f) Differentiation Property:

$$\mathcal{Z}\left\{ \frac{\partial}{\partial a}\big(f(t,a)\big) \right\} = \frac{\partial}{\partial a}\big(\mathcal{F}(z,a)\big)$$

g) Convolution Properties:

$$\mathcal{Z}\{ f(t) * g(t) \} = \mathcal{F}(z) \cdot \mathcal{G}(z)$$

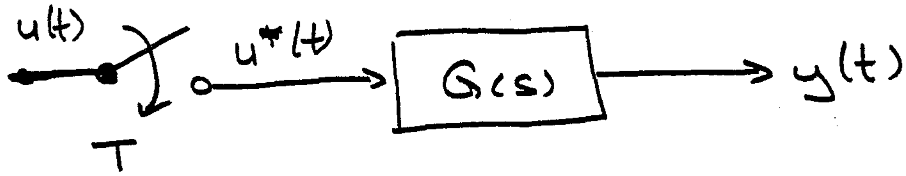
$$\mathcal{Z}\{ f(t) \cdot g(t) \} = \mathcal{F}(z) * \mathcal{G}(z)$$

$\Rightarrow$ We can really compute with z-Transforms exactly the same way as we are used to do with Laplace-Transforms:

$$\mathcal{U}(z) \longrightarrow \boxed{\mathcal{G}(z)} \longrightarrow \mathcal{Y}(z)$$

$$\Rightarrow \mathcal{Y}(z) = \mathcal{G}(z) \cdot \mathcal{U}(z)$$

However, this requires some words of interpretation.



$$\Rightarrow \quad u^*(t) = \sum_{k=0}^{\infty} u(kT) \cdot \delta(t - kT)$$

Due to linearity, we can superimpose the individual Dirac responses.

$$\Rightarrow \quad y(t) = \sum_{k=0}^{\infty} u(kT) \cdot g(t - kT)$$

If $y(t)$ is only considered at the sampling times, we obtain:

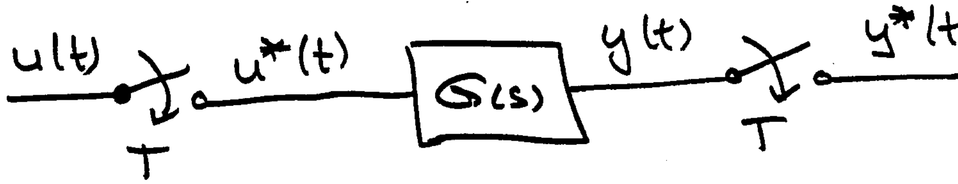
$$y(mT) = \sum_{k=0}^{\infty} u(kT) \cdot g(mT - kT)$$

$\Rightarrow$ This is a discrete convolution of u with g

$$\Rightarrow y(mT) = y^*(t) = g^*(t) * u^*(t)$$

$$\Rightarrow \mathcal{Y}(z) = \mathcal{G}(z) \cdot \mathcal{U}(z)$$

$\Rightarrow$ To be able to use this convenient calculus, we need to sample (but not hold) <u>all</u> signals:



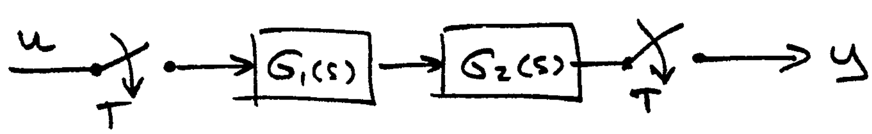
$$\Rightarrow \quad \mathcal{Y}(z) = \mathcal{Z}\{y^*(t)\} = \mathcal{G}(z) \cdot \mathcal{U}(z)$$

<u>where</u>: $\mathcal{U}(z) = \mathcal{Z}\{u^*(t)\}$

<u>and</u>: $\mathcal{G}(z) = \mathcal{Z}\{g^*(t)\}$

<u>where</u>: $g(t) = \mathcal{L}^{-1}\{G(s)\}$

$\Rightarrow$

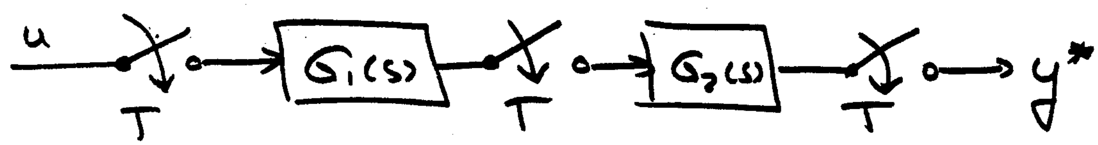


$$\Rightarrow \quad \mathcal{Y}(z) = \mathcal{G}_{12}(z) \cdot \mathcal{U}(z)$$

<u>where</u>: $\mathcal{G}_{12}(z) = \mathcal{Z}\{g_{12}^*(t)\}$

<u>where</u>: $g_{12}(t) = \mathcal{L}^{-1}\{G_1(s) \cdot G_2(s)\}$

but:



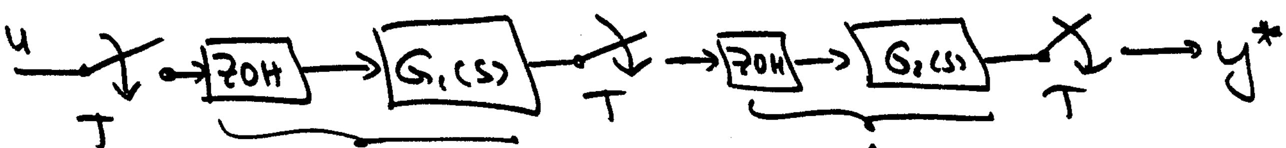
$$\mathcal{Y}(z) = \mathcal{G}_2(z) \cdot \mathcal{G}_1(z) \cdot \mathcal{U}(z)$$

and:

$$\mathcal{G}_2(z) \cdot \mathcal{G}_1(z) \neq \mathcal{G}_{12}(z)$$

If there is a ZOH inbetween, just add its transfer function in the Laplace domain to the plant.

Thus:



$$\hat{G}_1(s) = G_1(s) \cdot G_R(s) \; ; \quad \hat{G}_2(s) = G_2(s) \cdot G_R(s)$$

$$\hat{G}_1(s) \rightarrow \hat{g}_1(t) \rightarrow \hat{\mathcal{G}}_1(z) \qquad \hat{G}_2(s) \rightarrow \hat{g}_2(t) \rightarrow \hat{\mathcal{G}}_2(z)$$

$$\Rightarrow \mathcal{Y}(z) = \hat{\mathcal{G}}_2(z) \cdot \hat{\mathcal{G}}_1(z) \cdot \mathcal{U}(z)$$

A slight simplification is possible due to the shifting theorem:

$$G_h(s) = \frac{1-e^{-Ts}}{s}$$

$$\Rightarrow \hat{G}_1(s) = G_1(s) \cdot G_h(s) = \frac{G_1(s)}{s} - e^{-Ts} \cdot \frac{G_1(s)}{s}$$

$$\Rightarrow \hat{g}_1(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G_1(s)}{s}\right\}\right\} - \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{e^{-Ts} \cdot \frac{G_1(s)}{s}\right\}\right\}$$

$$= \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G_1(s)}{s}\right\}\right\} - z^{-1} \cdot \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G_1(s)}{s}\right\}\right\}$$

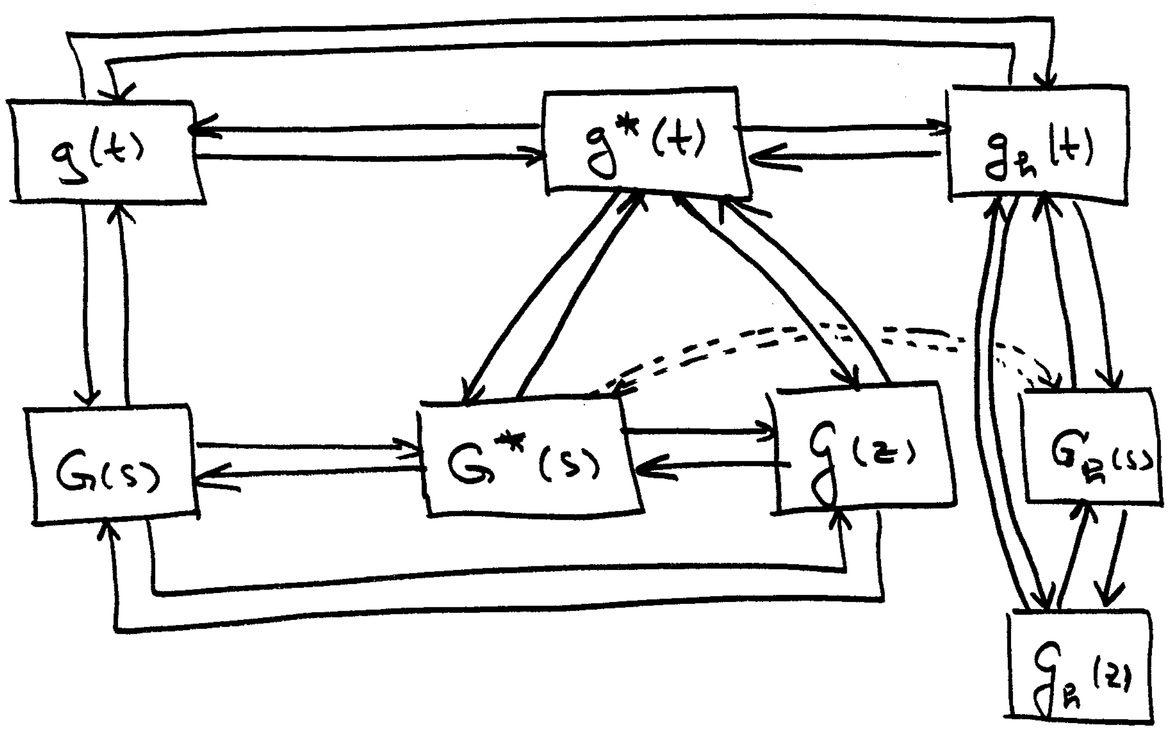
$$\Rightarrow \underline{\underline{\hat{g}_1(z) = (1-z^{-1}) \cdot \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G_1(s)}{s}\right\}\right\}}}$$

For simplification, we are now going to define:

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\{G(s)\}\right\} \equiv \mathcal{Z}\{G(s)\}$$

$$\Rightarrow \underline{\underline{\hat{g}_1(z) = (1-z^{-1}) \cdot \mathcal{Z}\left\{\frac{G_1(s)}{s}\right\}}}$$

Transformations Between the Data Representations Met So far.



Questions: How do these different representations belong together? How can we find one of them if another one of them is given? How does the transformation work mathematically, and how does it work physically?

1) 

a) mathematically: evaluate $g(t)$ at the time instants kT

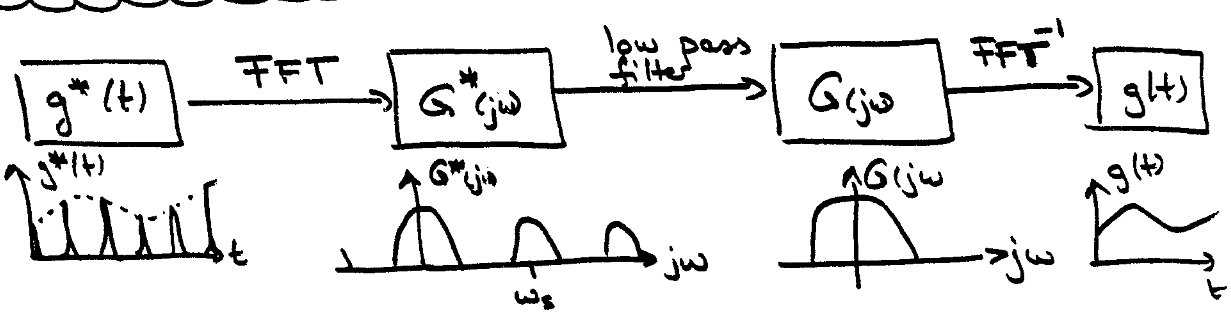
t	$g(t)$
0	$g(0)$
T	$g(T)$
$2T$	$g(2T)$
$3T$	$g(3T)$
$\vdots$	$\vdots$

$\Leftarrow g^*(t)$

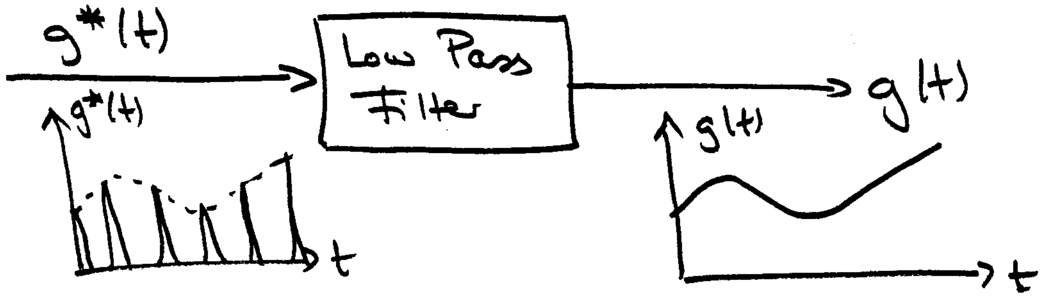
b) physically: by a PAM device.

2) $\boxed{g^*(t)} \longrightarrow \boxed{g(t)}$

a) mathematically:



b) physically:

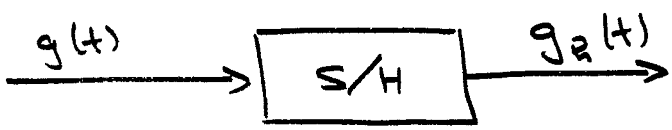


③

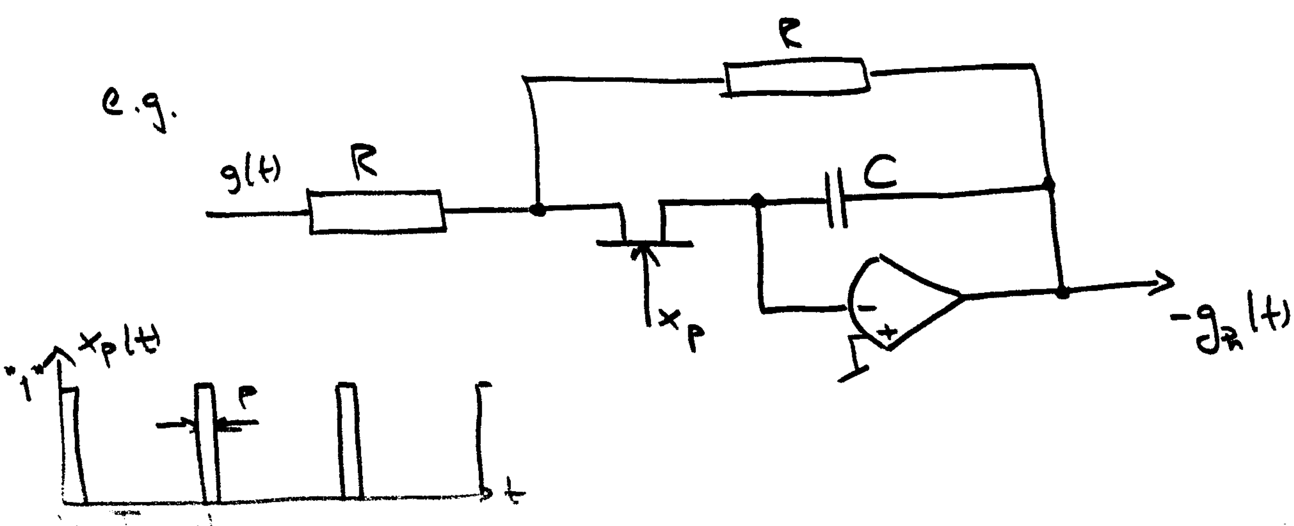
a) mathematically:

same as ①

b) physically:



e.g.



$X_p =$ "1" : MOSFET switch is closed $\Longrightarrow$ The capacitor is charged to $g(kT)$ with a time constant $T_c = R \cdot C$

$X_p =$ "0" : MOSFET switch is open $\Longrightarrow$ The amplifier works as an integrator with open input $\Longrightarrow$ output stays constant. C is discharged through the (high) input impedance of the amplifier. ($T_c^* = R_A \cdot C$)
large

Choose: $R \cdot C \ll p \ll T \ll R_A \cdot C$

4 $\boxed{g_R(t)} \longrightarrow \boxed{g(t)}$

a) mathematically:
Same as 2

b) Physically:

$$\xrightarrow{g_R(t)} \boxed{\text{PAM}} \xrightarrow{g^*(t)} \boxed{\text{Low Pass Filter}} \xrightarrow{g(t)}$$

5) $\boxed{g(t)} \longrightarrow \boxed{G(s)}$

a) mathematically:

$$\xrightarrow{g(t)} \boxed{\mathcal{L}} \longrightarrow G(s)$$

cf. ECE 441

b) physically:

α) $g(t)$ represents a signal:

$$\xrightarrow{g(t)} \boxed{\text{FFT}} \longrightarrow G(j\omega)$$

Fast Fourier Transform is a discrete convolution. There exist very efficient special purpose chips for that purpose.

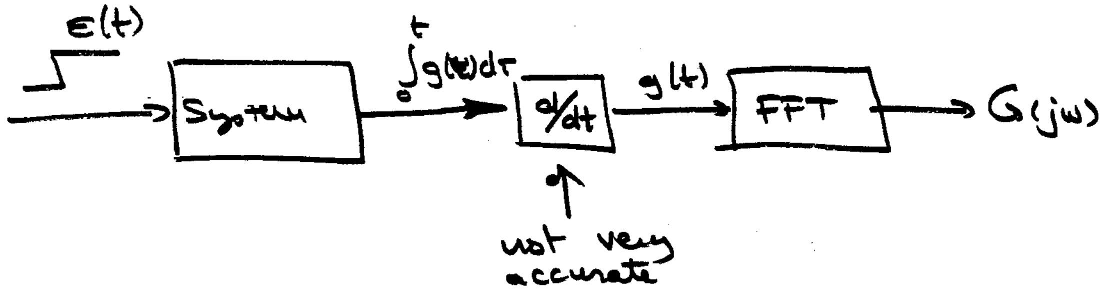
β) $g(t)$ represents the impulse response of a system.

Solution: Several ways are possible:

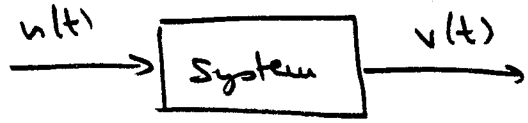
(1) $\xrightarrow{\delta(t)} \boxed{\text{System}} \xrightarrow{g(t)} \boxed{\text{FFT}} \longrightarrow G(j\omega)$

never very accurate

(2)



(3)



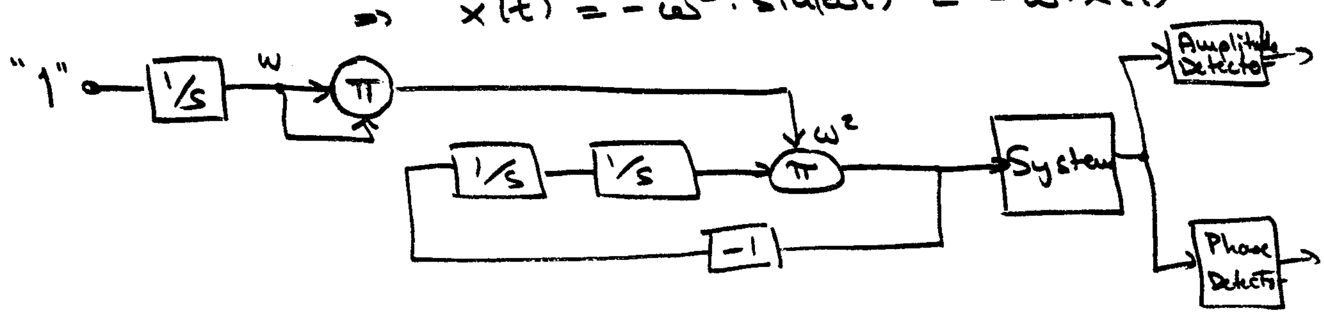
$n(t)$: white, gaussian noise
$v(t)$: colored noise

$$\varrho(n,v) \equiv g(t) \qquad (\mu\text{-processor})$$

The cross correlation between the input and the output is a very accurate $g(t)$ $\Rightarrow$ Then FFT.

(4) Direct determination of the Bode diagram:

$$x(t) = \sin(\omega t)$$
$$\Rightarrow \quad \ddot{x}(t) = -\omega^2 \cdot \sin(\omega t) = -\omega^2 \cdot x(t)$$



Some companies are specialized in computerized equipment for this sort of signal processing (e.g. Hewlett Packard).

6) $\boxed{G^*(s)} \longrightarrow \boxed{G(z)}$

simply replace $e^{-Ts} \longrightarrow z^{-1}$

7) $\boxed{G(z)} \longrightarrow \boxed{G^*(s)}$

simply replace $z^{-1} \longrightarrow e^{-Ts}$

8) $\boxed{G(s)} \longrightarrow \boxed{G(z)}$

a) Use formulae directly as given before.

b) do partial fraction expansion of $G(s)$, and map each term separately.

$$\frac{k}{s+a} \longrightarrow \frac{kz}{z-b} \; ; \; b = e^{-aT}$$

$$\frac{k}{(s+a)^2} \longrightarrow \frac{kTbz}{(z-b)^2} \; ; \; b = e^{-aT}$$

etc.

(9) $$\boxed{G(z)} \longrightarrow \boxed{G(s)}$$

Best way is to use now a partial fraction expansion of $G(z)$

but:
$$\frac{k}{z+a} \longrightarrow \; ?$$

As seen from previous page, we know that:

$$\frac{kz}{z+a} \longrightarrow \frac{k}{s-b} \; ; \quad b = \frac{1}{T}\ln(-a)$$

Trick:
$$\hat{G}(z) = \frac{G(z)}{z}$$

We decompose $\hat{G}(z)$ by partial fraction expansion:

e.g. $$\hat{G}(z) = \frac{A}{z+a} + \frac{B}{z+b} + \frac{C_1}{z+c} + \frac{C_2}{(z+c)^2}$$

$$\Rightarrow G(z) = z \cdot \hat{G}(z) = \frac{Az}{z+a} + \frac{Bz}{z+b} + \frac{C_1 z}{z+c} + \frac{C_2 z}{(z+c)^2}$$

Now we know how to map back.