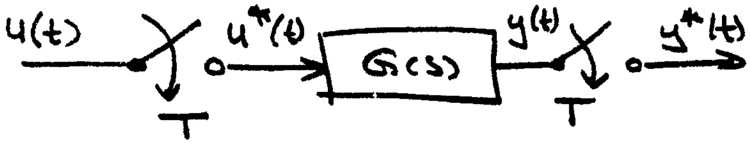
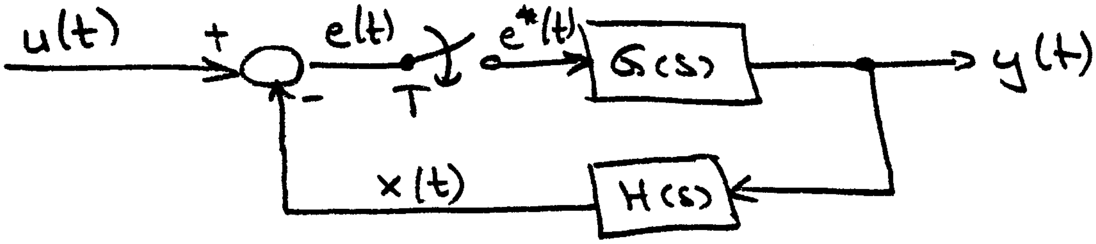
Description of System Topologies by the z-Transform:

So far, we looked only at unstructured systems of the form:



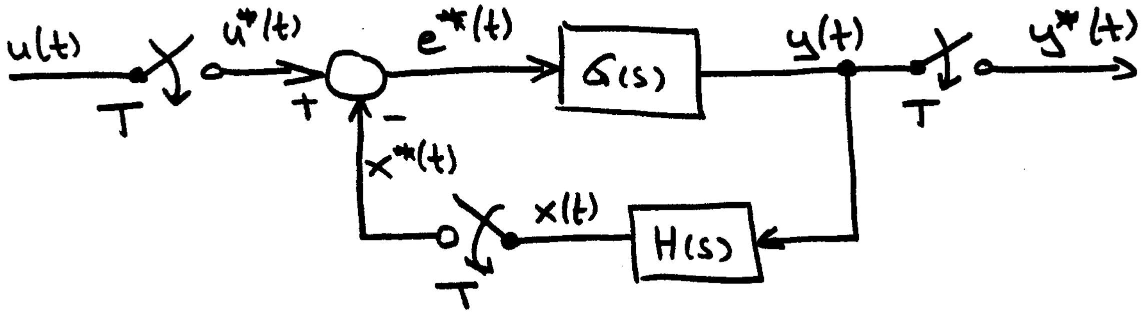
We need to study what happens in the case of composite systems such as:



We start by introducing additional samplers where they do not harm.

E.g. : $e(t) = u(t) - x(t)$

$$\Longrightarrow e^*(t) = u^*(t) - x^*(t)$$



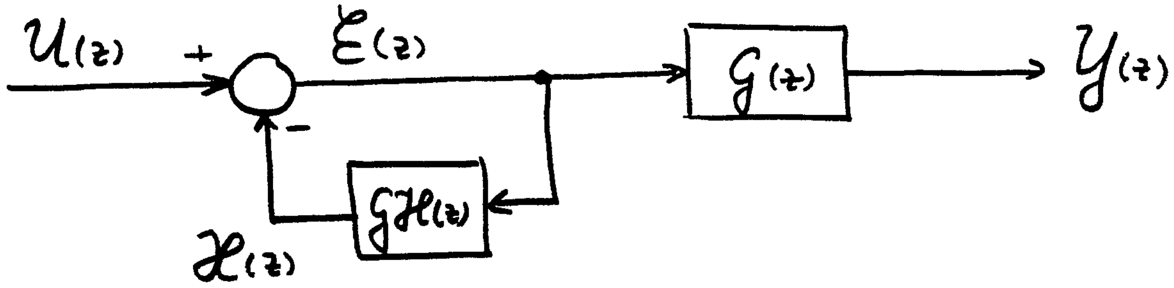
Obviously:

$$\mathcal{Y}(z) = \mathcal{G}(z) \cdot \mathcal{E}(z)$$

$$\mathcal{X}(z) = \mathcal{GH}(z) \cdot \mathcal{E}(z)$$

$$\mathcal{E}(z) = \mathcal{U}(z) - \mathcal{X}(z)$$

where: $\mathcal{GH}(z)$ is the z-Transform of $G(s) \cdot H(s)$



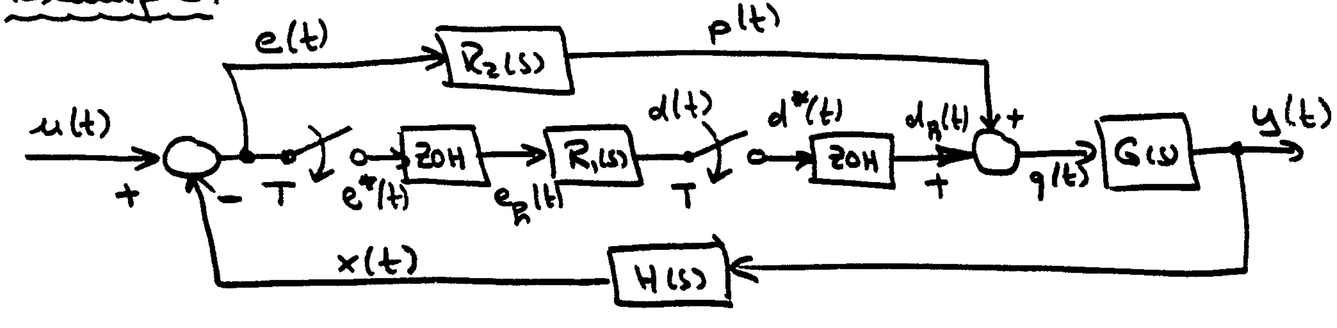
We have been able to transform the mixed-signal block-diagram into one that contains discrete signals only.

From here, we can apply all the known techniques again, such as Mason's rule, etc., and we find e.g. for our example:

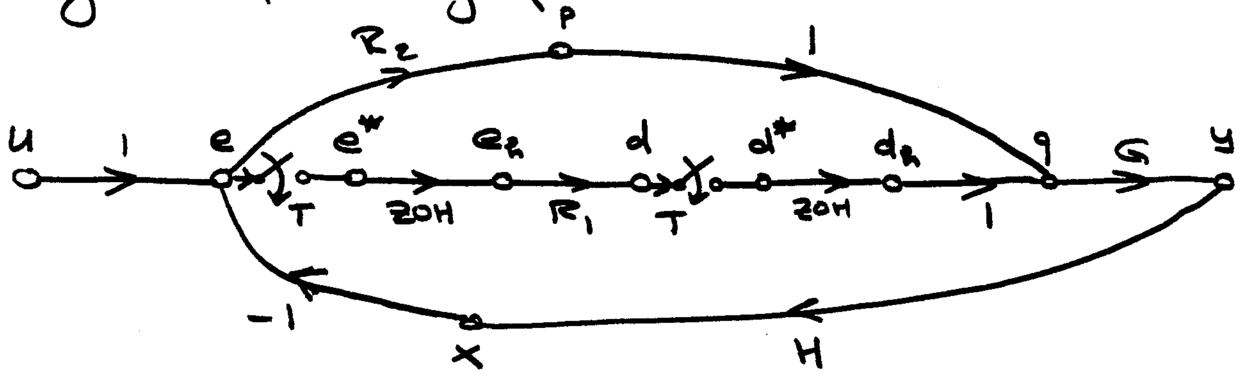
$$\mathcal{Y}(z) = \underbrace{\frac{\mathcal{G}(z)}{1 + \mathcal{GH}(z)}}_{\mathcal{G}_{tot}(z)} \cdot \mathcal{U}(z)$$

This technique is always possible, as long as all samplers operate synchronously with the same frequency.

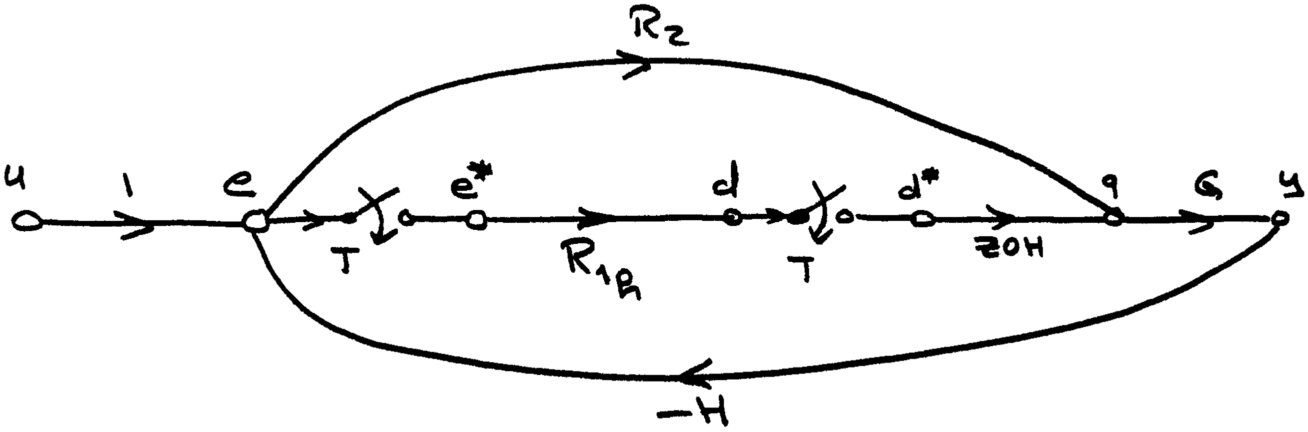
Example:



In more complicated systems, it is sometimes easier to operate on a signal flow graph:



We simplify by eliminating all cascades except when they involve samplers:



$$\left|\begin{array}{l} E(s) = U(s) - H(s)\cdot Y(s) \\ D(s) = R_{1_h}(s)\cdot E^*(s) \\ Q(s) = G_h(s)\cdot D^*(s) + R_2(s)\cdot E(s) \\ Y(s) = G(s)\cdot Q(s) \end{array}\right|$$

$$\Rightarrow E(s) = U(s) - G(s) \cdot H(s) \cdot Q(s)$$

$$= U(s) - G(s) \cdot H(s) \cdot \left[G_h(s) \cdot D^*(s) + R_2(s) \cdot E(s) \right]$$

$$\Rightarrow \left[1 + G(s) \cdot H(s) \cdot R_2(s) \right] E(s) = U(s) - G(s) \cdot H(s) \cdot G_h(s) \cdot D^*(s)$$

$$\Rightarrow E(s) = \frac{1}{1 + R_2 GH} \cdot U(s) - \frac{GHG_h}{1 + R_2 GH} \cdot D^*(s)$$

At sampling points:

$$E^*(s) = \left[\frac{1}{1 + R_2 GH} \right]^* \cdot U^*(s) - \left[\frac{GHG_h}{1 + R_2 GH} \right]^* \cdot D^*(s)$$

$$D^*(s) = R_{1h}^* \cdot E^*(s)$$

$$Y(s) = G(s) \cdot Q(s)$$

$$= G(s) \cdot \left[G_h(s) \cdot D^*(s) + R_2(s) \cdot E(s) \right]$$

$$= GG_h \cdot D^*(s) + R_2 G \left[U(s) - H(s) \cdot Y(s) \right]$$

$$= GG_h \cdot D^*(s) + R_2 G \cdot U(s) - R_2 GH \cdot Y(s)$$

$$\Rightarrow \left[1 + GHR_2 \right] Y(s) = GG_h \cdot D^*(s) + GR_2 \cdot U(s)$$

$$\Rightarrow Y(s) = \frac{GG_h}{1 + GHR_2} \cdot D^*(s) + \frac{GR_2}{1 + GHR_2} \cdot U(s)$$

$$\Rightarrow Y^*(s) = \left[\frac{GG_h}{1 + GHR_2} \right]^* \cdot D^*(s) + \left[\frac{GR_2}{1 + GHR_2} \right]^* \cdot U^*(s)$$

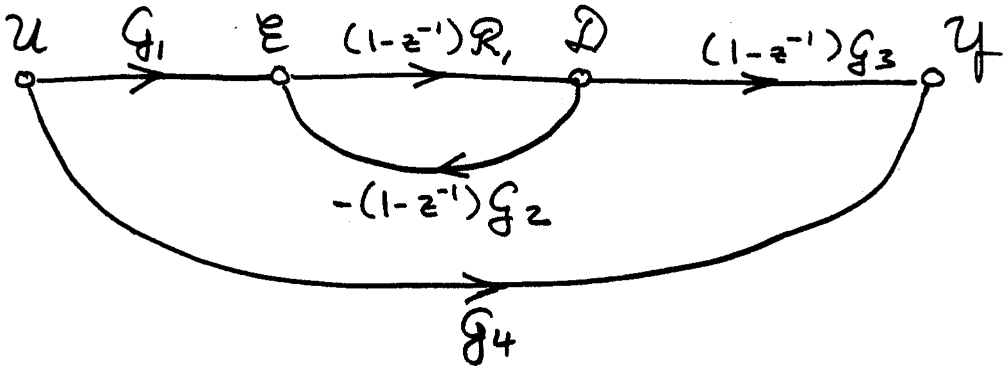
Recipe:

Use the continuous signals, and eliminate intermediate variables until all outputs are written as products of transfer functions with signals that are either true system inputs or outputs of samplers. Then, the entire set of equations can be sampled. Compute transfer functions for all true system outputs and for all inputs of samplers.

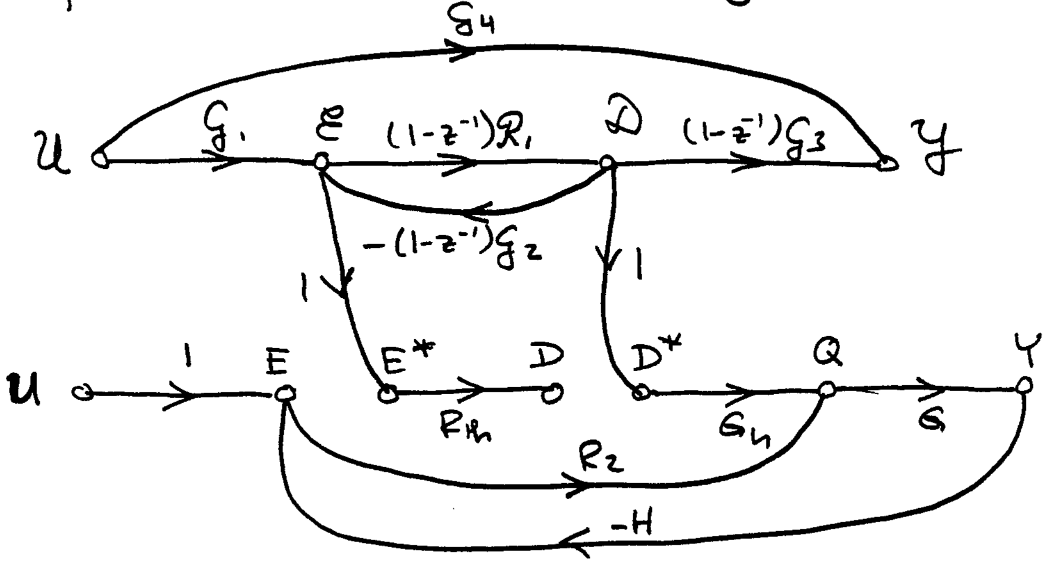
Let: $G_1(s) = \dfrac{1}{1+R_2GH}$; $G_2(s) = \dfrac{GH}{1+R_2GH}$

$G_3(s) = \dfrac{G}{1+R_2GH}$; $G_4(s) = \dfrac{GR_2}{1+R_2GH}$

$$\Rightarrow \left| \begin{aligned} &\mathcal{E}(z) = \mathcal{G}_1(z)\cdot\mathcal{U}(z) - (1-z^{-1})\cdot\mathcal{G}_2(z)\cdot\mathcal{D}(z) \\ &\mathcal{D}(z) = (1-z^{-1})\cdot\mathcal{R}_1(z)\cdot\mathcal{E}(z) \\ &\mathcal{Y}(z) = (1-z^{-1})\cdot\mathcal{G}_3(z)\cdot\mathcal{D}(z) + \mathcal{G}_4(z)\cdot\mathcal{U}(z) \end{aligned} \right|$$



Sometimes, it is desired to recuperate (in the time domain) the continuous signals. This can be achieved by a <u>composite flow graph</u>. Here, the continuous and the discrete flow graph are combined, the samplers are eliminated, and replaced by a feeding of the continuous flow graph from the discrete signals.



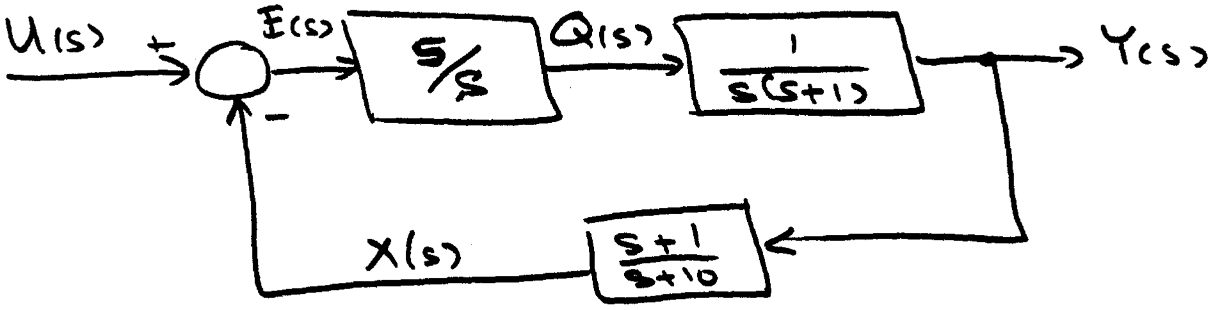
In a sufficiently large circuit, this technique is still too unorganized.

⟹ There exists a direct method (generalization of Mason's Rule) which is presented in Kuo (p. 127-137). This technique is systematic and algorithmic, but also messy. Moreover, it is very easy to forget some paths and/or loops.

⟹ I will not discuss this technique. Instead, I give you a better technique in the time domain.

Remember: For continuous systems, we were able to get immediately a state-space representation from the block diagram by simply denoting all outputs of integrators our state variables, as long as there were no algebraic loops (≡ loops with direct input/output coupling).

Example:



$$Q(s) = \frac{5}{s} \cdot E(s) \Rightarrow sQ(s) = 5E(s)$$

$$\Rightarrow \dot{q}(t) = 5e(t)$$

$$Y(s) = \frac{1}{s(s+1)} Q(s)$$

$$\Rightarrow Y(s)\left[s^2+s\right] = Q(s) \Rightarrow \ddot{y}(t) + \dot{y}(t) = q(t)$$

$$X(s) = \frac{s+1}{s+10} Y(s) = \left[1 - \frac{9}{s+10}\right] Y(s)$$

$$X(s) = X_a(s) + X_b(s)$$

$$X_a(s) = Y(s) \quad ; \quad X_b(s) = -\frac{9}{s+10} Y(s)$$

$$\Rightarrow X_b(s)\left[s+10\right] = -9Y(s)$$

$$\Rightarrow \dot{x}_b(t) + 10x_b(t) = -9\underline{y}(t)$$

$$\Rightarrow \quad \begin{aligned} x_1(t) &= q(t) \\ x_2(t) &= y(t) \\ x_3(t) &= \dot{y}(t) \\ x_4(t) &= x_b(t) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \dot{x}_1 = 5e &= -5x + 5u \\ &= -5x_a - 5x_b + 5u \\ &= -5x_2 - 5x_4 + 5u \end{aligned}$$

$$\begin{aligned} \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_3 + x_1 \\ \dot{x}_4 &= -10x_4 - 9x_2 \end{aligned}$$

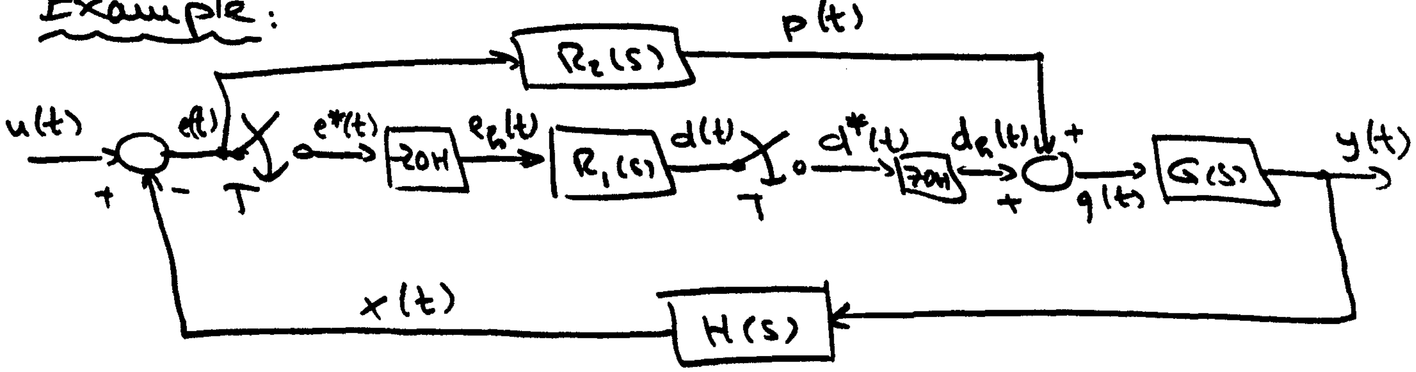
$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & -5 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -9 & 0 & -10 \end{bmatrix} \underline{x} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \underline{x} \end{aligned} \right|$$

Transformation into either controller-canonical form or observer-canonical form will immediately give us the coefficients of the transfer function.

Generalization:

a) Consider all sampled signals (eventually after a ZOH) as additional inputs, and all signals at the input of a sampler as additional outputs.

Example:



Inputs: $\underline{u} = \begin{bmatrix} u \\ e_h \\ d_h \end{bmatrix}$ True inputs first, followed by samplers without ZOH, followed by samplers with ZOH

Outputs: $\underline{y} = \begin{bmatrix} y \\ e \\ d \end{bmatrix}$ True outputs first, followed by samplers in same sequence as above.

This allows us to create a state-space description (continuous) of this MIMO system.

Let $R_1(s) = 5/s$; $R_2(s) = \frac{s}{s+1}$

$G(s) = \frac{1}{s(s+1)}$; $H(s) = \frac{s+1}{s+10}$

$$\Rightarrow \quad \dot{d} = 5 e_R$$

$$P(s) = \frac{s}{s+1} \cdot E(s) = \left[1 - \frac{1}{s+1} \right] E(s) = (P_1(s) + P_2(s)$$

$$\Rightarrow \quad P_2(s)[s+1] = -E(s)$$

$$\Rightarrow \quad \dot{p}_2 + p_2 = -e = -u + x$$

$$p_1 = e = u - x$$

$$x = x_a + x_b$$

$$x_a = y$$

$$\dot{x}_b + 10 x_b = -9y$$

$$\ddot{y} + \dot{y} = q$$

$$q = d_R + p$$

We choose: $\underline{x} = \begin{bmatrix} d \\ p_2 \\ x_b \\ y \\ \dot{y} \end{bmatrix}$

$$\Rightarrow \dot{d} = \dot{x}_1 = 5u_2$$

$$\dot{p}_2 = \dot{x}_2 = -p_2 - u + x = -p_2 - u + x_a + x_b$$

$$= -x_2 - u_1 + \cancel{x_4} + x_3$$

$$\dot{x}_b = \dot{x}_3 = -10x_b - 9y = -10x_3 - 9x_4$$

$$\dot{y} = \dot{x}_4 = x_5$$

$$\ddot{y} = \dot{x}_5 = -\dot{y} + q = -\dot{y} + d_h + p$$

$$= -\dot{y} + d_h + p_1 + p_2$$

$$= -x_5 + u_3 + u_1 - x_4 - x_3 + x_2$$

$$y = y_1 = x_4$$

$$e = y_2 = u - x = u_1 - x_4 - x_3$$

$$d = y_3 = x_1$$

$$\left|\begin{aligned}\dot{\underline{x}} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -10 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 5 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underline{u} \\ \underline{y} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{u} \end{aligned}\right|$$

$$\Rightarrow \left|\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u}\end{aligned}\right|$$

We cut $\underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} \begin{matrix} \Leftarrow \text{without ZOH} \\ \Leftarrow \text{with ZOH} \end{matrix}$

$$\Rightarrow \left|\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B_1\underline{u}_1 + B_2\underline{u}_2 \\ \underline{y} &= C\underline{x} + D_1\underline{u}_1 + D_2\underline{u}_2\end{aligned}\right|$$

<u>Example</u>: $\underline{u}_1 = u$; $\underline{u}_2 = \begin{bmatrix} e_h \\ d_h \end{bmatrix}$

$$\Rightarrow B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad B_2 = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we can sample:

$$F = e^{AT}$$

$$G_1 = e^{AT} \cdot B_1$$

$$G_2 = \left[e^{AT} - I\right] A^{-1} B_2$$

$$G = \left[G_1 \;\vdots\; G_2 \right]$$

$$\Rightarrow \left| \begin{array}{l} \underline{x}(k+1) = F\,\underline{x}(k) + G\,\underline{u}(k) \\ \underline{y}(k) = H\,\underline{x}(k) + I\,\underline{u}(k) \end{array} \right|$$

$$(H \equiv C \; ; \; I \equiv D)$$

Now, we cut: $\underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} \begin{array}{l} \leftarrow \text{true inputs} \\ \leftarrow \text{samplers} \end{array}$

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \begin{array}{l} \leftarrow \text{true outputs} \\ \leftarrow \text{samplers} \end{array}$$

$$\Longrightarrow \underline{u}_2 \equiv \underline{y}_2$$

$$\left(\begin{aligned} &\underline{x}(k+1) = F\cdot\underline{x}(k) + \left[G_1 \vdots G_2\right]\left[\begin{array}{c}\underline{u}_1\\ \hline \underline{u}_2\end{array}\right] \\ &\left[\begin{array}{c}\underline{y}_1(k)\\ \hline \underline{y}_2(k)\end{array}\right] = \left[\begin{array}{c}H_1\\ \hline H_2\end{array}\right]\underline{x}(k) + \left[\begin{array}{c|c}I_{11} & I_{12}\\ \hline I_{21} & I_{22}\end{array}\right]\cdot\left[\begin{array}{c}\underline{u}_1\\ \hline \underline{u}_2\end{array}\right] \end{aligned} \right)$$

$$\underline{x}(k+1) = F\,\underline{x}(k) + G_1\,\underline{u}_1(k) + G_2\,\underline{u}_2(k)$$

$$\underline{y}_1(k) = H_1\,\underline{x}(k) + I_{11}\,\underline{u}_1(k) + I_{12}\,\underline{u}_2(k)$$

$$\underline{y}_2(k) = H_2\,\underline{x}(k) + I_{21}\,\underline{u}_1(k) + I_{22}\,\underline{u}_2(k)$$

$$\underline{u}_2(k) = \underline{y}_2(k) = H_2\,\underline{x}(k) + I_{21}\,\underline{u}_1(k) + I_{22}\,\underline{u}_2(k)$$

$$\Rightarrow \left[I^{(s)} - I_{22}\right)\underline{u}_2(k) = H_2\,\underline{x}(k) + I_{21}\,\underline{u}_1(k)$$

$$\begin{aligned} \Rightarrow \underline{u}_2(k) = {} & \left[I^{(s)} - I_{22}\right]^{-1} H_2\,\underline{x}(k) \\ & + \left\{I^{(s)} - I_{22}\right]^{-1} I_{21}\,\underline{u}_1(k) \end{aligned}$$

<u>Plug in above:</u>

$$\Rightarrow \quad \underline{x}(k+1) = \left[F + G_2\left(I^{(s)} - I_{22}\right)^{-1} H_2\right]\underline{x}(k) + \left[G_1 + G_2\left(I^{(s)} - I_{22}\right)^{-1} I_{21}\right]\underline{u}_1(k)$$

$$\underline{y}_1(k) = \left[H_1 + I_{12}\left(I^{(s)} - I_{22}\right)^{-1} H_2\right]\underline{x}(k) + \left[I_{11} + I_{12}\left(I^{(s)} - I_{22}\right)^{-1} I_{21}\right]\underline{u}_1(k)$$

This is the final discrete state-space representation. This can now be brought into either controller-canonical or observer-canonical form, and then the coefficients of $G(z)$ can be read out directly.

Example: (assume: $\underline{\underline{T = 0.1}}$)

$$F = e^{AT} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.905 & 0.0596 & 0.0631 & 0.0035 \\ 0 & -0.0011 & 0.3688 & -0.568 & -0.0319 \\ 0 & 0.0047 & -0.0034 & 0.9965 & 0.095 \\ 0 & 0.0904 & -0.0561 & -0.0596 & 0.9014 \end{bmatrix}$$

$$G_1 = F \cdot B_1 = \begin{bmatrix} 0 \\ -0.9014 \\ -0.0308 \\ 0.0904 \\ 0.8111 \end{bmatrix}$$

$$G_2 = e^{AT} \cdot \int_0^T e^{-A\sigma} d\sigma \cdot B_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.0001 \\ 0 & -0.0012 \\ 0 & 0.0048 \\ 0 & 0.095 \end{bmatrix}$$

Input splitting remains the same, even for next step.

$$\Rightarrow H_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad I_{11} = \begin{bmatrix} 0 \end{bmatrix}; I_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad I_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; I_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow I^{(2)} - I_{22} \equiv I^{(2)} \Rightarrow \left[I^{(2)} - I_{22} \right]^{-1} = I^{(2)}$$

$$\Rightarrow F_{new} = F + G_2 H_2 = \begin{bmatrix} 1 & 0 & -0.5 & -0.5 & 0 \\ 0.0001 & 0.905 & 0.0596 & 0.0631 & 0.0035 \\ -0.0012 & -0.0011 & 0.3688 & -0.568 & -0.0319 \\ 0.0048 & 0.0047 & -0.0034 & 0.9965 & 0.095 \\ 0.095 & 0.0904 & -0.0561 & -0.0596 & 0.9014 \end{bmatrix}$$

$$G_{new} = G_1 + G_2 I_{21} = \begin{bmatrix} 0.5 \\ -0.9014 \\ -0.0308 \\ 0.0904 \\ 0.8111 \end{bmatrix}$$

$$H_{new} = H_1 + I_{12} H_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$I_{new} = I_{11} + I_{12} I_{21} = [0]$$

$$\Rightarrow \left| \begin{array}{l} \underline{x}(k+1) = F_{new}\, \underline{x}(k) + \underline{g}_{new}\, u(k) \\ y(k) = \underline{h}'_{new}\, \underline{x}(k) + i_{new}\, u(k) \end{array} \right|$$

⇒ Transform into controller-canonical form:

$$\Rightarrow \quad \underline{\xi}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.3001 & -2.0806 & 5.433 & -6.8241 & 4.1716 \end{bmatrix} \underline{\xi}(k)$$

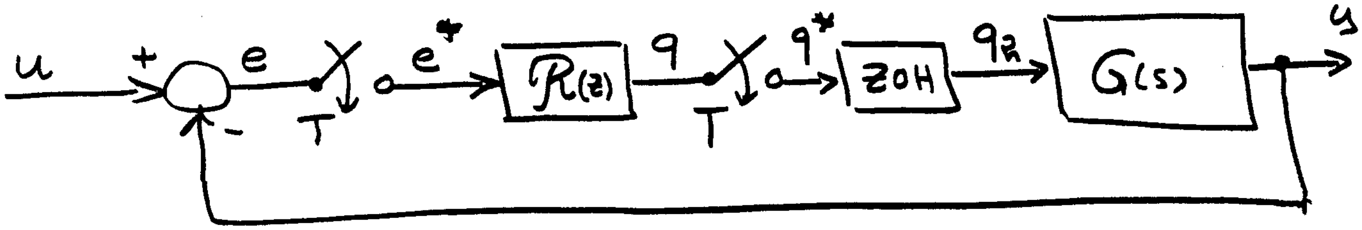
$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.0008 & -0.0354 & 0.1561 & -0.2115 & 0.0904 \end{bmatrix} \underline{\xi}(k)$$

$$\Rightarrow \quad G(z) = \frac{0.0904z^4 - 0.2115z^3 + 0.1561z^2 - 0.0354z + 0.0008}{z^5 - 4.1716z^4 + 6.8241z^3 - 5.433z^2 + 2.0806z - 0.3001} \;//$$

Example:

Given the system:



$$G(s) = \frac{1}{s(s+1)}$$

$$R(z) = \frac{a_2 + a_1 z^{-1}}{1 + b_1 z^{-1}} = \frac{a_2 z + a_1}{z + b_1}$$

Find a state-space representation as a function of the parameters $\{a_1, a_2, b_1, T\}$.