Sampling of Arbitrary Signals

Given $f(t) \Rightarrow$ What is $F^*(s)$?

Example: $f(t) = \varepsilon(t) \Rightarrow F(s) = \frac{1}{s}$

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\,\delta(t-kT) = \sum_{k=0}^{\infty} \delta(t-kT)$$

$$\Rightarrow F^*(s) = \sum_{k=0}^{\infty} e^{-kTs} = 1 + e^{-Ts} + e^{-2Ts} + \dots$$

$$= \frac{1}{1-e^{-Ts}}$$

Question: Is there a simple way to compute $F^*(s)$ out of $F(s)$?

Answer: For poles of $F(s)$ with multiplicity 1 (simple poles):

$$F(s) = \frac{P(s)}{Q(s)} \Rightarrow F^*(s) = \sum_{n=1}^{\overset{\#\text{ of poles}}{k}} \frac{P(s_n)}{Q'(s_n)} \cdot \frac{1}{1-e^{-T(s-s_n)}}$$

Looks like coefficients of the partial fraction expansion.

example: $F(s) = \frac{1}{s}$ $\Rightarrow$ one pole at zero

$$\Rightarrow P(s) = 1 \; ; \; Q(s) = s \Rightarrow Q'(s) = 1$$

$$\Rightarrow F^*(s) = \sum_{n=1}^{1} \frac{1}{1} \cdot \frac{1}{1 - e^{-T(s-0)}} = \frac{1}{1 - e^{-Ts}}$$

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example: $F(s) = \frac{1}{s(s+1)}$ $\Rightarrow$ two poles, one at zero, one at (-1)

$$P(s) = 1 \; ; \; Q(s) = s^2 + s \Rightarrow Q'(s) = 2s+1$$

$$s_1 = 0 \; ; \; s_2 = -1$$

$$F^*(s) = \frac{P(s_1)}{Q'(s_1)} \cdot \frac{1}{1 - e^{-T(s-s_1)}} + \frac{P(s_2)}{Q'(s_2)} \cdot \frac{1}{1 - e^{-T(s-s_2)}}$$

$$= \frac{1}{1} \cdot \frac{1}{1 - e^{-Ts}} + \frac{1}{(-1)} \cdot \frac{1}{1 - e^{-T(s+1)}}$$

$$= \frac{1}{(1 - e^{-Ts})} - \frac{1}{(1 - e^{-T} \cdot e^{-Ts})}$$

$$= \frac{1 - e^{-T} \cdot e^{-Ts} - 1 + e^{-Ts}}{(1 - e^{-Ts}) \cdot (1 - e^{-T} \cdot e^{-Ts})} = \frac{(1 - e^{-T}) e^{-Ts}}{(1 - e^{-Ts})(1 - e^{-T} \cdot e^{-Ts})}$$

It turns out that, in both examples, $F^*(s)$ is a function of the term (e^{-Ts}).

The rule is pretty obvious and easy to derive. We use the partial fraction expansion on $F(s)$:

$$F(s) = \sum_{n=1}^{k} \frac{P(s_n)}{Q'(s_n)} \cdot \frac{1}{s - s_n}$$

When going from $F(s) \longrightarrow F^*(s)$, sums map into sums (as this happens also when going through the time domain). $\Longrightarrow$ We need only to find $F^*(s)$ for $\frac{a}{s-b}$, and then we know the general formulae for all $F(s)$ with simple poles.

$$F(s) = \frac{a}{s-b} \Rightarrow F^*(s) = \frac{a}{1 - e^{-T(s-b)}}$$

as can be easily verified by looking at the infinite series.

Similar for multiple poles (multiplicity of pole n is m_n):

$$F^*(s) = \sum_{n=1}^{k} \cdot \sum_{i=1}^{m_n} \frac{(-1)^{m_n - i} K_{ni}}{(m_n - i)!} \cdot \frac{\partial^{m_n - i} \Delta T(s)}{\partial s^{m_n - i}} \Bigg|_{s=s-s_n}$$

where: $$K_{ni} = \frac{1}{(i-1)!} \cdot \frac{\partial^{i-1}\left[(s-s_n)^{m_n} F(s)\right]}{\partial s^{i-1}} \Bigg|_{s=s_n}$$

and: $$\Delta T(s) = \frac{1}{1 - e^{-Ts}}$$

example: $$F(s) = \frac{2}{(s+a)^3}$$

$$\Rightarrow \quad s_1 = -a \; ; \; m_1 = 3$$

$$K_{11} = \frac{1}{1} \cdot (s+a)^3 F(s) \Big|_{s=-a} = 2 \Big|_{s=-a} = 2$$

$$K_{12} = \frac{1}{1} \cdot \frac{\partial}{\partial s}(2) \Big|_{s=-a} = 0$$

$$K_{13} = \frac{1}{2} \frac{\partial^2 (2)}{\partial s^2} \Big|_{s=-a} = 0$$

Only the first term gives a contribution
$\Longrightarrow$

$$F^*(s) = \frac{(-1)^2 \cdot 2}{(3-1)!} \cdot \frac{\partial^2 \Delta T(s)}{\partial s^2}\Bigg|_{s=s-s_n}$$

$$\frac{\partial}{\partial s}\Delta T(s) = \frac{\partial}{\partial s}\left(\frac{1}{1-e^{-Ts}}\right) = \frac{-Te^{-Ts}}{(1-e^{-Ts})^2}$$

$$\frac{\partial^2}{\partial s^2}\Delta T(s) = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\Delta T(s)\right) = \frac{\partial}{\partial s}\left(\frac{-Te^{-Ts}}{(1-e^{-Ts})^2}\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

$$u = -Te^{-Ts} \Rightarrow u' = T^2 e^{-Ts}$$

$$v = (1-e^{-Ts})^2 \Rightarrow v' = 2(1-e^{-Ts})Te^{-Ts}$$

$$\Rightarrow \frac{\partial}{\partial s}\left(\frac{-Te^{-Ts}}{(1-e^{-Ts})^2}\right) = \frac{T^2e^{-Ts}(1-e^{-Ts})^{\cancel{2}} + Te^{-Ts}\cdot 2\cancel{(1-e^{-Ts})}Te^{-Ts}}{(1-e^{-Ts})^{\cancel{4}\,3}}$$

$$= \frac{T^2 e^{-Ts}(1+2e^{-Ts})}{(1-e^{-Ts})^3}$$

evaluate at $s = s - s_n = s + a$:

$$F^*(s) = \frac{T^2 e^{-T(s+a)}\left(1+2e^{-T(s+a)}\right)}{\left(1-e^{-T(s+a)}\right)^3}$$

$$F^*(s) = \frac{T^2 e^{-Ts} e^{-aT}\left(1+2e^{-aT}\cdot e^{-Ts}\right)}{\left(1-e^{-aT}\cdot e^{-Ts}\right)^3}$$

Again, we obtained an expression in e^{-Ts}.

Let us set: $z = e^{Ts} \Longleftrightarrow z^{-1} = e^{-Ts}$

$$F^*(s) \equiv \mathcal{F}(z^{-1}) = \frac{T^2 z^{-1} e^{-aT}\left(1+2e^{-aT}z^{-1}\right)}{\left(1-e^{-aT}z^{-1}\right)^3}$$

$$\equiv \mathcal{F}(z) = \frac{T^2 z e^{-aT}\left(z+2e^{-aT}\right)}{\left(z-e^{-aT}\right)^3}$$

$\Rightarrow$ $f(t) \circ\!\!-\!\!\bullet F(s)$ Laplace Transform

or: $f(t) \circ\!\!-\!\!\bullet \tilde{F}(s^{-1})$ Reverse Laplace Transform

$f(t) \circ\!\!-\!\!\bullet F(z)$ Z-Transform

or: $f(t) \circ\!\!-\!\!\bullet \tilde{F}(z^{-1})$ Reverse Z-Transform

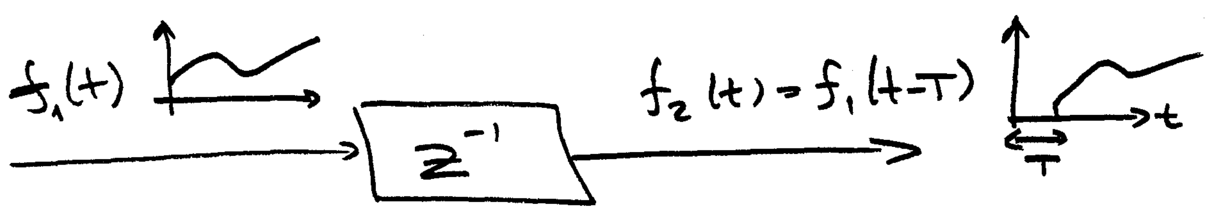
ike the Laplace transform, also the z-transform has a physical meaning:

$$s \;\hat{=}\; \text{Derivative } \left(\text{"}\tfrac{d}{dt}\text{"}\right)$$

$$\tfrac{1}{s} = s^{-1} \;\hat{=}\; \text{Integration } (\text{"}\textstyle\int\text{"})$$

$$z \;\hat{=}\; \text{Left Shifting by } T$$

$$\tfrac{1}{z} = z^{-1} \;\hat{=}\; \text{Right Shifting by } T$$



Signal got delayed by T.

Like in the Laplace transform, physicality requires higher order denominators than numerators.

Computational Rules for the z-Transform:

(Proof is straight forward from the definition)

Linearity:

$$\mathcal{Z}\{a\cdot f(t)+b\cdot g(t)\} = a\mathcal{Z}\{f(t)\}+b\mathcal{Z}\{g(t)\}$$

Shifting Property:

$$\mathcal{Z}\{f(t-kT)\} = z^{-k}\cdot\mathcal{Z}\{f(t)\}$$

$$\mathcal{Z}\{f(t+kT)\} = z^{k}\cdot\mathcal{Z}\{f(t)\} + \text{I.C.}$$

Damping Property:

$$\mathcal{Z}\{e^{-at}\cdot f(t)\} = \mathcal{F}(ze^{aT})$$

where: $\mathcal{Z}\{f(t)\} = \mathcal{F}(z)$

1) <u>Initial Value Theorem</u>:

$$\lim_{t \to 0} f(t) \equiv \lim_{z \to \infty} \mathcal{F}(z)$$

if the right limes exists

<u>Proof</u>:

$$\mathcal{F}(z) = \sum_{k=0}^{\infty} f(kT) \cdot z^{-k}$$

$$= f_0 + f_1 \cdot z^{-1} + f_2 \cdot z^{-2} + \cdots$$

$$\{ f_0, \ f_1, \ f_2, \ \cdots \ \}$$

$$\equiv \{ f(0), \ f(T), \ f(2T), \ \cdots \ \}$$

<u>Obviously</u>: $\lim_{z \to \infty} \mathcal{F}(z) = f_0 = f(0)$

if the series converges

Final Value Theorem:

$$\lim_{t\to\infty} f(t) \equiv \lim_{z\to 1} \frac{z-1}{z}\mathcal{F}(z)$$

if the left limes exists.

Proof:

Let us look at the two finite series:

$$(1) \qquad \mathcal{F}_a(z) = \sum_{k=0}^{n} f(kT)z^{-k} = f_0 + f_1 z^{-1} + \dots + f_n z^{-n}$$

$$(2) \qquad \mathcal{F}_b(z) = \sum_{k=1}^{n} f((k-1)T)z^{-k} = f_0 z^{-1} + f_1 z^{-2} + \dots + f_{n-1} z^{-n}$$

Obviously, $\mathcal{F}_b(z)$ can also be written as:

$$\mathcal{F}_b(z) = z^{-1}\sum_{k=0}^{n-1} f(kT)z^{-k}$$

Now, we look at:

$$\lim_{z\to 1}\left\{\mathcal{F}_a(z) - \mathcal{F}_b(z)\right\}$$

$$= \lim_{z\to 1}\left\{\sum_{k=0}^{n} f(kT)z^{-k} - z^{-1}\sum_{k=0}^{n-1} f(kT)z^{-k}\right\}$$

$$= \lim_{z\to 1} \left\{ \sum_{k=0}^{n} f(kT) - \sum_{k=0}^{n-1} f(kT) \right\}$$

$$= f(nT)$$

$$\lim_{t\to\infty} f(t) \equiv \lim_{t\to\infty} f^*(t) \equiv \lim_{n\to\infty} f(nT)$$

$$\lim_{n\to\infty} f(nT) = \lim_{n\to\infty} \left\{ \lim_{z\to 1} \left\{ \sum_{k=0}^{n} f(kT) z^{-k} - z^{-1} \sum_{k=0}^{n-1} f(kT) z^{-k} \right\}\right\}$$

$$= \lim_{z\to 1} \left\{ \lim_{n\to\infty} \left\{ \sum_{k=0}^{n} f(kT) z^{-k} - z^{-1} \sum_{k=0}^{n-1} f(kT) z^{-k} \right\}\right\}$$

$$= \lim_{z\to 1} \left\{ \sum_{k=0}^{\infty} f(kT) z^{-k} - z^{-1} \cdot \sum_{k=0}^{\infty} f(kT) z^{-k} \right\}$$

$$= \lim_{z\to 1} \left\{ (1 - z^{-1}) \mathcal{F}(z) \right\}$$

$$= \lim_{z\to 1} \left\{ \frac{z-1}{z} \cdot \mathcal{F}(z) \right\} = \lim_{z\to 1} \left\{ (z-1) \mathcal{F}(z) \right\}$$

q.e.d.

Physical Interpretation:

$$z = e^{sT}$$

$$\Rightarrow \quad s \to 0 \iff z \to 1$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

s: differentiation operator

What is a "difference" operator:

$$\mathcal{Z}\{f(t)\} = \mathcal{F}(z)$$

$$\Rightarrow \quad \mathcal{Z}\{f(t-T)\} = z^{-1}\mathcal{F}(z)$$

$$\Rightarrow \quad \mathcal{Z}\{\underbrace{f(t)-f(t-T)}_{\Delta f(t)}\} = \mathcal{F}(z) - z^{-1}\mathcal{F}(z)$$

$$= (1-z^{-1})\mathcal{F}(z)$$

$$= \frac{z-1}{z}\mathcal{F}(z)$$