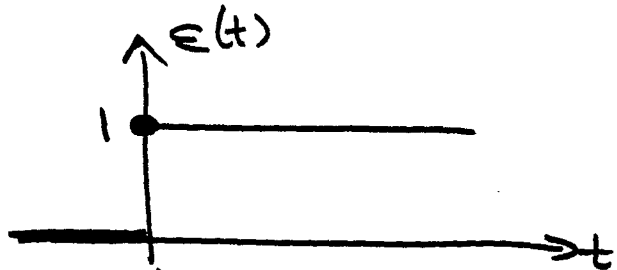
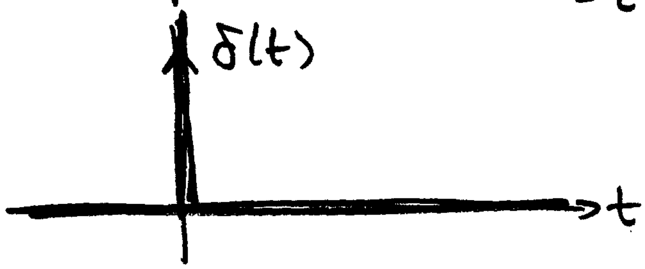
Analysis of sampled data:

ef:



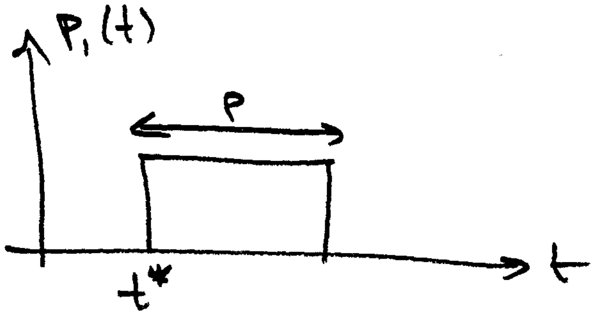
step function



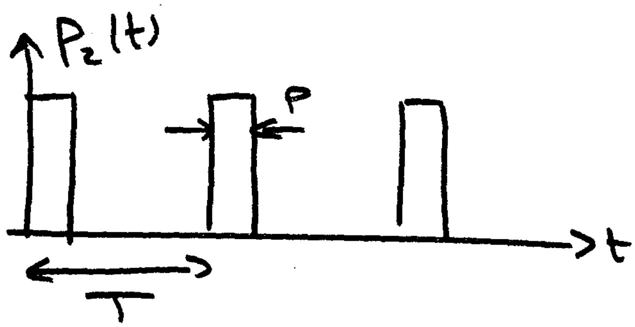
Impulse function

$$\int_{-\infty}^{\infty} \delta(\tau)\,d\tau = 1$$

⇒



$$\Rightarrow \quad p_1(t) = \varepsilon(t - t^*) - \varepsilon(t - t^* - p)$$



$$p_2(t) = \sum_{k=0}^{\infty} \left\{ \varepsilon(t - kT) - \varepsilon(t - kT - p) \right\} ; \; p < T$$

Sometimes, it is preferred to start the pulse train at time $-\infty$:

$$p_3(t) = \sum_{k=-\infty}^{\infty} \{ \varepsilon(t-kT) - \varepsilon(t-kT-p) \} ;$$

$$p<T$$

$p_3(t)$ is periodic in T, thus:

$$p_3(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t} \quad ; \quad \omega_s = \frac{2\pi}{T}$$

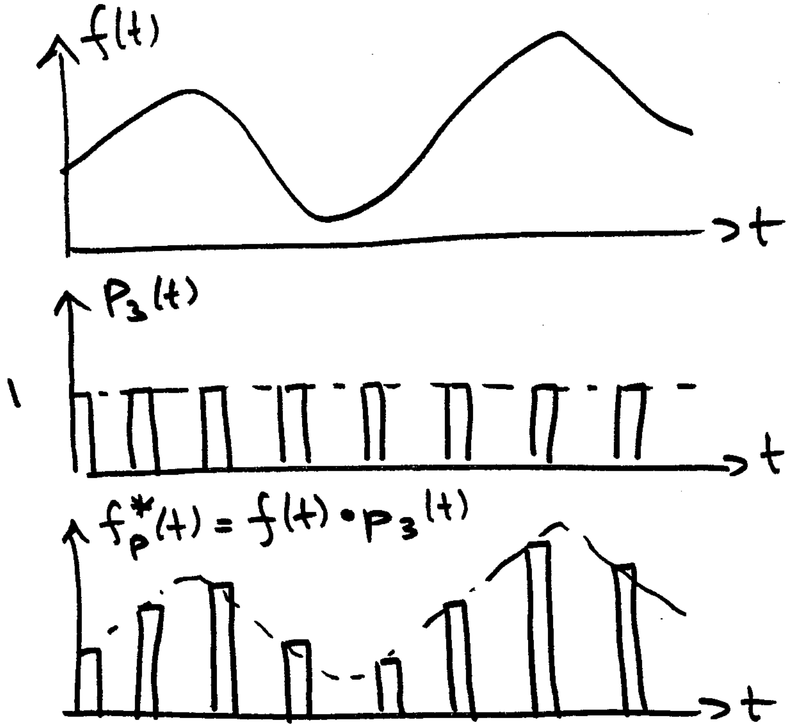
$$C_n = \frac{1}{T} \int_0^T p(\tau) e^{-jn\omega_s \tau} d\tau$$

(Fourier decomposition)

$$\Rightarrow C_n = \frac{1}{T} \int_0^p e^{-jn\omega_s \tau} d\tau = \frac{1 - e^{-jn\omega_s p}}{jn\omega_s T}$$

$$= \left(\frac{p}{T}\right) \cdot \text{sinc}\left(n\omega_s \frac{p}{2}\right) e^{-jn\omega_s \frac{p}{2}}$$

$$\Rightarrow p_3(t) = \left(\frac{p}{T}\right) \sum_{n=-\infty}^{\infty} \text{sinc}\left(n\omega_s \frac{p}{2}\right) e^{jn\omega_s \left(t - \frac{p}{2}\right)}$$



$$f_p^*(t) = f(t)\cdot p_3(t) = \sum_{n=-\infty}^{\infty} c_n f(t) e^{jn\omega_s t}$$

$$\Rightarrow F_p^*(j\omega) = \int_{-\infty}^{\infty} f_p^*(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n f(\tau) e^{jn\omega_s \tau} \right) e^{-j\omega\tau} d\tau$$

$$= \sum_{n=-\infty}^{\infty} c_n \left\{ \int_{-\infty}^{\infty} f(\tau) e^{jn\omega_s \tau} \cdot e^{-j\omega\tau} d\tau \right\}$$

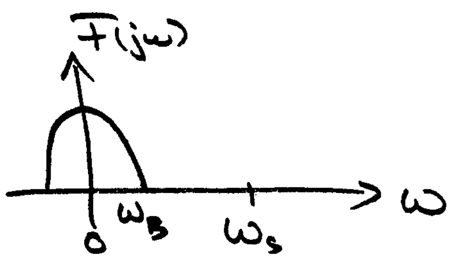
but: $\quad F(j\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau$

$$\Rightarrow \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} \cdot e^{jn\omega_s\tau} d\tau = F(j\omega - jn\omega_s)$$

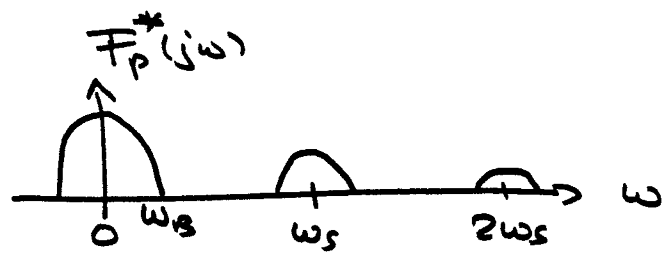
(Shifting Theorem)

$$F_p^*(j\omega) = \sum_{n=-\infty}^{\infty} C_n F(j\omega - jn\omega_s)$$

$$\left[= \sum_{n=-\infty}^{\infty} C_n F(j\omega + jn\omega_s) \right]$$



$\Rightarrow$



A low pass filter can recuperate the original signal <u>iff</u>

$$\underline{\underline{\omega_s \geq 2\omega_B}}$$

$\Rightarrow$ Sampling Theorem.

f(t) contains "information". If we do not want to loose any information, we must sample sufficiently fast:

$$\underline{\underline{T}} = \frac{2\pi}{\omega_s} \leq \frac{2\pi}{2\omega_B} = \underline{\underline{\frac{\pi}{\omega_B}}}$$

where ω_B is the bandwidth of $f(t)$.

⇒ The sampling period is limited from both sides:

$$T_{min} \leq T \leq T_{max}$$

T_{min} ↑ Technology (Converters, Computer Speed)

T_{max} ↑ Signals / Systems (Information) (Stability)

For control purposes, the Laplace transform is more useful than the Fourier analysis

$$F_p^*(s) = \mathcal{L}\{f_p^*(t)\} = \mathcal{L}\{f(t)\cdot p_2(t)\}$$

$$\Rightarrow F_p^*(s) = F(s) * P_2(s)$$

$$p_2(t) = \sum_{k=0}^{\infty}\{\varepsilon(t-kT) - \varepsilon(t-kT-p)\}$$

$$\Rightarrow P_2(s) = \sum_{k=0}^{\infty}\left\{\frac{1}{s}e^{-kTs} - \frac{1}{s}e^{-(kT+p)s}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{1-e^{-ps}}{s}\cdot e^{-kTs}$$

$$= \frac{1-e^{-ps}}{s}\left[\underbrace{1 + e^{-Ts} + e^{-2Ts} + \cdots}_{x}\right]$$

$$x = 1 + e^{-Ts} + e^{-2Ts} + \cdots$$

$$e^{-Ts} \cdot x = e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots$$

$$x - e^{-Ts} \cdot x = 1$$

$$\Rightarrow x\left(1 - e^{-Ts}\right) = 1$$

$$\Rightarrow x = \frac{1}{1 - e^{-Ts}}$$

$$\Rightarrow P_2(s) = \frac{1 - e^{-ps}}{s\left(1 - e^{-Ts}\right)}$$

The convolution integral is a little messy. It seems easier to carry out the multiplication in the time domain:

$$f_p^*(t) \cong \sum_{k=0}^{\infty} f(kT)\left\{\varepsilon(t-kT) - \varepsilon(t-kT-p)\right\}$$

assuming that $p \ll T$.

$$\Rightarrow F_p^*(s) = \sum_{k=0}^{\infty} \frac{1-e^{-ps}}{s} f(kT)\, e^{-kTs}$$

$$= \frac{1-e^{-ps}}{s} \sum_{k=0}^{\infty} f(kT)\, e^{-kTs}$$

With an ideal sampler:

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\,\delta(t-kT)$$

$$\Rightarrow F^*(s) = \sum_{k=0}^{\infty} f(kT)\, e^{-kTs}$$

$$\Rightarrow \boxed{F_p^*(s) = \frac{1-e^{-ps}}{s} F^*(s)}$$

Simplification if $p \ll T$:

$$1-e^{-ps} = 1-\left[1-ps+\frac{p^2s^2}{2}-\frac{p^3s^3}{6}\cdots\right]$$

$$\cong ps$$

$$\Rightarrow \boxed{\begin{array}{l} F_p^*(s) \approx p F^*(s) \\ \Longleftrightarrow \; f_p^*(t) \approx p f^*(t) \end{array}}$$

For most purposes, we shall neglect the influence of the finite value of p.

- Let us now look what happens to the signal $f_h(t)$ after the hold:

$$f_h(t) = \sum_{k=0}^{\infty} f(kT) \left\{ \varepsilon(t-kT) - \varepsilon(t-(k+1)T) \right\}$$

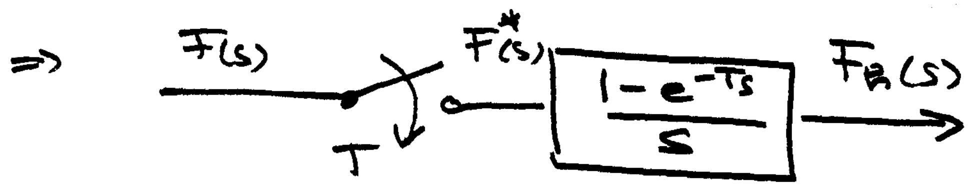
$$\Rightarrow F_h(s) = \sum_{k=0}^{\infty} f(kT) \left\{ \frac{1}{s} e^{-kTs} - \frac{1}{s} e^{-(k+1)Ts} \right\}$$

$$= \sum_{k=0}^{\infty} f(kT) \cdot \frac{1}{s} \left(1 - e^{-Ts} \right) e^{-kTs}$$

$$= \frac{1 - e^{-Ts}}{s} \sum_{k=0}^{\infty} f(kT) e^{-kTs}$$

$$\Rightarrow \quad F_h(s) = \frac{1-e^{-Ts}}{s} \cdot F^*(s)$$

$\Rightarrow$



Laplace transform of ZOH

$$G_h(s) = \mathcal{L}\{ZOH\} = \frac{1-e^{-Ts}}{s}$$

$\Rightarrow$ <u>Linear operation</u>

Notice:

$$g_h(t) = \mathcal{L}^{-1}\{G_h(s)\} = \varepsilon(t) - \varepsilon(t-T)$$