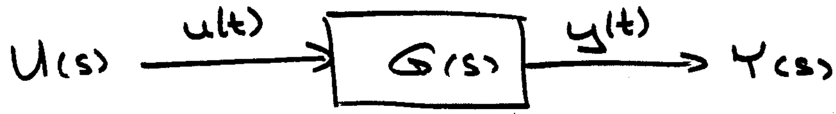
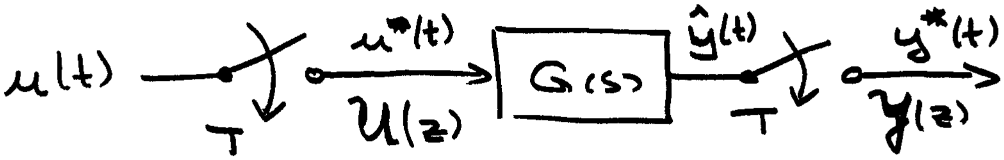
Let us recapture:



$$\Rightarrow \quad G(s) = \frac{Y(s)}{U(s)}$$

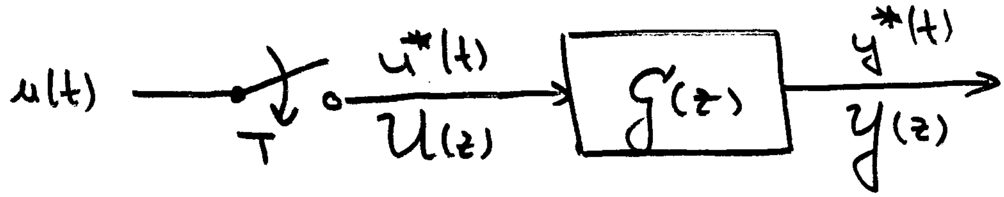
$$U(s) = \mathcal{L}\{u(t)\}$$
$$Y(s) = \mathcal{L}\{y(t)\}$$



$$\hat{y}(t) \neq y(t)$$

$$\mathcal{U}(z) = \mathcal{Z}\{u^*(t)\}$$
$$\mathcal{Y}(z) = \mathcal{Z}\{y^*(t)\}$$

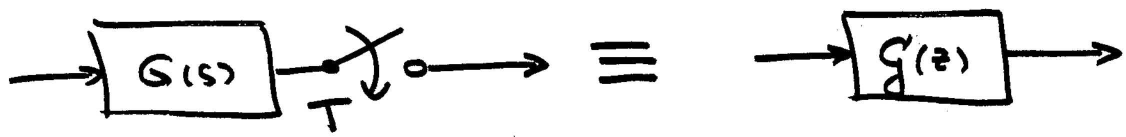
$$\equiv$$



$$\mathcal{G}(z) = \frac{\mathcal{Y}(z)}{\mathcal{U}(z)}$$

<u>Notice</u>: $u^*(t)$ is sampled $u(t)$
$y^*(t)$ is sampled $\hat{y}(t) \neq y(t)$

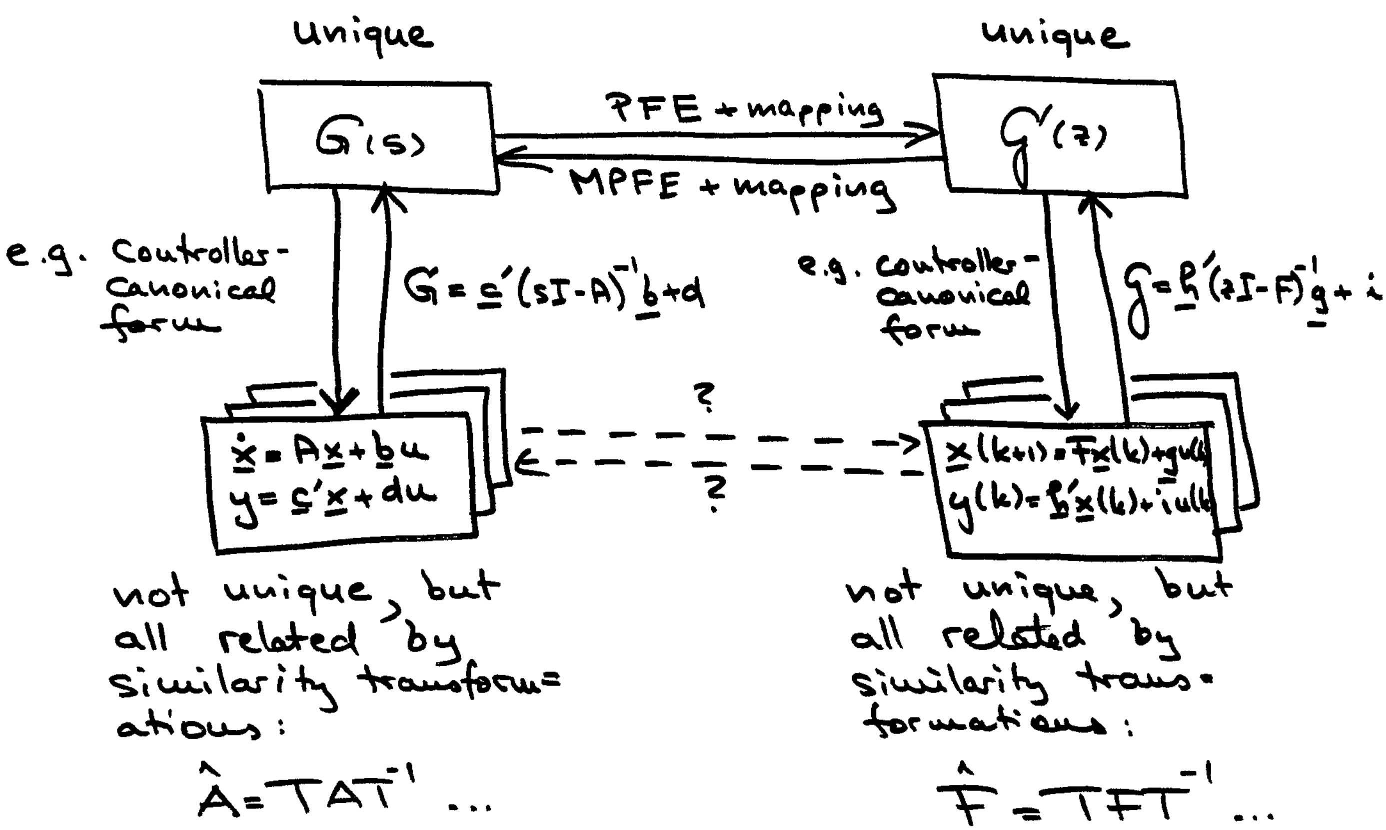
The identity:



is <u>exact</u>, iff

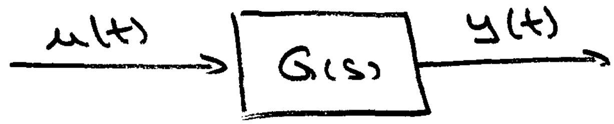
→ the sampler is "ideal", that is: $p \ll T$, and

→ the quantization error can be neglected, that is $2^{-n} \cdot V_{FS} \ll V_{FS}$.



What relation exists between $A \longleftrightarrow F$, etc. ?

- Let us start with the continuous system:



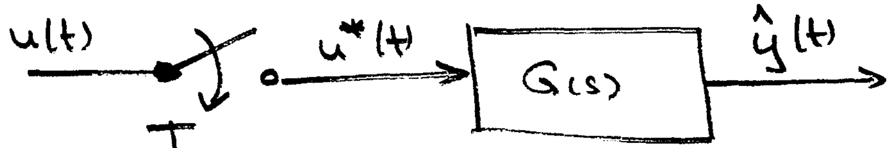
$$\equiv$$

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}\,u \\ y = \underline{c}'\underline{x} + d\,u \end{array} \right|$$

$$\Rightarrow \underline{x}(t) = e^{A(t-t_0)} \cdot \underline{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\, \underline{b}\, u(\tau)\, d\tau$$

(usually simplified for the special case: $t_0 \equiv 0$)

- Let us now introduce a sampler:



$$\Rightarrow \hat{\underline{x}}(t) = e^{A(t-t_0)} \cdot \underline{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\, \underline{b}\, u^*(\tau)\, d\tau$$

We choose:

$$t_0 := kT$$
$$t := kT + \Delta t \quad ; \quad \Delta t \in [0, T]$$

$$\Rightarrow \hat{\underline{x}}(kT+\Delta t) = e^{A\Delta t} \cdot \underline{x}(kT) + \int_{kT}^{kT+\Delta t} e^{A(kT+\Delta t-\tau)} \underline{b}\, u^*(\tau)$$

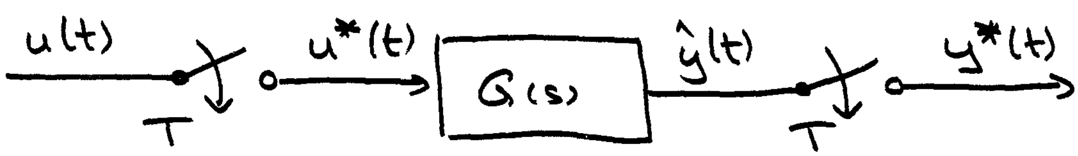
We substitute: $\sigma = \tau - kT$

$$\Rightarrow \tau = \sigma + kT$$
$$d\tau = d\sigma$$

τ	σ
kT	0
$kT+\Delta t$	Δt

$$\Rightarrow \hat{\underline{x}}(kT+\Delta t) = e^{A\Delta t} \cdot \underline{x}(kT) + \int_{0}^{\Delta t} e^{A(\Delta t-\sigma)} \underline{b}\, u^*(\sigma)\, d\sigma$$

- Now, we introduce a second sampler:



We choose: $\Delta t := T$

$$\Rightarrow \underline{x}^*\big((k+1)T\big) = e^{AT} \cdot \underline{x}^*(kT) + \int_0^T e^{A(T-\sigma)}\, \underline{b}\, u^*(\sigma)\, d\sigma$$

for $\Delta t \in [0, T]$

$$\Rightarrow u^*(\tau) = u(kT) \cdot \delta(\tau - kT)$$

$$\Rightarrow u^*(\sigma) = u(kT) \cdot \delta(\sigma)$$

Due to the blending property of the Dirac function:

$$\int_0^T e^{A(T-\sigma)} \cdot \underline{b} \cdot u(kT) \cdot \delta(\sigma) \cdot d\sigma$$

$$\equiv e^{AT} \cdot \underline{b} \cdot u(kT)$$

$$\Rightarrow \underline{x}^*\big((k+1)T\big) = e^{AT} \cdot \underline{x}^*(kT) + e^{AT} \cdot \underline{b} \cdot u(kT)$$

$$\equiv F \cdot \underline{x}^*(kT) + \underline{g} \cdot u(kT)$$

Output equation:

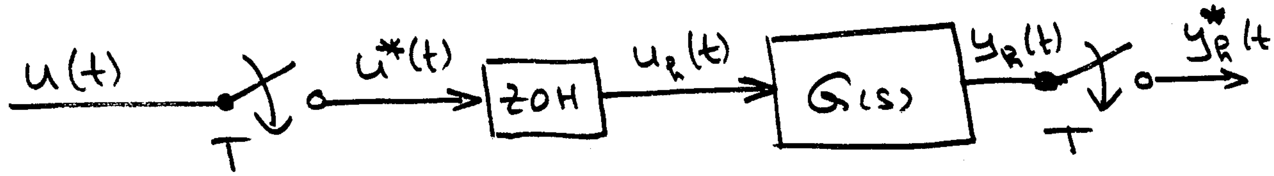
$$y(t) = \underline{c}' \underline{x}(t) + d\, u(t)$$

$$\Rightarrow y^*(kT) = \underline{c}' \underline{x}^*(kT) + d\, u(kT)$$

$$\equiv \underline{h}' \underline{x}^*(kT) + i\, u(kT)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \underline{F} &= e^{AT} \\ \underline{g} &= e^{AT} \cdot \underline{b} \\ \underline{h}' &= \underline{c}' \\ i &= d \end{aligned}}$$

How about introducing a ZOH:



for $\Delta t \in [0, T]$

$$u_h(\tau) = u(kT) = u_h(\sigma)$$

$$\Rightarrow \underline{x}^*((k+1)T) = e^{AT} \cdot \underline{x}^*(kT) + \underbrace{\int_0^T e^{A(T-\sigma)} \underline{b}\, u(kT)\, d\sigma}$$

$$\int_0^T e^{A(T-\sigma)} \underline{b}\, u(kT)\, d\sigma \equiv \int_0^T e^{AT} \cdot e^{-A\sigma} \cdot \underline{b} \cdot u(kT)\, d\sigma$$

$$\equiv e^{AT} \cdot \underbrace{\int_0^T e^{-A\sigma} d\sigma} \cdot \underline{b} \cdot u(kT)$$

Remember: $$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

$$\Rightarrow \frac{d}{dt}(e^{At}) = A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \dots$$

$$\equiv A \cdot \left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right]$$

$$\equiv \left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right] \cdot A$$

$$\Rightarrow \frac{d}{dt}(e^{At}) \equiv A \cdot e^{At} \equiv e^{At} \cdot A$$

Correspondingly:

$$\int e^{At} dt \equiv A^{-1} \cdot e^{At} \equiv e^{At} \cdot A^{-1}$$

$$\Rightarrow \int_0^T e^{-A\sigma} d\sigma = e^{-A\sigma}\Big|_0^T \cdot (-A)^{-1} = \left[e^{-AT} - I\right](-A)^{-1}$$

$$= \left[I - e^{-AT}\right] \cdot A^{-1}$$

$$\Rightarrow e^{AT} \cdot \int_0^T e^{-A\sigma} d\sigma \cdot \underline{b} \cdot u(kT) = e^{AT} \cdot \left[I - e^{-AT}\right] \cdot A^{-1} \cdot \underline{b} \cdot u(k$$

$$= \left[e^{AT} - I\right] \cdot A^{-1} \cdot \underline{b} \cdot u(kT)$$

$$\Rightarrow \underline{x}^*((k+1)T) = e^{AT}\cdot \underline{x}^*(kT) + \left[e^{AT} - I\right]\cdot A^{-1}\cdot \underline{b}\cdot u(kT)$$
$$= F\cdot \underline{x}^*(kT) + \underline{g}\cdot u(kT)$$

$$\Rightarrow \boxed{\begin{aligned} F &= e^{AT} \\ \underline{g} &= \left[e^{AT} - I\right]\cdot A^{-1}\cdot \underline{b} \\ \underline{h}' &= \underline{c}' \\ i &= d \end{aligned}}$$

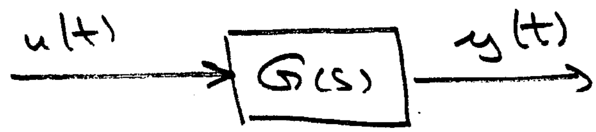
Notice: If A is in diagonal form
$\Longrightarrow F$ will be in diagonal form

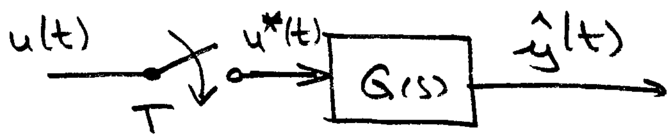
but: A is in controller-canonical form
$\Longrightarrow F$ will not be in controller-canonical form, although there may exist a similarity transformation $\hat{F} = TFT^{-1}$ which gets $\hat{F}$ back into controller-canonical form.

<u>Notice</u>:

$$\left| \begin{aligned} \underline{x}(k+1) &= F\underline{x}(k) + \underline{g}\, u(k) \\ y(k) &= \underline{h}'\underline{x}(k) + i\, u(k) \end{aligned} \right|$$

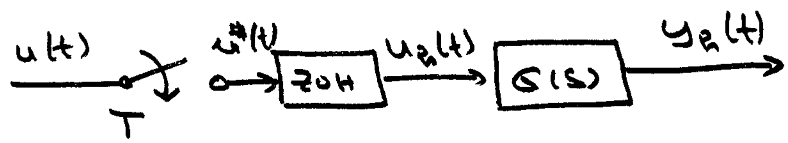
While $u(k)$ denotes $u(t)$ sampled at $t=kT$, $y(k)$ does <u>not</u> denote $y(t)$ sampled at $t=kT$, but rather $\hat{y}(t)$ sampled at $t=kT$:





$$\hat{y}(t) \neq y(t)$$

<u>However</u>:



As $u_R(t) \approx u(t) \implies y_R(t) \approx y(t)$

$$\implies y_R^*(t) \implies y_R(k) \approx y^*(t) \implies y(k)$$

<u>Warning:</u> A may be singular (pole at origin)

$$\Rightarrow \int e^{At}\,dt \neq A^{-1}e^{At}\,,$$ as A^{-1} does not exist.

However, the integral does perfectly exist. We just need another method to compute:

$$\Rightarrow \quad \underline{g} \neq \left[e^{AT} - I\right]\cdot A^{-1}\cdot \underline{b}$$

(cf. ECE 544 for more detail).

<u>Example</u>: Given $G(s) = \dfrac{2(s+3)}{s(s+1)^2(s+5)}$